CHAPTER V

ON THE RAMSEY NUMBER N(4,4;3)

5.0 Introduction

From Remark 2.5.4 we obtain the relation $N(4,4;3) \leq N(q_1', q_2'; 2)+1$ where $q_1' = N(3,4;3)$ and $q_2' = N(4,3;3)$. Since $N(3,4;3) = N(4,3;3) = 4$, then $N(4,4;3) \leq N(4,4;2)+1$. By Theorem 3.4.3, we have $N(4,4;2) = 18$. Hence we obtain $N(4,4;3) \leq 19$. In this chapter we shall show that $12 \leq N(4,4;3) \leq 18$.

5.1 Face-Coloring of 3-Graphs

By an r-graph G we mean an ordered pair (S,E), where S is a finite set and E is a set of r-subsets of S. According to this definition, graphs considered in the previous chapter are 2-graphs. Elements of S will be refered to as points and elements of E will be refered to as r-faces. For convenience, we shall refer to any 3-face simply by a face. If $(S,E),(S_1,E_1)$ are r-graphs such that $S_1 \subseteq S$, $E_1 \subseteq E$, we say that (S_1,E_1) is an r-subgraph of (S,E).

Any 3-graph (S,E) can be represented geometrically by representing points of S by points in space and each face $\{x,y,z\}$ in E by a triangle with x,y,z as vertices. The following Fig.5.1 shows geometrical representation of the 3-graph (S,E), where $S = \{1,2,3,4,5\}$ and $E = \{\{1,2,3\}, \{1,3,4\}, \{1,4,5\}\}$.

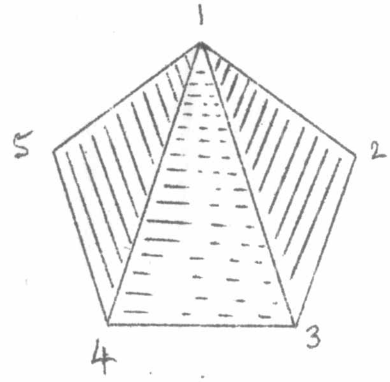

Fig. 5.1

By a complete 3-graph we mean 3-graph (S,E) in which E consists of all 3-subsets of S. If S contains n elements, we shall denote any complete 3-graph (S,E) by $K_n^{(3)}$.

If each face of the complete 3-graph $K_n^{(3)}$ is colored by red or blue. Let E_1 consist of all red faces and E_2 consist of all blue faces. Thus a red-blue coloring of faces of $K_n^{(3)}$ corresponds to a partition (E_1,E_2) of $P_3(S)$. By a coloring of a complete 3-graph $(S,P_3(S))$ we mean an ℓ-tuple $((S,E_1),(S,E_2),(S,E_3),\ldots,(S,E_\ell))$ where $(E_1,E_2,\ldots,E_\ell)$ forms a partition of $P_3(S)$. A complete 3-graph $(S,P_3(S))$ together with a coloring will be refered to as a chromatic 3-graph. By a $(q_1,q_2,\ldots,q_\ell;2)$-coloring of a complete 3-graph $(S,P_3(S))$ we mean a coloring $((S,E_1),(S,E_2),\ldots,(S,E_\ell))$ in which each (S,E_i) does not contain any complete 3-graph $K_{q_i}^{(3)}$ as its 3-subgraph. Hence a $(p,q;3)$-coloring of a complete 3-graph $(S,P_3(S))$ is a coloring $((S,E_1),(S,E_2))$ in which (S,E_1) does not contain a complete 3-graph $K_p^{(3)}$ as its 3-subgraph and (S,E_2) does not contain a complete 3-graph $K_q^{(3)}$ as its 3-subgraph. In what follows we shall refer to faces in E_1 as red faces and faces in E_2 as blue faces.

Geometrically, we can think of a$(p,q;3)$-coloring of a complete 3-graph $(S,P_3(S))$ as a coloring of the faces (triangles) formed by all combinations of 3 points of S in such a way that no complete 3-subgraph $K_p^{(3)}$ of $(S,P_3(S))$ has all its faces colored red and no complete 3-subgraph $K_q^{(3)}$ of $(S,P_3(S))$ has all its faces colored blue.

