CHAPTER IV

ON $(p,q\ ;\ 2)$ - COLORING OF K_n

4.0 Introduction

In this chapter some structures of $(p,q\ ;\ 2)$- coloring of K_n are derived for later uses. A method for constructing a $(p,q\ ;\ 2)$-coloring of K_n will be discussed.

4.1 Some Structural Theorems for $(p,q;2)$-Coloring of K_n

4.1.1 Theorem Let $((S,E_1),(S,E_2))$ be a $(p,q;2)$-chromatic graph. For any point $v_0 \in S$ let X be the set of all points of $((S,E_1),(S,E_2))$ which are joined to v_0 by red lines, and Y be the set of all points of $((S,E_1),(S,E_2))$ which are joined to v_0 by blue lines, i.e.

$$X = \left\{ v/\ v \in S, \{v,v_0\} \in E_1 \right\} , \text{ and } Y = \left\{ v/\ v \in S, \{v,v_0\} \in E_2 \right\}.$$

Let $\mathcal{X} = ((X,E_1'),(X,E_2'))$, $\mathcal{Y} = ((Y,E_1''),(Y,E_2''))$ be the chromatic subgraphs of $((S,E_1),(S,E_2))$ induced by X and Y, respectively. Let n,x,y denote the numbers of points of S,X,Y, respectively. Then

(1) $\mathcal{X}$ is a $(p-1,q;2)$-chromatic subgraph of $((S,E_1),(S,E_2))$,

(2) $\mathcal{Y}$ is a $(p,q-1;2)$-chromatic subgraph of $((S,E_1),(S,E_2))$,

and

(3) $x + y + 1 = n$.

Proof : First, we shall show that $\mathcal{X}$ is a $(p-1,q;2)$-chromatic subgraph of $((S,E_1),(S,E_2))$. Suppose that there exists a $(p-1)$-subset of X which forms a red $K_{(p-1)}$ in $\mathcal{X}$. Thus this $(p-1)$-subset together with v_0 will give a p-subset of S which forms a red K_p in $((S,E_1),(S,E_2))$, which contradicts to hypothesis. Hence there does not exist a (p-1)-subset of X which forms a red $K_{(p-1)}$ in the chromatic subgraph $\mathcal{H}$. Since there does not exist a q-subset of S which forms a blue K_q in $((S,E_1),(S,E_2))$. Thus there does not exist a q-subset of X which forms a blue K_q in the chromatic subgraph $\mathcal{H}$. Therefore, a coloring of the chromatic subgraph $\mathcal{H}$ of $((S,E_1),(S,E_2))$ is a (p-1,q;2)-coloring. Hence $\mathcal{H}$ is a (p-1,q;2)-chromatic subgraph of $((S,E_1),(S,E_2))$.

Next, we shall show that $\mathcal{Y}$ is a (p,q-1;2)-chromatic subgraph of $((S,E_1),(S,E_2))$. Suppose that there exists a (q-1)-subset of Y which forms a blue $K_{(q-1)}$ in $\mathcal{Y}$. Thus this (q-1)-subset together with v_0 will give a q-subset of S which forms a blue K_q in $((S,E_1),(S,E_2))$, which contradicts to hypothesis. Hence there does not exist a(q-1)-subset of Y which forms a blue $K_{(q-1)}$ in the chromatic subgraph $\mathcal{Y}$. Since there does not exist a p-subset of S which forms a red K_p in $((S,E_1),(S,E_2))$. Thus there does not exist a p-subset of Y which forms a red K_p in the chromatic subgraph $\mathcal{Y}$. Therefore, a coloring of the chromatic subgraph $\mathcal{Y}$ is a (p,q-1;2)-coloring. Hence $\mathcal{Y}$ is a (p,q-1;2)-chromatic subgraph of $((S,E_1),(S,E_2))$. It is clear that $x + y + 1 = n$.

Q.E.D.

4.1.2 Remark From Theorem 4.1.1 we may conclude that a(p,q;2)-chromatic graph with n points exists, then it must contain chromatic subgraphs $\mathcal{H}$, $\mathcal{Y}$ with x,y points, respectively, where

(1) $\mathcal{H}$ is a (p-1,q;2)-chromatic subgraph,

(2) $\mathcal{Y}$ is a (p,q-1;2)-chromatic subgraph, and

(3) $x + y + 1 = n$.

