กฎอย่างเข้มของเลขจำนวนมากสำหรับแถวลำดับสองมิติของตัวแปรสุ่มที่ไม่อิสระเชิงลบทุกคู่

นายกานต์ สุระกำแหง

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STRONG LAW OF LARGE NUMBERS FOR A 2-DIMENSIONAL ARRAY OF PAIRWISE NEGATIVELY DEPENDENT RANDOM VARIABLES

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
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ในงานนี้ เราพิสูจน์สองทฤษฎีบทหลักของกฎอย่างเข้มของเลขจำนวนมากสำหรับแถวลำดับสองมิติของตัวแปรสุ่มสองทฤษฎีบท ทฤษฎีบทแรกคือกฎอย่างเข้มของเลขจำนวนมากสำหรับตัวแปรสุ่มที่ไม่เป็นอิสระเชิงลบทุกคู่ ซึ่งไม่จำเป็นต้องมีการแจกแจงแบบเดียวกัน ทฤษฎีบทที่สอง เราได้กฎอย่างเข้มของเลขจำนวนมากสำหรับผลต่างของตัวแปรสุ่มที่ไม่ต้องการเงื่อนไขความเป็นอิสระและการแจกแจงเดียวกัน ในการศึกษา เราพิจารณาค่าลิมิต เมื่อ $m \times n$ มีค่าเข้าสู่อนันต์ สำหรับจำนวนธรรมชาติ m, n ใดๆ ซึ่งเป็นเงื่อนไขที่เข้มกว่าเงื่อนไขเมื่อ m, n มีค่าเข้าสู่อนันต์

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KARN SURAKAMHAENG : STRONG LAW OF LARGE NUMBERS FOR A 2-DIMENSIONAL ARRAY OF PAIRWISE NEGATIVELY DEPENDENT RANDOM VARIABLES. ADVISOR : ASST.PROF.NATTAKARN CHAIDEE, Ph.D., CO-ADVISOR : PROF.KRITSANA NEAMMANEE, Ph.D., 33 pp.

In this work, we obtain two main theorems of strong law of large numbers for a 2-dimensional array of random variables. The first theorem is the strong law of large numbers for pairwise negatively dependent random variables which are not necessary identically distributed. The second theorem, we obtain strong law of large numbers for the difference of random variables which independent and identically distributed conditions are regarded. In this study, we use the limit as $m \times n$ tends to infinity instead of using the limit as m, n tends to infinity when m, n are natural numbers which is stronger.

Department : Mathematics and Computer Science   Student's Signature ....

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CONTENTS

CHAPTER I

INTRODUCTION

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables in a probability space $(\Omega, \mathcal{F}, P)$ and for each $n \in \mathbb{N}$,

$$S_n = X_1 + X_2 + \cdots + X_n.$$

We say that $(X_i)_{i\in\mathbb{N}}$ satisfies the **strong law of large numbers (SLLN)** if there exist sequences of real numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$\frac{S_n - a_n}{b_n} \xrightarrow{\text{a.s.}} 0 \text{ as } n \to \infty,$$

the abbreviation a.s. stands for almost sure convergence.

Now, to study the strong law of large numbers, a simple question comes in mind. When does the sequence $(X_i)_{i\in\mathbb{N}}$ satisfy the SLLN? Many conditions of the sequence $(X_i)_{i\in\mathbb{N}}$ have been found under this question.

In 1983, Etemadi[5] found a sufficient condition of the sequence of nonnegative random variables with finite variances to obtain the SLLN.

Theorem 1.1. [5] *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of nonnegative random variables with finite variances such that*

(i) $\sup_{i\in\mathbb{N}} E(X_i) < \infty$,

(ii) $E(X_iX_j) \leqslant E(X_i)E(X_j)$ *for any* $i < j$, *and*

(iii) $\displaystyle\sum_{i=1}^{\infty} \frac{Var(X_i)}{i^2} < \infty$.

Then

$$\frac{S_n - E(S_n)}{n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

In the same year, he found another conditions for nonnegative random variables possessing SLLN.

Theorem 1.2. [6] *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers and* $b_n = \sum_{i=1}^{n} a_i$ *such that*

$$\frac{a_n}{b_n} \to 0 \quad and \quad b_n \to \infty \ as \ n \to \infty.$$

Let $(X_i)_{i\in\mathbb{N}}$ *be a sequence of nonnegative random variables with finite variances such that*

(i) $\sup_{i\in\mathbb{N}} E(X_i) < \infty$, *and*

(ii) $\displaystyle\sum_{j=1}^{\infty}\sum_{i=1}^{j} \frac{a_i a_j Cov^+(X_i, X_j)}{b_j^2} < \infty$

where $Cov^+(X_i, X_j) = \max\{Cov(X_i, X_j), 0\}$. *Then*

$$\frac{W_n - E(W_n)}{b_n} \xrightarrow{a.s.} 0 \ as \ n \to \infty$$

where $W_n = \sum_{i=1}^{n} a_i X_i$.