Note that the Ramsey number $N(p,q;3)$ is the smallest integer such that if $n \geq N(p,q;3)$ there exists no $(p,q;3)$-coloring of $K_n^{(3)}$.

5.2 Induced Line-Coloring

Assume that we are given a chromatic 3-graph $((S,P_3(S))\ ;\ ((S,E_1),(S,E_2),\ldots,(S,E_\ell)))$. For any $v_0 \in S$ let $S_0 = S - \{v_0\}$. We define the <u>induced line-coloring of $(S_0,P_2(S_0))$</u> to be the coloring $((S_0,E_1'),(S_0,E_2'),\ldots,(S_0,E_\ell'))$ of $(S_0,P_2(S_0))$, where $(E_1',E_2',\ldots,E_\ell')$ is a partition of $P_2(S_0)$ induced by $(E_1,E_2,\ldots,E_\ell)$. In what follows we shall refer to colorings of complete 3-graphs and complete graphs as <u>face-colorings</u> and <u>line-colorings</u>, respectively.

5.2.1 <u>Lemma</u> Let $((S,E_1),(S,E_2))$ be a $(4,4;3)$-coloring of $(S,P_3(S))$. For any $v_0 \in S$ let $S_0 = S - \{v_0\}$, then the induced line-coloring of $(S_0,P_2(S_0))$ is a $(4,4;2)$-coloring.

<u>Proof</u> : Let v_i,v_j,v_k be any points of S_0. If the lines $\{v_i,v_j\}$, $\{v_j,v_k\}$ and $\{v_i,v_k\}$ are red, then the faces $\{v_0,v_i,v_j\}$, $\{v_0,v_j,v_k\}$ and $\{v_0,v_i,v_k\}$ are red. Therefore, $K_4^{(3)}$ with points v_0,v_i,v_j,v_k has all its faces colored red unless the face $\{v_i,v_j,v_k\}$ is blue.

Hence, if $\{v_i, v_j, v_k\}$ is a red triangle in the line-coloring, the face $\{v_i, v_j, v_k\}$ must be blue in the face-coloring. It follows that the induced line-coloring of $(S_0, P_2(S_0))$ can not contain any 4-subset which forms a red K_4, otherwise this 4-subset will form a complete 3-graph $K_4^{(3)}$ with all its faces colored blue. Similary, the induced line-coloring of $(S_0, P_2(S_0))$ can not contain any 4-subset which forms a blue K_4. Hence the induced line-coloring of $(S_0, P_2(S_0))$ is a (4,4;2)-coloring.

Q.E.D.

5.3 An Improved Upper Bound of N(4,4;3)

5.3.1 Theorem $N(4,4;3) \leq 18$.

Proof : Let $S = \{v_0, v_1, v_2, \ldots, v_{17}\}$. Suppose that there exists a (4,4;3)-coloring of $(S, P_3(S))$. By Lemma 5.2.1, v_0 induces a (4,4;2)-coloring on $(S_0, P_2(S_0))$, where $S_0 = S - \{v_0\}$. By Theorem 4.1.6, v_1 is joined by red lines to 8 points. We may assume that these 8 points are $v_2, v_3, \ldots, v_9$. The red lines interjoining these 8 points must have the configuration G_3 :

G_3 :

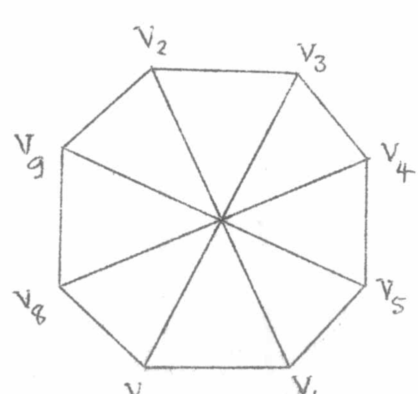

Fig. 5.2

Thus the blue lines interjoining the points $v_2, v_3, \ldots, v_9$ must have the configuration G_6 :