This fact can be used as a basis for constructing a (p,q;2)-coloring of K_n as follows :

(1) Determine all positive integers x,y such that

$$x + y + 1 = n$$

$$x \leq N(p-1,q;2)-1,$$

$$y \leq N(p,q-1;2)-1.$$

(2) Construct (p-1,q;2)-chromatic subgraph $\mathcal{H}$ with x points and (p,q-1;2)-chromatic subgraph $\mathcal{Y}$ with y points.

(3) Construct the complete graph K_n by taking the points of $\mathcal{H}$, $\mathcal{Y}$ and an extra point v_0 as points of K_n. Let the lines of the chromatic subgraphs $\mathcal{H}$, $\mathcal{Y}$ have the original coloring. Let each line from v_0 to points of $\mathcal{H}$ be colored red and each line from v_0 to points of $\mathcal{Y}$ be colored blue.

Then we try to color the lines joining $\mathcal{H}$ and $\mathcal{Y}$, one at a time, in such a way that no red K_p or blue K_q occurs as a subgraph of K_n.

This method of constructing (p,q;2)-coloring K_n is rather cumbersome for large values of n. However, when n is not so large, this method give us all non-isomorphic (p,q;2)-colorings of K_n.

As an illustration, let us apply the above method to obtain all (3,3;2)-colorings of K_5.

First, we look for positive integers x,y such that $x \leq N(2,3;2)-1$, $y \leq N(3,2;2)-1$, and $x + y + 1 = 5$. Since $N(2,3;2) = N(3,2;2) = 3$, thus $x = 2$, $y = 2$. Hence $\mathcal{H}$, $\mathcal{Y}$ which are chromatic graphs with 2 points is the only possibility. Next, we color $\mathcal{H}$, $\mathcal{Y}$ so that $\mathcal{H}$ is a (2,3;2)-chromatic graph and $\mathcal{Y}$ is a (3,2;2)-chromatic graph. The only possible colorings of $\mathcal{H}$ and $\mathcal{Y}$ are shown below.

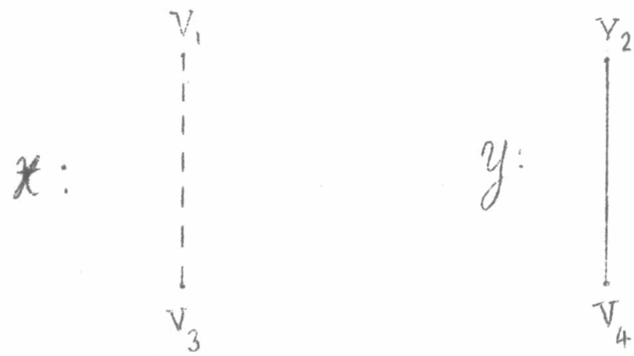

Fig. 4.1

In our diagrams red lines will be represented by heavy lines and blue **lines** will be represented by dotted lines. Since v_0 is joined to points of $\mathcal{X}$ by red lines and joined to points of $\mathcal{Y}$ by blue lines. Hence in our (3,3;2)-coloring of K_5 the coloring of the lines $\{v_1, v_3\}$, $\{v_2, v_4\}$, $\{v_0, v_1\}$, $\{v_0, v_3\}$, $\{v_0, v_2\}$, $\{v_0, v_4\}$ must be shown in the following figure.

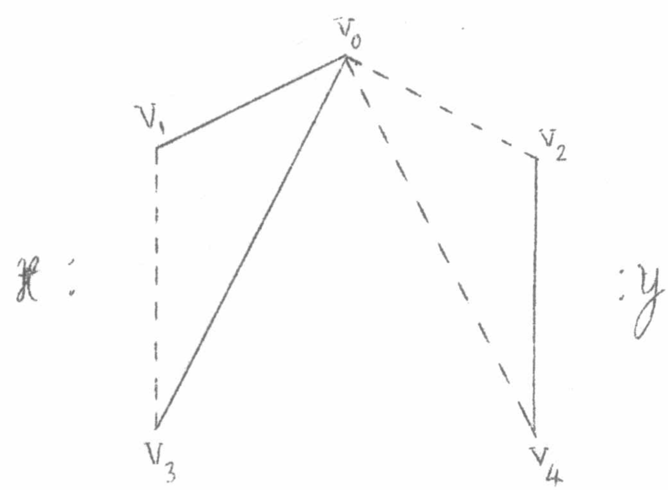

Fig. 4.2.