In 1992, Tapas, Chandra and Goswami[16] also proved the SLLN of the sequence of nonnegative random variables with finite variances.

Theorem 1.3. [16] *Let* $(a_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive real numbers such that*

$$a_n \to \infty \ as \ n \to \infty.$$

Let $(X_i)_{i\in\mathbb{N}}$ *be a sequence of nonnegative random variables with finite variances. Assume that*

(i) $\displaystyle\sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \frac{E(X_i)}{a_n} < \infty$,

(ii) *there exists a double sequence* $(\rho_{i,j})_{i,j\in\mathbb{N}}$ *of nonnegative real numbers such that for each* $n \in \mathbb{N}$,

$$Var(S_n) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j},\ and$$

(iii) $\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\rho_{i,j}}{a^2_{(i\vee j)}} < \infty \quad where \quad i \vee j = \max\{i, j\}.$

Then

$$\frac{S_n - E(S_n)}{a_n} \xrightarrow{a.s.} 0 \ as\ n \to \infty.$$

The random variables in Theorem 1.1-1.3 are nonnegative. The followings are SLLN's for random variables which nonnegative condition is replaced by dependence properties.

A sequence $(X_i)_{i\in\mathbb{N}}$ of random variables is said to be **pairwise positively dependent (pairwise PD)** if for any $a, b \in \mathbb{R}$,

$$P(X_i > a, X_j > b) \geqslant P(X_i > a)P(X_j > b) \text{ for } i \neq j$$

and it is said to be **pairwise negatively dependent (pairwise ND)** if for any $a, b \in \mathbb{R}$,

$$P(X_i > a, X_j > b) \leqslant P(X_i > a)P(X_j > b) \text{ for } i \neq j.$$

In 1989, Birkel[2] derived a SLLN for a sequence of PD random variables as follow.

Theorem 1.4. [2] *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of pairwise PD random variables with finite variances. Assume that*

(i) $\sup_{i\in\mathbb{N}} E|X_i - E(X_i)| < \infty$, *and*

(ii) $\displaystyle\sum_{i=1}^{\infty}\frac{Cov(X_i, S_i)}{i^2} < \infty.$

Then

$$\frac{S_n - E(S_n)}{n} \xrightarrow{a.s.} 0 \ as\ n \to \infty.$$

On the other hand, Matula[12] derived a SLLN for pairwise ND in the year 1992.

Theorem 1.5. [12] *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of identically distributed and pairwise ND random variables. Then*

$$\frac{S_n}{n} \xrightarrow{a.s.} a \text{ as } n \to \infty \text{ for some constant } a \in \mathbb{R}$$

if and only if

$$E(|X_1|) < \infty$$

and if $E(|X_1|) < \infty$, then $a = E(X_1)$.

The followings are SLLN's for pairwise ND random variables which were obtained between 2003-2011.

Theorem 1.6. (Azarnoosh[1], 2003)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of pairwise ND random variables with finite variances. Assume that

(i) $\sup_{i\in\mathbb{N}} E|X_i| < \infty$, *and*

(ii) $\displaystyle\sum_{i=1}^{\infty} \frac{Var(X_i)}{i^2} < \infty$.

Then

$$\frac{S_n - E(S_n)}{n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

Theorem 1.7. (Azarnoosh[1], 2003)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of pairwise ND random variables such that

(i) $\sup_{i\in\mathbb{N}} E|X_i| < \infty$,

(ii) $\displaystyle\sum_{i=1}^{\infty} P\{|X_i| > i\} < \infty$,

(iii) $\displaystyle\sum_{i=1}^{n} \frac{E(|X_i|I_{\{|X_i|>i\}})}{n} \xrightarrow{a.s.} 0$ *as* $n \to \infty$, *and*

(iv) $\displaystyle\sum_{i=1}^{\infty} \frac{E(|X_i^2|I_{\{|X_i|\leqslant i\}})}{i^2} < \infty$.