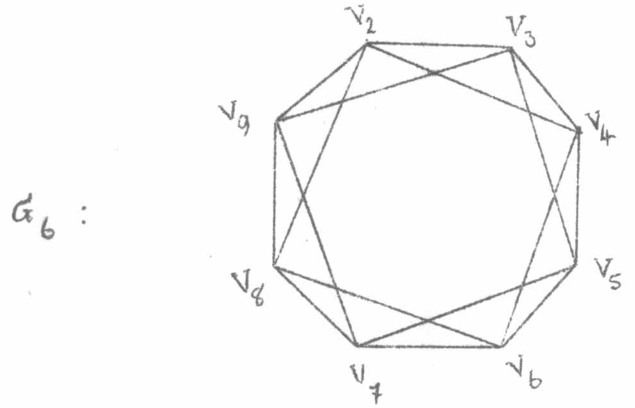

Fig. 5.3

The following Fig. 5.4 shows the red lines interjoining $v_1, v_2, \ldots, v_9$.

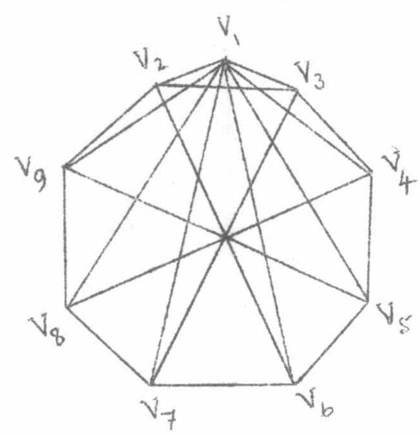

Fig. 5.4

Observe that the followings are the only red triangles of this graph :

$\{v_1,v_2,v_3\}, \{v_1,v_3,v_4\}, \{v_1,v_4,v_5\}, \{v_1,v_5,v_6\}, \{v_1,v_6,v_7\}, \{v_1,v_7,v_8\},$

$\{v_1,v_8,v_9\}, \{v_1,v_9,v_2\}, \{v_1,v_2,v_6\}, \{v_1,v_3,v_7\}, \{v_1,v_4,v_8\}, \{v_1,v_5,v_9\}.$

In the proof of Lemma 5.2.1 we see that any triangle which is red in the line-coloring must be blue in the face-coloring. Thus the faces

$\{v_1,v_2,v_3\},\{v_1,v_3,v_4\},\{v_1,v_4,v_5\},\{v_1,v_5,v_6\},\{v_1,v_6,v_7\},\{v_1,v_7,v_8\},$

$\{v_1,v_8,v_9\},\{v_1,v_9,v_2\},\{v_1,v_2,v_6\},\{v_1,v_3,v_7\},\{v_1,v_4,v_8\},\{v_1,v_5,v_9\}$

are blue.

By Lemma 5.2.1, v_1 also induces a (4,4;2)-coloring on $(S_1,P_2(S_1))$ where $S_1 = S - \{v_1\}$. Since the face $\{v_1,v_2,v_3\}$ is blue, hence $\{v_2,v_3\}$ is a blue line in the induced line-coloring of $(S_1,P_2(S_1))$. By the same reason it can be seen that

$\{v_3,v_4\},\{v_4,v_5\},\{v_5,v_6\},\{v_6,v_7\},\{v_7,v_8\},\{v_8,v_9\},\{v_9,v_2\},\{v_2,v_6\},$

$\{v_3,v_7\},\{v_4,v_8\},\{v_5,v_9\}$ are also blue lines in the induced line-coloring of $(S_1,P_2(S_1))$. Observe that these blue lines are precisely those red lines interjoining $v_2,v_3,\dots,v_9$ in the induced line-coloring of $(S_0,P_2(S_0))$. Hence they form a graph isomorphic to G_3. Therefore, the blue lines interjoining points $v_2,v_3,\dots,v_9$ in the induced line-coloring of $(S_1,P_2(S_1))$ contain a subgraph isomorphic to G_3. But the blue lines interjoining points $v_2,v_3,\dots,v_9$ must have the configuration G_6. Thus G_6 contains a subgraph isomorphic to G_3. But G_6 is a subgraph of G_4 in Fig.4.9. Therefore, G_4 contains a subgraph isomorphic to G_3, this contradicts to Theorem 4.1.7. Thus there does not exist a (4,4;3)-coloring of $(S,P_3(S))$ where S consists of 18 points. Hence $N(4,4;3) \leq 18$.