There are two possibilities for coloring the line $\{v_1, v_2\}$, Case I : The line $\{v_1, v_2\}$ is colored red :

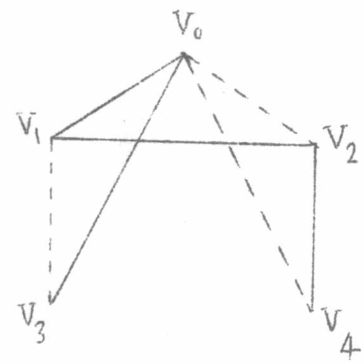

Fig. 4.3

If $\{v_1, v_4\}$ is a red line, then $\{v_1, v_2, v_4\}$ is a red triangle. Hence $\{v_1, v_4\}$ can not be red. Therefore, $\{v_1, v_4\}$ must be blue line :

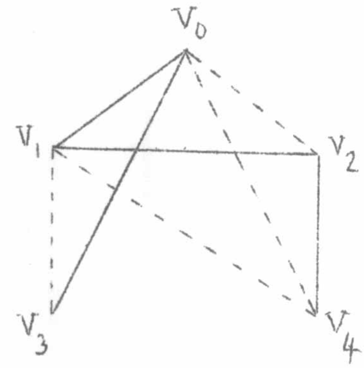

Fig. 4.4

By a similar argument, it follows that $\{v_3, v_4\}$ must be red line :

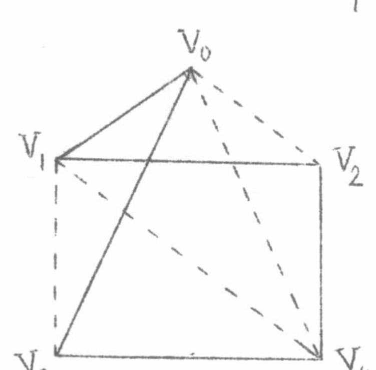

Fig. 4.5

Finally, we see that the line $\{v_2, v_3\}$ must be blue. Hence the coloring of K_5 must be as shown in the following Fig.4.6.

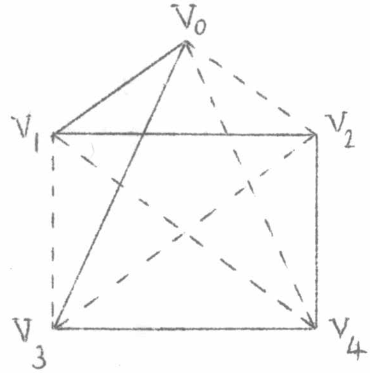

Fig. 4.6

In fact, this is a (3,3;2)-coloring of K_5.

<u>Case II</u> : The line $\{v_1, v_2\}$ is blue :

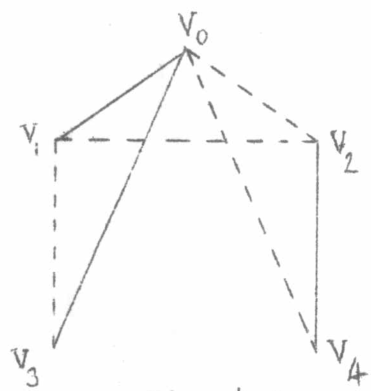

Fig. 4.7

By arguments similar to Case I, we obtain a coloring of K_5 as shown in the following Fig. 4.8

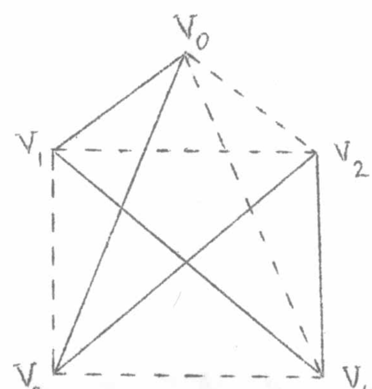

Fig. 4.8

This is also a (3,3;2)-coloring of K_5. However, it can be seen to be isomorphic to the (3,3;2)-coloring obtained in Case I. Hence there is a unique (3,3;2)-coloring of K_5.