Then

$$\frac{S_n - E(S_n)}{n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

Theorem 1.8. (Nili Sani, Azarnoosh and Bozorgnia[15], 2004)

Let $(a_n)_{n\in\mathbb{N}}$ be a positive and increasing sequence such that

$$a_n \to \infty \text{ as } n \to \infty.$$

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of pairwise ND random variables with finite variances such that

(i) $\sup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} \frac{E(|X_i - E(X_i)|)}{a_n} \right) < \infty$, *and*

(ii) $\sum_{i=1}^{\infty} \frac{Var(X_i)}{a_i^2} < \infty.$

Then

$$\frac{S_n - E(S_n)}{a_n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

In case of 2-dimensional array of random variables, we let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a double sequence of random variables and

$$S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{i,j} \quad \text{for } m,n \in \mathbb{N}.$$

We say that $(X_{i,j})_{i,j\in\mathbb{N}}$ satisfies the **strong law of large numbers(SLLN)** if there exist double sequences of real numbers $(a_{m,n})_{m,n\in\mathbb{N}}$ and $(b_{m,n})_{m,n\in\mathbb{N}}$ such that

$$\frac{S_{m,n} - a_{m,n}}{b_{m,n}} \xrightarrow{\text{a.s.}} 0 \text{ as } m,n \to \infty.$$

Note that for $a \in \mathbb{R}$, the notation $a_{m,n} \to a$ as $m,n \to \infty$ means

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall m,n \in \mathbb{N} \left[m,n \geqslant N_\epsilon \to |a_{m,n} - a| < \epsilon \right]. \tag{1.1}$$

A double sequence $(X_{i,j})_{i,j\in\mathbb{N}}$ is said to be **pairwise positively dependent (pairwise PD)** if for any $a,b \in \mathbb{R}$,

$$P(X_{i,j} > a, X_{k,l} > b) \geqslant P(X_{i,j} > a)P(X_{k,l} > b) \quad \text{for } (i,j) \neq (k,l),$$

and it is said to be **pairwise negatively dependent (pairwise ND)** if for any real numbers $a,b \in \mathbb{R}$,

$$P(X_{i,j} > a, X_{k,l} > b) \leqslant P(X_{i,j} > a)P(X_{k,l} > b) \quad \text{for } (i,j) \neq (k,l).$$

In 1998, a SLLN for a 2-dimensional array of pairwise PD random variables was proved by Kim, Beak and Seo[10] which extended the works of Etemadi[6] and Birkel[2].

Theorem 1.9. [10] *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise PD random variables with finite variances. Assume that*

(i) $\sup_{i,j\in\mathbb{N}} E|X_{i,j} - E(X_{i,j})| < \infty$, *and*

(ii) $$\sum_{\substack{i,\,j\\ i\times j\geqslant 1}} \left(\sum_{\substack{k,\,l\\ k\times l\leqslant i\times j}} \frac{Cov(X_{i,j}, X_{k,l})}{(i\times j)^2} \right) < \infty.$$

Then

$$\frac{S_{m,n} - E(S_{m,n})}{m\times n} \xrightarrow{a.s.} 0 \text{ as } m\times n \to \infty.$$

For $a \in \mathbb{R}$, the notation $a_{m,n} \to a$ as $m \times n \to \infty$ means

$$\forall\epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall m,n \in \mathbb{N}\, [m\times n \geqslant N_\epsilon \to |a_{m,n} - a| < \epsilon]\,. \tag{1.2}$$

Note that the convergence of $(a_{m,n})_{m,n\in\mathbb{N}}$ as $m\times n \to \infty$ implies the convergence of $(a_{m,n})_{m,n\in\mathbb{N}}$ as $m,n \to \infty$. In other words, if $a_{m,n} \to a$ as $m\times n \to \infty$, we have $a_{m,n} \to a$ as $m,n \to \infty$, to see this, we assume (1.2) holds. Let $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that for $m,n \in \mathbb{N}$, we have $m\times n \geqslant N_\epsilon$ implies $|a_{m,n} - a| < \epsilon$. Let $m,n \in \mathbb{N}$ such that $m,n \geqslant N_\epsilon$ then we have

$$m\times n \geqslant N_\epsilon \times N_\epsilon \geqslant N_\epsilon \text{ implies } |a_{m,n} - a| < \epsilon.$$

This proves (1.1). By the way, the following example show that the converse is not true in general.

Example 1.1. $a_{m,n} = \dfrac{(-1)^{m+n}(m+n)}{m\times n}$ *for* $m,n \in \mathbb{N}$.