Q.E.D.

5.4 A Lower Bound of N(4,4;3).

Assume that we are given a complete 3-graph $(S,P_3(S))$ where S consists of points $1,2,\dots,n$. If $i < j < k$, the triple $(j-i,\ k-j,\ n+i-k)$ will be called the shape of the face $\{i,j,k\}$. It can be shown that (a,b,c) is a shape if and only if a,b,c are positive integers such that $a+b+c = n$. We say that the shapes (a,b,c) and (a',b',c') are congruent if and only if $(a',b',c') = (a,b,c)$ or (b,c,a) or (c,a,b). If the faces $\{i,j,k\}$ and $\{i',j',k'\}$ have congruent shapes, we say that they are congruent. It is clear from the definition that congruent is an equivalence relation.

In what follows we let P denote the set of all shapes of the faces of $(S,P_3(S))$. By a partition of P we mean an ordered pair (P_1,P_2) where P_1,P_2 are disjoint subsets of P such that $P_1 \cup P_2 = P$. To each partition (P_1,P_2) of P we let E_1 be the set of all faces whose shapes are in P_1 and E_2 be the set of all faces whose shapes are in P_2. Then (E_1,E_2) is a coloring of $(S,P_3(S))$. This coloring will be refered to as the coloring induced by the partition (P_1,P_2).

5.4.1 Theorem Let $S = \{1,2,\dots,n\}$. For each 4-subset $\{i,j,k,m\}$ of S let $S(i,j,k,m)$ denote the set of shapes of faces $\{i,j,k\}$, $\{j,k,m\}$, $\{i,k,m\}$ and $\{i,j,m\}$. If (P_1,P_2) is a partition of P such that

(1) any shape in P_1 is not congruent to any shape in P_2,

(2) $P_1 \cap S(i,j,k,m) \neq \emptyset$ and $P_2 \cap S(i,j,k,m) \neq \emptyset$ for all 4-subset $\{i,j,k,m\}$ of S.

Then the coloring (E_1,E_2) of $(S,P_3(S))$ induced by (P_1,P_2) is a (4,4;3)-coloring.

Proof : Let $\{i,j,k,m\}$ be any 4-subset of S. Thus, by (2), $P_1 \cap S(i,j,k,m) \neq \emptyset$ and $P_2 \cap S(i,j,k,m) \neq \emptyset$. Therefore, at least one shape in S(i,j,k,m) must be in P_1 and at least one shape in S(i,j,k,m) must be in P_2. Hence the complete 3-graph $K_4^{(3)}$ with points i,j,k,m has at least one red face and at least one blue face. Thus every complete 3-subgraph $K_4^{(3)}$ of $(S,P_3(S))$ has both a red face and a blue face. It follows that no complete 3-subgraph $K_4^{(3)}$ of $(S,P_3(S))$ has all its faces colored red and no complete 3-subgraph $K_4^{(3)}$ of $(S,P_3(S))$ has all its faces colored blue. Hence (E_1,E_2) is a (4,4;3)-coloring of $(S,P_3(S))$.

Q.E.D.

For convenience, in what follows, we shall use the notation $i +_n j$ to mean the smallest positive integer k such that $i + j \equiv k \pmod{n}$.

5.4.2 Lemma Let $i < j < k \leq n$. Then the faces $\{i,j,k\}$ and $\{i +_n 1,\ j +_n 1,\ k +_n 1\}$ are congruent.

Proof : Case 1 Suppose that $k < n$. Thus $i + 1 < j + 1 < k + 1 \leq n$. Therefore, $i +_n 1 = i+1$, $j +_n 1 = j+1$, $k +_n 1 = k+1$. Hence the face $\{i +_n 1,\ j +_n 1,\ k +_n 1\}$ has shape (j-i, k-j, n+i-k). But the face $\{i,j,k\}$ has shape (j-i, k-j, n+i-k). Thus the faces $\{i,j,k\}$ and $\{i +_n 1,\ j +_n 1,\ k +_n 1\}$ are congruent.