By applying the method illustrated above to obtain all (3,4;2)-colorings of K_8 we obtain 3 non-isomorphic (3,4;2)-colorings of K_8. We state this result in the following.

4.1.3 <u>Lemma</u> Let $((S,E_1),(S,E_2))$ be a (3,4;2)-coloring of K_8 where S consists of 8 points. Then (S,E_1) must be isomorphic to one of the colorings shown as G_1,G_2,G_3 in Fig. 4.9. By counting the lines of G_1,G_2,G_3 we see that (S,E_1) must have at most 12 lines.

Observe that if (S,E_1) is isomorphic to G_1 in Fig.4.9, then (S,E_2) must be isomorphic to G_4 in Fig.4.9, the complement of G_1. Using this fact together with Lemma 4.1.3 we have

4.1.4 <u>Corollary</u> Let $((S,E_1),(S,E_2))$ be as in Lemma 4.1.3. Then (S,E_2) must be isomorphic to one of the graphs G_4,G_5,G_6 in Fig.4.9.

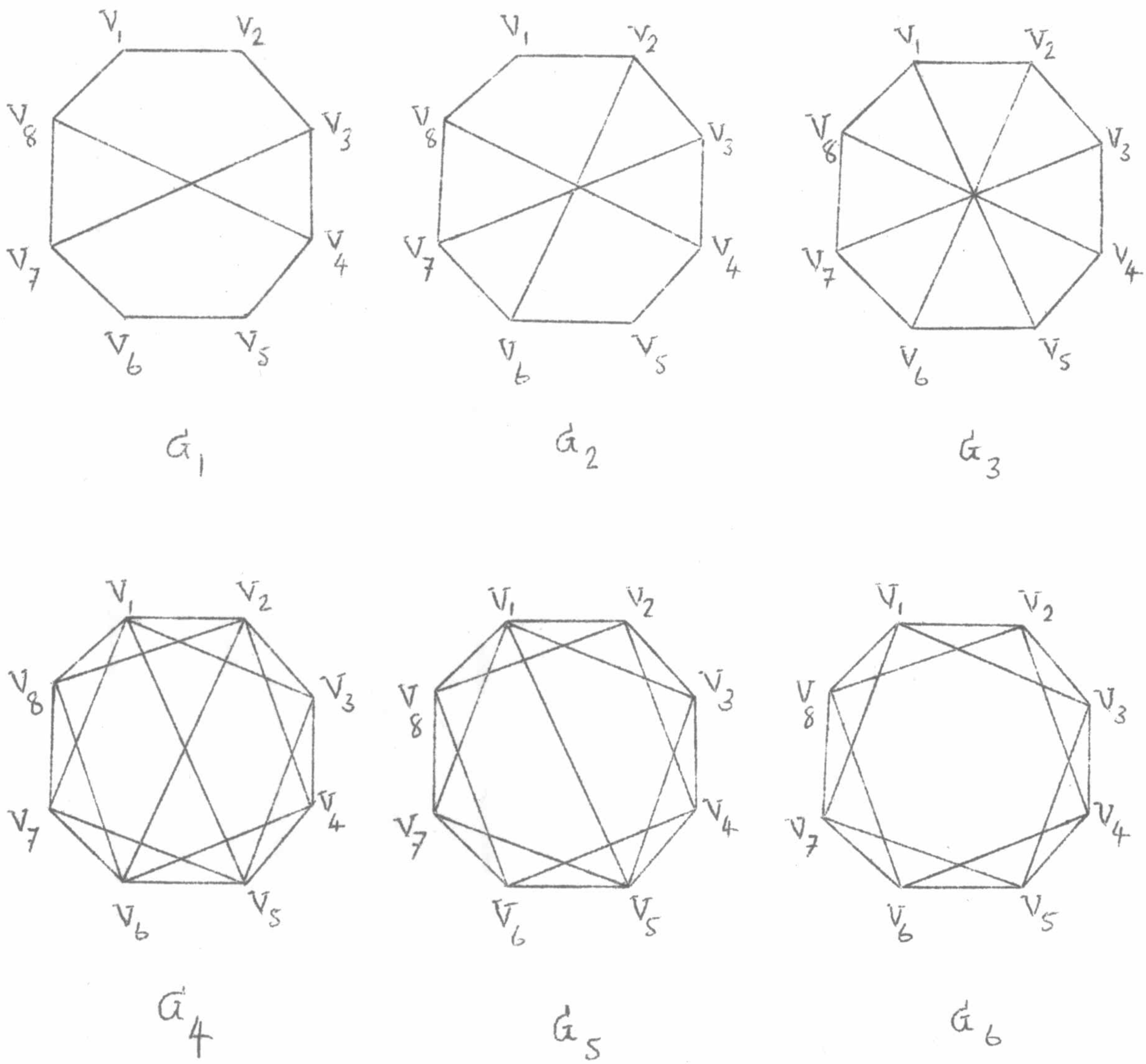

Fig. 4.9

4.1.5 Remark Observe that if $((S,E_1),(S,E_2))$ is a (4,3;2)-coloring of K_8, then $((S,E_2),(S,E_1))$ is a (3,4;2)-coloring of K_8. Hence (S,E_1) must be isomorphic to G_4 or G_5 or G_6 in Fig.4.9. By counting the lines of G_4, G_5, G_6 we can conclude that (S,E_1) must have at least 16 lines.

4.1.6 Theorem Let $((S,E_1),(S,E_2))$ be a (4,4;2)-chromatic K_{17} where S consists of 17 points. Let x, y, X, Y, $\mathcal{X} = ((X,E_1'),(X,E_2'))$, $\mathcal{Y} = ((Y,E_1''),(Y,E_2''))$ be as in Theorem 4.1.1. Then $x = y = 8$, and (X,E_1') must be isomorphic to G_3 in Fig.4.9.