We first show that $a_{m,n} \to 0$ as $m,n \to \infty$. Let $\epsilon > 0$. By Archimedean principle, choose $N \in \mathbb{N}$ with $N > \frac{2}{\epsilon}$. Then for $m,n > N$, we have

$$|a_{m,n} - 0| = |\frac{(-1)^{m+n}(m+n)}{m \times n}| \leqslant \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \frac{1}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Next, we will show that $a_{m,n} \nrightarrow 0$ as $m \times n \to \infty$. Suppose on the contrary that $a_{m,n} \to 0$ as $m \times n \to \infty$. Pick $\epsilon_0 = 1$. Then there exists $N_0 \in \mathbb{N}$ such that for $m,n \in \mathbb{N}$,

$$m \times n \geqslant N_0 \Longrightarrow |a_{m,n} - 0| < \epsilon_0.$$

But for $m = 1, n = N_0$ we have $m \times n \geqslant N_0$ but

$$|a_{m,n} - 0| = \left| \frac{(-1)^{1+N_0}(1+N_0)}{N_0} \right| = \frac{1+N_0}{N_0} \not< 1.$$

This is a contradiction. Therefore $a_{m,n} \nrightarrow 0$ as $m \times n \to \infty$.

One year later, in 1999, Kim, Baek and Han[9] generalized Theorem 1.9 to a weighted sum of 2-dimensional array of pairwise PD random variables.

Theorem 1.10. [9] *Let $(a_{i,j})_{i,j \in \mathbb{N}}$ be a 2-dimensional array of positive numbers and $b_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$ such that*

$$\frac{a_{m,n}}{b_{m,n}} \to 0 \quad \textit{and} \quad b_{m,n} \to \infty \textit{ as } m,n \to \infty.$$

Let $(X_{i,j})_{i,j \in \mathbb{N}}$ be a 2-dimensional array of pairwise PD random variables with finite variances such that

(i) $\sup_{i,j \in \mathbb{N}} E|X_{i,j} - E(X_{i,j})| < \infty$, *and*

(ii) $$\sum_{\substack{i,j \\ i \times j \geqslant 1}} \left(\sum_{\substack{k,l \\ k \times l \leqslant i \times j}} \frac{a_{i,j} a_{k,l} Cov(X_{i,j}, X_{k,l})}{b_{i,j}^2} \right) < \infty.$$

Then

$$\frac{W_{m,n} - E(W_{m,n})}{b_{m,n}} \xrightarrow{a.s.} 0 \textit{ as } m \times n \to \infty.$$

where $W_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} X_{i,j}$.

The purpose of our work is to obtain SLLN for a 2-dimensional array of random variables. The followings are our results.

Theorem 1.11. *Let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be increasing sequences of positive numbers such that* $a_m, b_n \geqslant e$ *and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ *be a 2-dimensional array of random variables with finite variances. Assume that*

(i) there exists a double sequence $(\rho_{i,j})_{i,j\in\mathbb{N}}$ *of nonnegative real numbers such that for each* $m, n \in \mathbb{N}$,

$$Var(S_{m,n}) \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} \rho_{i,j}, \text{ and}$$

(ii) there exist positive real numbers p, q *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\rho_{i,j}}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty.$$

Then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ *such that* $c_{m,n} \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}$ *for every* $m, n \in \mathbb{N}$,

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Theorem 1.12. *Let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be increasing sequences of positive numbers such that* $a_m, b_n \geqslant e$ *and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ *be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist positive real numbers* p, q *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty,$$

then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ *such that* $c_{m,n} \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}$ *for every* $m, n \in \mathbb{N}$,

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Corollary 1.13. *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be increasing sequences of positive numbers such that $a_m, b_n \geqslant e$ and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist $p, q \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty, \tag{1.3}$$

then for any natural number $k \geqslant p + q$,

$$\frac{S_{m,n} - E(S_{m,n})}{(a_m + b_n)^k} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Corollary 1.14. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise ND random variables with finite variances. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{(i \times j)^2} < \infty,$$

then

$$\frac{S_{m,n} - E(S_{m,n})}{81(m \times n)^2} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Theorem 1.15. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ and $(Y_{i,j})_{i,j\in\mathbb{N}}$ be 2-dimensional arrays of random variables on a probability space $(\Omega, \mathcal{F}, P)$. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X_{i,j} \neq Y_{i,j}\} < \infty,$$

then

$$\frac{1}{m \times n}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j} - Y_{i,j}) \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic knowledge in probability which will be used in our work.