Case 2 Suppose that $k = n$. In this case we have $i +_n 1 = i+1$, $j +_n 1 = j+1$, $k +_n 1 = 1$, Therefore, the face $\{i +_n 1,\ j +_n 1,\ k +_n 1\}$ has shape (i,j-i,n-j). But the face $\{i,j,k\}$ has shape (j-i,n-j,i).

We see that the shape of face $\{i,j,k\}$ is congruent to the shape of face $\{i +_n 1, j +_n 1, k +_n 1\}$. Hence the faces $\{i,j,k\}$ and $\{i +_n 1, j +_n 1, k +_n 1\}$ are congruent.

Q.E.D.

5.4.3 Corollary Let $i < j < k \leq n$. For any positive integer s, the faces $\{i,j,k\}$ and $\{i +_n s, j +_n s, k +_n s\}$ are congruent.

5.4.4 Lemma Let S, P, S(i,j,k,m) be as in Theorem 5.4.1. Let (P_1, P_2) be a partition of P such that any shape in P_1 is not congruent to any shape in P_2. For any positive integer s if $P_1 \cap S(i,j,k,m) \neq \phi$ and $P_2 \cap S(i,j,k,m) \neq \phi$, then $P_1 \cap S(i +_n s, j +_n s, k +_n s, m +_n s) \neq \phi$ and $P_2 \cap S(i +_n s, j +_n s, k +_n s, m +_n s) \neq \phi$

Proof : Assume that $P_1 \cap S(i,j,k,m) \neq \phi$. We may assume also that $i < j < k < m$. Hence there exists a face $\{x, y, z\} \in P_3(\{i,j,k,m\})$ whose shape is in P_1. By Corollary 5.4.3, the face

$\{x +_n s, y +_n s, z +_n s\} \in P_3(\{i +_n s, j +_n s, k +_n s, m +_n s\})$ is congruent to the face $\{x,y,z\}$. Hence its shape must be in P_1. Therefore, $P_1 \cap S(i +_n s, j +_n s, k +_n s, m +_n s) \neq \phi$. We have shown that $P_1 \cap S(i,j,k,m) \neq \phi$ implies $P_1 \cap S(i +_n s, j +_n s, k +_n s, m +_n s) \neq \phi$. Similary, we can show that $P_2 \cap S(i,j,k,m) \neq \phi$ implies $P_2 \cap S(i +_n s, j +_n s, k +_n s, m +_n s) \neq \phi$.

Q.E.D.

5.4.5 Theorem $N(4,4;3) \geqslant 12$.

Proof : Let $S = \{1,2,3,\ldots,11\}$. Then $P = \{(1,1,9),(1,9,1),(9,1,1),$
$(1,2,8),(2,8,1),(8,1,2),(3,1,7),(1,7,3),(7,3,1),(2,4,5),(4,5,2),(5,2,4),$
$(3,2,6),(2,6,3),(6,3,2),(1,5,5),(5,5,1),(5,1,5),(2,1,8),(1,8,2),(8,2,1),$
$(4,1,6),(1,6,4),(6,4,1),(2,3,6),(3,6,2),(6,2,3),(3,3,5),(3,5,3),(5,3,3),$
$(1,4,6),(4,6,1),(6,1,4),(2,5,4),(5,4,2),(4,2,5),(3,4,4),(4,4,3),(4,3,4),$
$(2,2,7),(2,7,2),(7,2,2),(1,3,7),(3,7,1),(7,1,3)\}$.

Let $P_1 = \{(1,1,9),(1,9,1),(9,1,1),(1,2,8),(2,8,1),(8,1,2),(1,3,7),$
$(3,7,1),(7,1,3),(1,6,4),(6,4,1),(4,1,6),(2,6,3),(6,3,2),(3,2,6),$
$(2,4,5),(4,5,2),(5,2,4),(2,5,4),(5,4,2),(4,2,5),(3,3,5),(3,5,3),$
$(5,3,3)\}$, and

$P_2 = \{(1,5,5),(5,5,1),(5,1,5),(2,2,7),(2,7,2),(7,2,2),(1,8,2),$
$(8,2,1),(2,1,8),(1,7,3),(7,3,1),(3,1,7),(1,4,6),(4,6,1),(6,1,4)$
$(2,3,6),(3,6,2),(6,2,3),(3,4,4),(4,4,3),(4,3,4)\}$.