Proof : By Theorem 4.1.1, $\mathcal{X}$ is a (3,4;2)-chromatic subgraph of $((S,E_1),(S,E_2))$, $\mathcal{Y}$ is a (4,3;2)-chromatic subgraph of $((S,E_1),(S,E_2))$ and $x + y + 1 = 17$. Thus $x \leq N(3,4;2)-1$, $y \leq N(4,3;2)-1$. Since $N(3,4;2) = N(4,3;2) = 9$, hence $x = y = 8$. From this,it follows that v_0 is incident with 8 red lines. Since v_0 is arbitrary, hence every point of S is incident with 8 red lines. Assume that $\mathcal{X}$ has r red lines. Therefore, there are $8.8 - 2.r$ red lines from X to the points outside X. Among these lines, 8 of them are the lines joined to v_0. Thus there are $8.8 - 2r - 8$ red lines from $\mathcal{X}$ to $\mathcal{Y}$. Since every point of Y is incident with 8 red lines. Therefore, (Y,E_1'') has $[8.8 - (8.8 - 2.r - 8)]/2$ red lines. By Remark 4.1.5, (Y,E_1'') has at least 16 red lines. Hence we have $[8.8 - (8.8 - 2.r - 8)]/2 \geq 16$. Thus $r \geq 12$. By Lemma 4.1.3, (X,E_1') has at most 12 red lines. Hence (X,E_1') has exactly 12 red lines. Thus (X,E_1') is isomorphic to G_3 in Fig.4.9.

Q.E.D.

4.1.7 Theorem Let G_3, G_4 be as shown in Fig.4.9. Then G_4 contains no subgraph isomorphic to G_3.

Proof : Suppose that six lines can be removed from G_4 to obtain a graph isomorphic to G_3. Now G_3 does not contain a triangle or an independent set of four points, furthermore every point of G_3 has degree 3. By Remark 3.1., the six lines must be removed from G_4 in such a way that in the resulting graph there does not exist a triangle or an independent set of four points and every point has degree 3.