2.1 Basic Knowledge in Probability

Definition 2.1. *Let Ω be a set and $\mathcal{F}$ be a σ-algebra. Let $P : \mathcal{F} \to [0,1]$ be a measure such that $P(\Omega) = 1$. Then $(\Omega, \mathcal{F}, P)$ is called a* ***probability space*** *and P, a* ***probability measure****. The set Ω is the* ***sure event*** *and the elements of $\mathcal{F}$ are called* ***events****.*

A ***random variable*** *is a function $X : \Omega \to \mathbb{R}$ with the property that for every Borel set B in $\mathbb{R}$,*

$$X^{-1}(B) := \{\, \omega \in \Omega \mid X(\omega) \in B \,\} \in \mathcal{F}.$$

Note that the events $\{\, \omega \in \Omega \mid X(\omega) > a \,\}$ are always abbreviated by $(X > a)$.

Let E be an event on Ω. A function $I_E : \Omega \to \mathbb{R}$ defined by

$$I_E(\omega) = \begin{cases} 1, & \text{if } \ \omega \in E, \\ 0, & \text{if } \ \omega \notin E \end{cases}$$

is a random variable which is called an ***indicator random variable****.*

Proposition 2.1. *Let X, Y be random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then $X + Y, XY$ and cX are random variables on $(\Omega, \mathcal{F}, P)$ for any $c \in \mathbb{R}$.*

Theorem 2.1. *Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. Then $\liminf\limits_{n\to\infty} X_n, \limsup\limits_{n\to\infty} X_n$ and $\lim\limits_{n\to\infty} X_n$ (if it exists) are random variables.*

Note that the abbreviation

$$\text{“}X_n \xrightarrow{\text{a.s.}} \mu \text{ as } n \to \infty\text{”}$$

means there exists a set $M \subset \Omega$ such that $P(M) = 0$ and for every $\omega \in \Omega - M$,

$$X_n(\omega) \to \mu \text{ as } n \to \infty.$$

Let $A_1, A_2, A_3, ...$ be a sequence of events. We define

$$\limsup_{n\to\infty} A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$$

and we often write "A_n *i.o.*" to be a representative of $\limsup_{n\to\infty} A_n$ where i.o. is the abbreviation for "infinitely often".

Theorem 2.2. [3] *Let* $(X_n)_{n\geqslant 1}$ *be a sequence of random variables on a probability space* $(\Omega, \mathcal{F}, P)$*. Then*

$$X_n \xrightarrow{a.s.} 0 \textit{ as } n \to \infty$$

if and only if for any $\epsilon > 0$,

$$P\{\ |X_n| > \epsilon \ \text{ i.o. }\} = 0$$

Theorem 2.3. [3](Borel-Cantelli Lemma).
Suppose that $A_1, A_2, A_3, ...$ *is a sequence of events such that* $\sum_{n=1}^{\infty} P(A_n) < \infty$*. Then* $P(\limsup_{n\to\infty} A_n) = 0$.

2.2 Expectation Variance and Covariance

Definition 2.2. *Let* X *be a random variable on a probability space* $(\Omega, \mathcal{F}, P)$*. Denote*

$$E(X) := \int_{\Omega} X dP$$

and we say that a random variable X *has a finite expectation if* $E(|X|) < \infty$*. Otherwise we say that the expectation of* X *does not exist and we call* $E(X)$ *an* ***expected value of*** X.

Theorem 2.4. *Let A be an event and I_A be an indicator random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then*

1. $E(I_A) = P(A)$ and

2. if $E(|X|) < \infty$, then

2.1 $|E(X)| \leqslant E(|X|)$ and

2.2 $E(|X|I_A) = 0$ if and only if $P(A) = 0$ or $X = 0$ almost surely on A.

Proposition 2.2. *Let X and Y be random variables and $a, b \in \mathbb{R}$. Then the followings are true.*

1. If $E(X), E(Y) < \infty$ then $E(aX + bY) = aE(X) + bE(Y)$.

2. If $X \leqslant Y$, then $E(X) \leqslant E(Y)$.

3. $|E(X)| \leqslant E(|X|)$.

Definition 2.3. *Let X be a random variable such that $E(|X|) < \infty$. Denote*

$$Var(X) := E[X - E(X)]^2$$

*and we say that a random variable X has a finite variance if $Var(X) < \infty$. Otherwise we say that the variance of X does not exist and we call $Var(X)$ the **variance** of X. For any random variables X, Y, we denote*

$$Cov(X, Y) := E[(X - E(X))(Y - E(Y))]$$

*and we call $Cov(X, Y)$ the **covariance** of X and Y. Note that*

$$0 \leqslant Cov(X, X) = Var(X).$$

Proposition 2.3. *Let X be a random variable such that $E(X^2) < \infty$. Then*

$$Var(X) = E(X^2) - E^2(X).$$

Theorem 2.5. (Chebyshev's inequality).