Note that (P_1,P_2) forms a partition of P. We shall verify that any shape in P_1 is not congruent to any shape in P_2. First, we observe that $S(1,2,3,4) = \{(1,1,9),(1,1,9),(2,1,8),(1,2,8)\}$. By inspection, we see that $P_1 \cap S(1,2,3,4) \neq \phi$ and $P_2 \cap S(1,2,3,4) \neq \phi$. By Lemma 5.4.4, we have

$P_1 \cap S(2,3,4,5) \neq \phi$ and $P_2 \cap S(2,3,4,5) \neq \phi$,

$P_1 \cap S(3,4,5,6) \neq \phi$ and $P_2 \cap S(3,4,5,6) \neq \phi$,

$P_1 \cap S(4,5,6,7) \neq \phi$ and $P_2 \cap S(4,5,6,7) \neq \phi$,

$P_1 \cap S(5,6,7,8) \neq \phi$ and $P_2 \cap S(5,6,7,8) \neq \phi$,

$P_1 \cap S(6,7,8,9) \neq \phi$ and $P_2 \cap S(6,7,8,9) \neq \phi$,

$P_1 \cap S(7,8,9,10) \neq \phi$ and $P_2 \cap S(7,8,9,10) \neq \phi$,

$P_1 \cap S(8,9,10,11) \neq \phi$ and $P_2 \cap S(8,9,10,11) \neq \phi$,

$P_1 \cap S(9,10,11,1) \neq \phi$ and $P_2 \cap S(9,10,11,1) \neq \phi$,

$P_1 \cap S(10,11,1,2) \neq \phi$ and $P_2 \cap S(10,11,1,2) \neq \phi$,

$P_1 \cap S(11,1,2,3) \neq \phi$ and $P_2 \cap S(11,1,2,3) \neq \phi$.

By using this type of arguments, it suffices to show that $P_1 \cap S(i,j,k,m) \neq \phi$ and $P_2 \cap S(i,j,k,m) \neq \phi$ for the following combinations $\{i,j,k,m\}$.

$\{1,2,3,5\}$, $\{1,2,3,6\}$, $\{1,2,3,7\}$, $\{1,2,3,8\}$, $\{1,2,3,9\}$, $\{1,2,3,10\}$,

$\{1,2,4,5\}$, $\{1,2,4,6\}$, $\{1,2,4,7\}$, $\{1,2,4,8\}$, $\{1,2,4,9\}$, $\{1,2,4,10\}$,

$\{1,2,5,6\}$, $\{1,2,5,7\}$, $\{1,2,5,8\}$, $\{1,2,5,9\}$, $\{1,2,5,10\}$, $\{1,2,6,7\}$,

$\{1,2,6,8\}$, $\{1,2,6,9\}$, $\{1,2,6,10\}$, $\{1,2,7,9\}$, $\{1,2,7,10\}$, $\{1,2,8,10\}$,

$\{1,3,5,7\}$, $\{1,3,5,8\}$, $\{1,3,5,9\}$, $\{1,3,6,8\}$, $\{1,3,6,9\}$.

These can be done in the same way as for the combination $\{1,2,3,4\}$. In doing so, we see that for all 4-subset $\{i,j,k,m\}$ of S $P_1 \cap S(i,j,k,m) \neq \phi$ and $P_2 \cap S(i,j,k,m) \neq \phi$. It follows from Theorem 5.4.1 that (P_1,P_2) induces a (4,4;3)- coloring of $K_{11}^{(3)}$. Therefore, there exists a (4,4;3)- coloring of $K_{11}^{(3)}$. Hence $N(4,4;3) \geq 12$.

Q.E.D.

APPENDIX

In this appendix, we prove two theorems which justify the inductions used in the proof of Theorem 2.5.3 and Corollary 3.3.3 in Chapters II and III.