For convenience, we denote the lines $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_4,v_5\}$, $\{v_5,v_6\}$, $\{v_6,v_7\}$, $\{v_7,v_8\}$, $\{v_8,v_1\}$, $\{v_1,v_3\}$, $\{v_2,v_4\}$, $\{v_3,v_5\}$, $\{v_4,v_6\}$, $\{v_5,v_7\}$, $\{v_6,v_8\}$, $\{v_7,v_1\}$, $\{v_8,v_2\}$, $\{v_1,v_5\}$ and $\{v_2,v_6\}$ of G_4 by $p_1,p_2,\ldots,p_8,s_1,s_2,\ldots,s_8$, d_1 and d_2, respectively (see Fig.4.10 below).

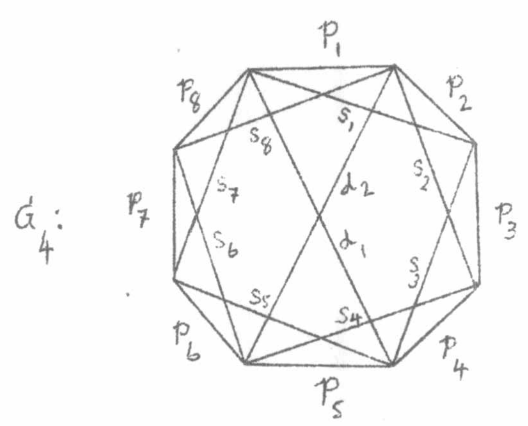

Fig. 4.10

First, let us suppose that the lines d_1, d_2 are among the six lines removed from G_4. Then we obtain the graph G_6 as shown in Fig.4.11.

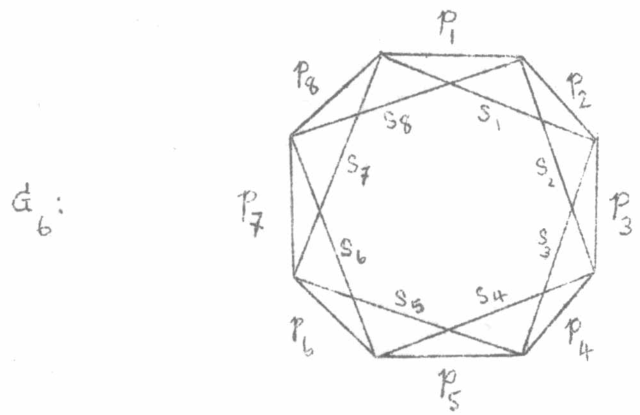

Fig. 4.11

The other four lines must be removed from G_6 to obtain G_3.

Suppose further that the line s_7 can be among the four lines removed from G_6. Thus we have the graph as in Fig.4.12.

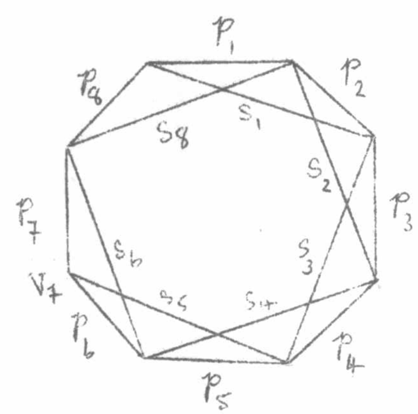

Fig. 4.12

We see that the point v_7 has degree 3, so the lines s_5, p_6, p_7 can not be removed. In order that no triangle occurs in the resulting graph, the lines p_5, s_6 must be removed. But, if these lines are removed, the resulting graph (see Fig.4.13 below) contains a point of degree 2.

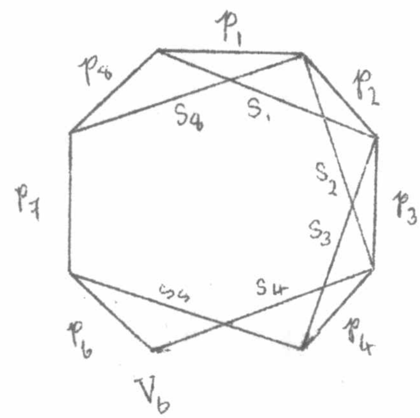

Fig. 4.13

Hence the line s_7 can not be among the removed lines. The same argument shows that none of the lines s_i can be among the removed lines.

If both of the lines p_1 and p_2 are removed from G_6, the resulting graph (see Fig.4.14 below) contains a point of degree 2.