Let X be any random variable. For any $c \in \mathbb{R}, b > 0$ and $m > 0$, we have

$$P(|X - c| \geqslant b) \leqslant \frac{E|X - c|^m}{b^m}.$$

In particular,

$$P(|X - E(X)| \geqslant b) \leqslant \frac{Var(X)}{b^2}.$$

CHAPTER III

STRONG LAW OF LARGE NUMBERS FOR 2-DIMENSIONAL ARRAY OF PAIRWISE NEGATIVELY DEPENDENT RANDOM VARIABLES

In this chapter, we prove a SLLN for a 2-dimensional array of pairwise ND random variables.

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of random variables. We denote

$$S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{i,j} \text{ for } m, n \in \mathbb{N}.$$

A double sequence $(X_{i,j})_{i,j\in\mathbb{N}}$ is said to be **pairwise negatively dependent (pairwise ND)** if for any $a, b \in \mathbb{R}$ and $i, j, k, l \in \mathbb{N}$ such that $(i, j) \neq (k, l)$,

$$P(X_{i,j} > a, X_{k,l} > b) \leqslant P(X_{i,j} > a)P(X_{k,l} > b).$$

Throughout this work, C stands for a constant which may be different in each appearance.

3.1 Auxiliary Results

Proposition 3.1. ([7] pp.313) *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a double sequence of pairwise ND random variables. Then*

$$Cov(X_{i,j}, X_{k,l}) \leqslant 0 \text{ for } (i, j) \neq (k, l).$$

Proposition 3.2. ([14] pp.42) *Let $(\lambda_{i,j})_{i,j\in\mathbb{N}}$ be a double sequence of positive numbers such that for all $i, j \in \mathbb{N}$,*

$$\lambda_{i+1,j} - \lambda_{i,j} \geqslant 0,\ \lambda_{i,j+1} - \lambda_{i,j} \geqslant 0,\ \lambda_{i+1,j+1} - \lambda_{i+1,j} - \lambda_{i,j+1} + \lambda_{i,j} \geqslant 0 \qquad (3.1)$$

and

$$\lambda_{i,j} \to \infty \text{ as } \max\{i,j\} \to \infty. \tag{3.2}$$

Let $(a_{i,j})_{i,j\in\mathbb{N}}$ *be a double sequence of real numbers. Assume that*

(i) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{\lambda_{i,j}} < \infty$ *and*

(ii) $\sum_{k=1}^{\infty}\frac{a_{i,k}}{\lambda_{i,k}} < \infty$ *for every* $i \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty}\frac{a_{k,j}}{\lambda_{k,j}} < \infty$ *for every* $j \in \mathbb{N}$.

Then

$$\frac{1}{\lambda_{m,n}}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{i,j} \to 0 \quad \text{as } \max\{m,n\} \to \infty.$$

We here note that for a double indexed sequence of real number $(a_{m,n})_{m,n\in\mathbb{N}}$ and $a \in \mathbb{R}$, the notation $a_{m,n} \to a$ as $\max\{m,n\} \to \infty$ means

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall m,n \in \mathbb{N}\left[\max\{m,n\} \geqslant N_\epsilon \to |a_{m,n} - a| < \epsilon\right].$$

The following proposition is a Borel-Cantelli lemma for a sequence of double indexed events .

Proposition 3.3. *Let* $(E_{i,j})_{i,j\in\mathbb{N}}$ *be a double sequence of events on a probability space* $(\Omega, \mathcal{F}, P)$. *Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P(E_{i,j}) < \infty \quad \textit{implies} \quad P\{E_{i,j} \text{ i.o.}\} = 0$$

where $\{E_{i,j} \text{ i.o.}\} = \bigcap_{k=1}^{\infty}\left(\bigcup_{\substack{i,j \\ i\times j \geqslant k}} E_{i,j}\right)$.