A-1 Theorem Let $S = \{ (q_1, q_2, \ldots, q_m) / q_i \geq 2,\ i = 1,2,\ldots, m \}$.
If T is a subset of S such that

(1) if $q_i \geq 2$ for all $i = 1,2,\ldots,m$ and $q_i = 2$ for some i, then $(q_1, q_2, \ldots, q_m)$ belongs to T, and

(2) for all $q_i \geq 2$, $i = 1,2,\ldots, m$ if $(q_1-1, q_2, \ldots, q_m)$, $(q_1, q_2-1, q_3, \ldots, q_m), \ldots, (q_1, \ldots, q_{m-1}, q_m-1)$ belong to T, then $(q_1, q_2, \ldots, q_m)$ belongs to T.

Then $T = S$.

Proof : Suppose $T \neq S$. Thus $S - T \neq \emptyset$. Let $U = S - T$. It follows from (1) that each $(q_1, q_2, \ldots, q_m)$ belongs to U satisfies $q_i > 2$ for all $i = 1,2,\ldots,m$. Choose $(q'_1, q'_2, \ldots, q'_m)$ from U such that $q'_1 + q'_2 + \ldots + q'_m$ is minimum. Then $(q'_1-1, q'_2, \ldots, q'_m), (q'_1, q'_2-1, q'_3, \ldots, q'_m), \ldots, (q'_1, q'_2, \ldots, q'_{m-1}, q'_m-1)$ belong to T. Therefore, by (2), $(q'_1, q'_2, \ldots, q'_m)$ belongs to T, which is a contradiction. Hence $T = S$.

Q.E.D.

A-2 Theorem Let $S = \{ (q_1, q_2, \ldots, q_m; r) / q_1 \geq r \geq 1,\ i = 1,2,\ldots,m \}$.
If T is a subset of S such that

(1) $(q_1, q_2, \ldots, q_m; 1)$ belongs to T for all $q_i \geq 1$, $i = 1,2,\ldots, m$,

(2) if $q_i \geq r$ for all $i = 1,2,\ldots,m$ and $q_i = r$ for some i, then $(q_1,q_2,\ldots, q_m; r)$ belongs to T, and

(3) for all $r \geq 2$ and all $q_i \geq r$, $i = 1,2,\ldots,m$ if $(\overset{*}{q}_1, \overset{*}{q}_2,\ldots,\overset{*}{q}_m; r-1)$ belongs to T for all $\overset{*}{q}_i \geq r-1$, $i = 1,2,\ldots,m$, and $(q_1-1, q_2,\ldots,q_m; r)$, $(q_1,q_2-1,q_3,\ldots,q_m;r),\ldots,$ $(q_1,\ldots,q_{m-1},q_m-1; r)$ belong to T, then $(q_1,q_2,\ldots,q_m; r)$ belongs to T.

Then T = S.

<u>Proof</u> : Suppose that $T \neq S$. Thus $S - T \neq \emptyset$. Let $U = S - T$. Let r_o be the smallest positive integer such that $(q_1,q_2,\ldots,q_m;r_o)$ belongs to U for some $q_1,q_2,\ldots,q_m$. By (1) we see that $r_o > 1$. Choose $(q_1',q_2',\ldots,q_m'; r_o)$ from U such that $q_1'+ q_2'+\ldots q_m'$ is minimum. By (2), we have $q_i' > r_o$ for all i. By the choice of r_o we see that $(\overset{*}{q}_1,\overset{*}{q}_2,\ldots,\overset{*}{q}_m;r_o-1)$ belongs to T for all $\overset{*}{q}_i \geq r_o-1$, $i = 1,2,\ldots,m$. By the choice of $q_1',q_2',\ldots,q_m'$ we see that $(q_1'-1, q_2',\ldots, q_m'; r_o)$, $(q_1',q_2'-1, q_3',\ldots,q_m'; r_o),\ldots,(q_1',q_2',\ldots,q_{m-1}',q_m'-1; r_o)$ belong to T. Therefore, by (3), $(q_1',q_2',\ldots,q_m'; r_o)$ belongs to T, which is a contradiction. Hence T = S.

<u>Q.E.D.</u>