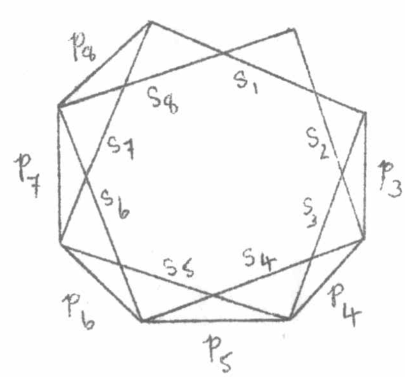

Fig. 4.14

Hence the lines p_1 and p_2 can not be both removed from G_6. Similary, we can show that no pair of adjacent lines p's can be both removed from G_6. So the only possibilities are that p_1, p_3, p_5, p_7 or p_2, p_4, p_6, p_8 are the four removed lines.

If the lines p_1, p_3, p_5, p_7 are the four lines removed from G_6, the resulting graph (see Fig.4.15 below) contains an independent set of four points. $\{v_3, v_4, v_7, v_8\}$ is such a set.

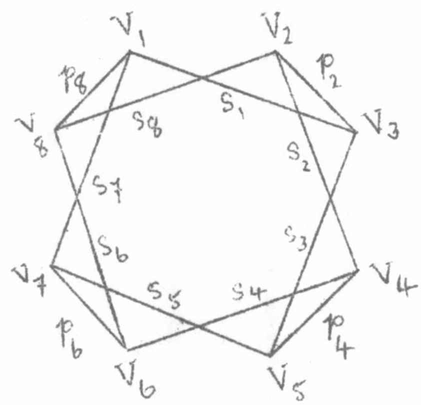

Fig 4.15

Hence the lines p_1, p_3, p_5, p_7 can not be the four removed lines. Similary, we can show that the lines p_2, p_4, p_6, p_8 can not be the four removed lines.

The above argument shows that not both of the lines d_1, d_2 can be among the removed lines. So at most one of di can be among the six removed lines.

Suppose that the line d_2 is removed from G_4. Thus we obtain the graph G_5 as shown in Fig.4.16.

G_5 :

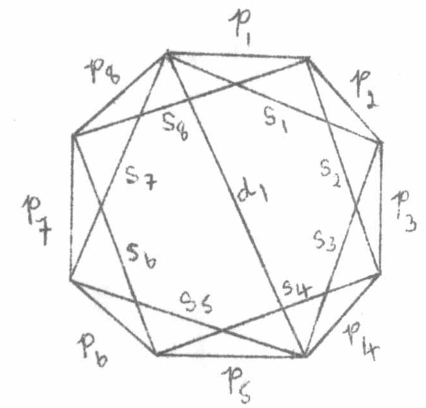

Fig. 4.16

In order that no triangle occurs, the line s_5 or s_7 must be removed.

If the line s_7 is removed. Thus we obtain the graph as shown in Fig.4.17

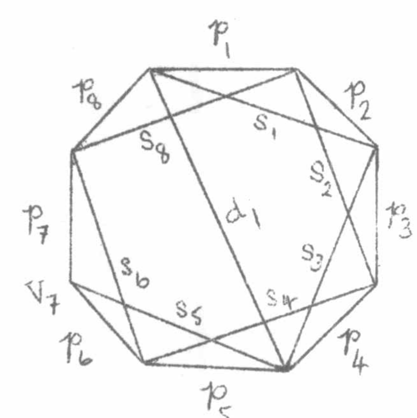

Fig. 4.17

We see that the point v_7 has degree 3, so the lines s_5, p_6, p_7 can not be removed. In order that no triangle occurs in the resulting graph, the lines p_5, s_6 must be removed. But, if these lines are removed, the resulting graph (see Fig.4.18) contains a point of degree 2.

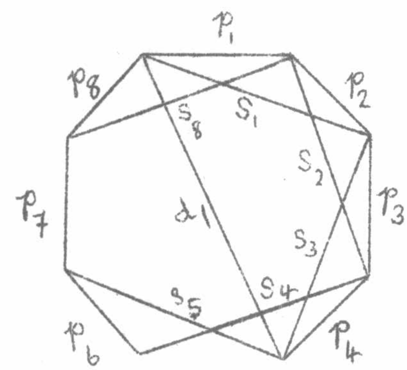

Fig. 4.18

Hence the line s_7 can not be removed. The same argument shows that the line s_5 can not be removed. In order that no triangle occurs in the resulting graph, the line d_1 must be removed, which is a contradiction. Therefore, the line d_2 can not be removed from G_4. Similary, we can show that the line d_1 can not be removed from G_4.

