CHAPTER I

INTRODUCTION

In [1] , the following elementary problem was posed :

Prove that a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.

It is convenient to translate this problem into an equivalent problem on coloring of lines of a complete graph. A complete graph of order n, denoted by K_n, consists of a set of n points joined together by lines such that each pair of distinct points is joined by one and only one line. The following figures (Fig.1.1,1.2) show the complete graphs K_3 and K_4 .

K_3 :

Fig.1.1

K_4 :

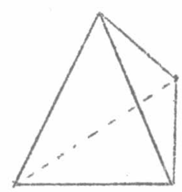

Fig 1.2

We can translate the above problem into a graphical one by representing people by points and their relationship by coloring of lines joining them. For example, we may paint the red lines if they join the points that represent acquaintances, and paint blue to those joining strangers. When this is done, a set of three mutual acquaintances will be represented by a triangle in which all its sides were painted red. Such triangle will be called red triangle. Similary, a set of three complete strangers will be represented by a blue triangle. Then the above problem becomes the following.

If the lines of a complete graph K_6 are colored, some red and the others blue, then some red or blue triangle must occur.

We may demonstrate this in the following way. We take one point P_o and consider the five lines from P_o to the remaining five points. Three of these lines must be of the same color. We may assume that these lines are red, for if they are blue we may reason in the same way with "red" and "blue" interchanged. Let three red lines be P_oP_1, P_oP_2, P_oP_3. If any one of the lines P_1P_2, P_1P_3 or P_2P_3 is red, say P_iP_j, this line P_iP_j together with P_oP_i and P_oP_j form a red triangle. If the lines P_1P_2, P_1P_3 and P_2P_3 are blue, they form a blue triangle. Hence a red or blue triangle must occur.

Observe that when $n \geq 6$, K_n contains K_6 as a subgraph. Hence it follows from the above result that if the lines of K_n are colored, some red and the others blue, then some red triangle or blue triangle must occur. However, when $n < 6$ the lines of K_n can be colored so that neither red nor blue triangle occurs. The following Fig.1.3 shows such a coloring of K_5.

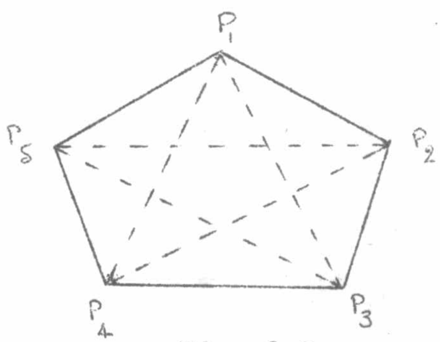

Fig. 1.3

P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_1 are colored red, and P_1P_3, P_1P_4, P_2P_4, P_2P_5, P_3P_5 are colored blue.

This example of coloring of lines of K_5 shows that 6 is the smallest number N such that if $n \geq N$, then in any red-blue coloring of lines of K_n a red K_3 or a blue K_3 must occur.

This problem may be generalized as follows : Find the smallest integer N such that if $n \geq N$, then in any red-blue coloring of lines of K_n a red subgraph K_p or a blue subgraph K_q must occur. Here p, q are any two given integers such that p, q $\geq$ 2. We shall show that such smallest integers, known as Ramsey numbers, always exist. This will be just a special case of a more general result proved in Chapter II. The remaining chapters deal with the determination of some Ramsey numbers.