Proof. Let $L \in \mathbb{R}$ be such that $L = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P(E_{i,j})$. First note that

$$\lim_{k\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{k}P(E_{i,j}) = \lim_{k\to\infty}\sum_{i=1}^{\lfloor\sqrt{k}\rfloor}\sum_{j=1}^{\lfloor\sqrt{k}\rfloor}P(E_{i,j}) = L,$$

where $\lfloor\sqrt{k}\rfloor$ denote the greatest integer smaller than or equal $\sqrt{k}$. Since $P(E_{i,j}) \geqslant 0$ for all $i, j \in \mathbb{N}$, we have for any $k \in \mathbb{N}$,

$$\sum_{i=1}^{\lfloor\sqrt{k}\rfloor}\sum_{j=1}^{\lfloor\sqrt{k}\rfloor} P(E_{i,j}) \leqslant \sum_{\substack{i,j \\ i\times j\leqslant k}} P(E_{ij}) \leqslant \sum_{i=1}^{k}\sum_{j=1}^{k} P(E_{i,j})$$

which implies that

$$L = \lim_{k\to\infty}\sum_{i=1}^{\lfloor\sqrt{k}\rfloor}\sum_{j=1}^{\lfloor\sqrt{k}\rfloor} P(E_{i,j}) \leqslant \lim_{k\to\infty}\sum_{\substack{i,j \\ i\times j\leqslant k}} P(E_{ij}) \leqslant \lim_{k\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{k} P(E_{i,j}) = L.$$

Hence $\lim\limits_{k\to\infty}\sum\limits_{\substack{i,j \\ i\times j\leqslant k}} P(E_{ij}) = L$. Therefore

$$\begin{aligned}
P\{E_{i,j} \text{ i.o.}\} &= \lim_{k\to\infty} P\Big(\bigcup_{\substack{i,j \\ i\times j\geqslant k}} E_{i,j}\Big) \\
&\leqslant \lim_{k\to\infty}\sum_{\substack{i,j \\ i\times j\geqslant k}} P(E_{i,j}) \\
&= \lim_{k\to\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(E_{i,j}) - \sum_{\substack{i,j \\ i\times j\leqslant k-1}} P(E_{i,j})\right) \\
&= L - \lim_{k\to\infty}\left(\sum_{\substack{i,j \\ i\times j\leqslant k-1}} P(E_{i,j})\right) \\
&= L - L \\
&= 0.
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then*

$$X_{i,j} \xrightarrow{a.s.} 0 \text{ as } i\times j \to \infty$$

if and only if

$$\forall\epsilon > 0, P\{|X_{i,j}| > \epsilon \text{ i.o.}\} = 0. \tag{3.3}$$

Proof. For $\epsilon > 0$ and $k \in \mathbb{N}$, we denote

$$A_k(\epsilon) = \bigcap_{\substack{i,j \\ i\times j\geqslant k}} \{|X_{i,j}| \leqslant \epsilon\} \text{ and } A(\epsilon) = \bigcup_{k=1}^{\infty} A_k(\epsilon).$$

We have

$$\{|X_{i,j}| > \epsilon \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \left(\bigcup_{\substack{i,j \\ i\times j\geqslant k}} \{|X_{i,j}| > \epsilon\} \right) = \bigcap_{k=1}^{\infty} A_k^c(\epsilon). \tag{3.4}$$

($\Rightarrow$) Assume that $X_{i,j} \xrightarrow{\text{a.s.}} 0$ as $i \times j \to \infty$. Then there exists $\Omega_0 \subseteq \Omega$ such that $P(\Omega_0^c) = 0$ and

$$\forall \omega \in \Omega_0, X_{ij}(\omega) \to 0 \text{ as } i \times j \to \infty.$$

Let $\epsilon > 0$ be given. For each $\omega \in \Omega_0$, the convergence of $\{X_{i,j}(\omega)\}$ to 0 as $i \times j \to \infty$ implies that, there exists $K_\epsilon \in \mathbb{N}$ such that

$$i \times j \geqslant K_\epsilon \implies |X_{i,j}(\omega)| \leqslant \epsilon.$$

This shows that $\omega \in A_{K_\epsilon}(\epsilon)$ and so $\Omega_0 \subseteq \bigcup_{k=1}^{\infty} A_k(\epsilon)$. From this facts and (3.4) we have that

$$P(\{|X_{i,j}| > \epsilon \text{ i.o.}\}) = P(\bigcap_{k=1}^{\infty} A_k^c(\epsilon)) \leqslant P(\Omega_0^c) = 0.$$

($\Leftarrow$) Assume that (3.3) holds. Let $\Omega_0 = \bigcap_{n=1}^{\infty} A(\frac{1}{n})$. It follows from (3.4) that

$$\begin{aligned} P(A(\frac{1}{n})) &= P(\bigcup_{k=1}^{\infty} A_k(\frac{1}{n})) \\ &= 1 - P(\bigcap_{k=1}^{\infty} A_k^c(\frac{1}{n})) \\ &= 1 - P(\{|X_{i,j}| > \frac{1}{n} \text{ i.o.}\}) \\ &= 1. \end{aligned} \tag{3.5}$$