Hence none of the lines d_i can be among the removed lines. So the other six lines are removed from G_4.

Suppose that the line s_8 is among the six lines removed from G_4. Thus we obtain the graph as shown in Fig.4.19

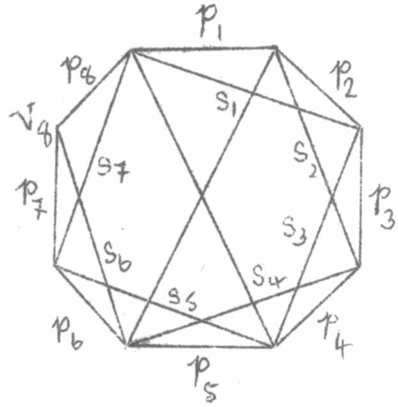

Fig. 4.19

We see that the point v_8 has degree 3, so the line s_6, p_7, p_8 can not be removed. In order that no triangle occurs in the resulting graph, the lines p_6, s_7 must be removed. But, if these lines are removed, the resulting graph (see Fig.4.20) contains a point of degree 2.

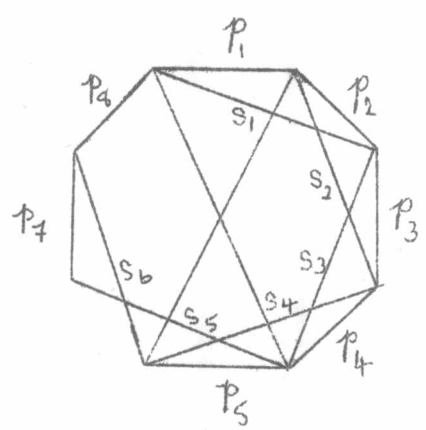

Fig. 4.20

Hence the line s_8 can not be among the removed lines. The same argument shows that the lines s_1, s_4, s_5 can not be among the removed lines.

Since d_2 and s_8 can not be among the removed lines, hence s_6 must be among the removed lines. Otherwise d_2, s_8 and s_6 would form a triangle. By the same reasoning we can conclude that s_2, s_3, s_7 must be removed. Thus we have the graph as shown in Fig.4.21.

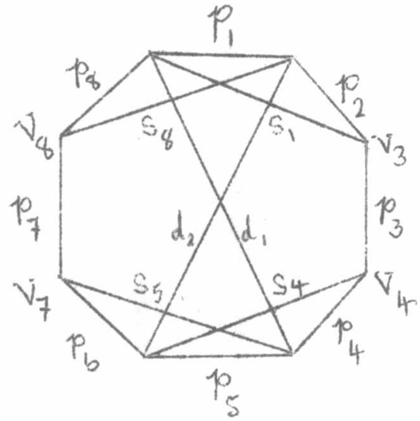

Fig.4.21

We see that each of the points v_3, v_4, v_7, v_8 has degree 3. So no more lines which are incident with the points v_3, v_4, v_7, v_8 can be removed. In order that no triangle occurs in the resulting graph, the lines p_1, p_5 must be removed. But, if these lines are removed, the resulting graph (see Fig.4.22) contains an independent set of four points. $\{v_1, v_2, v_4, v_7\}$ is such a set

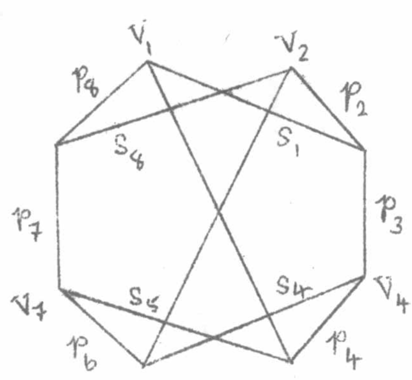

Fig.4.22

Hence no six lines can be removed from G_4 to obtain a graph isomorphic to G_3. Therefore, G_4 contains no subgraph isomorphic to G_3.

Q.E.D.