Since for each $n \in \mathbb{N}$, $\{|X_{i,j}| < \frac{1}{n+1}\} \subseteq \{|X_{i,j}| < \frac{1}{n}\}$, we have

$$A(\frac{1}{n+1}) \subseteq A(\frac{1}{n}) \text{ for all } n \in \mathbb{N}.$$

It follows from monotone property of probability measure and (3.5) that

$$P(\Omega_0) = P(\bigcap_{n=1}^{\infty} A(\frac{1}{n})) = \lim_{n\to\infty} P(A(\frac{1}{n})) = 1. \tag{3.6}$$

Let $\omega \in \Omega_0$. Then $\omega \in A(\frac{1}{n})$ for all $n \in \mathbb{N}$. In other word, for each $n \in \mathbb{N}$, there exists $K_n \in \mathbb{N}$ such that,

$$i \times j \geqslant K_n \Longrightarrow |X_{i,j}(\omega)| \leqslant \frac{1}{n}. \tag{3.7}$$

For arbitrary $\epsilon > 0$, by Archimedean property, there exists $n_\epsilon \in \mathbb{N}$ such that $\frac{1}{n_\epsilon} < \epsilon$. Then (3.7) holds for n_ϵ, that is, there exists $K_{n_\epsilon} \in \mathbb{N}$ such that

$$i \times j \geqslant K_{n_\epsilon} \Longrightarrow |X_{i,j}(\omega)| \leqslant \frac{1}{n_\epsilon} < \epsilon.$$

This means $\{X_{i,j}(\omega)\}$ converges to 0 for all ω in a set of probability 1 and this complete the proof. □

Theorem 3.1. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of random variables on a probability space $(\Omega, \mathcal{F}, P)$. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(|X_{i,j}| \geqslant \epsilon) < \infty \quad \textit{for all} \quad \epsilon > 0$$

then

$$X_{i,j} \xrightarrow{a.s.} 0 \textit{ as } i \times j \to \infty.$$

Proof. Follow directly from Proposition 3.3 and Proposition 3.4 with

$$E_{i,j}(\epsilon) = \{\ \omega \mid |X_{i,j}(\omega)| \geqslant \epsilon\ \}$$

for $\epsilon > 0$ and $i, j \in \mathbb{N}$. □

3.2 Main Results and Proofs

Theorem 3.2. *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be increasing sequences of real numbers such that $a_m, b_n \geqslant e$ and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ *be a 2-dimensional array of random variables with finite variances. Assume that*

(i) there exist a double sequence $(\rho_{i,j})_{i,j\in\mathbb{N}}$ *of nonnegative real numbers such that for each* $m, n \in \mathbb{N}$*,*

$$Var(S_{m,n}) \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} \rho_{i,j}, \text{ and}$$

(ii) there exist positive real numbers p, q *such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\rho_{i,j}}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty.$$

Then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ *such that* $c_{m,n} \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}$ *for every* $m, n \in \mathbb{N}$*,*

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \quad as \quad m \times n \to \infty.$$

Proof. Assume (i) and (ii) hold. Let $m, n \in \mathbb{N}$ and define

$$f(m) = \lfloor \ln a_m \rfloor \quad \text{and} \quad g(n) = \lfloor \ln b_n \rfloor.$$

Clearly, f and g are increasing and also

$$f(m) \leqslant \ln a_m < f(m) + 1 \quad \text{and} \quad g(n) \leqslant \ln b_n < g(n) + 1$$

which imply that

$$e^{f(m)} \leqslant a_m < e^{f(m)+1} \quad \text{and} \quad e^{g(n)} \leqslant b_n < e^{g(n)+1}.$$

Let $\epsilon > 0$ be given. Then by Chebyshev's inequality,

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\frac{|S_{m,n} - E(S_{m,n})|}{c_{m,n}} \geqslant \epsilon\right\} \\
&\qquad \leqslant C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{Var(S_{m,n})}{c_{m,n}^2} \\
&\qquad \leqslant C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{c_{m,n}^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\rho_{i,j} \\
&\qquad = C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{c_{m,n}^2} \\
&\qquad \leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{a_m^p \times b_n^q} \\
&\qquad \leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{1}{e^{(pf(m)+qg(n))}}. \qquad (3.8)
\end{aligned}$$

We need the following facts to obtain the next inequality. For each $i, j \in \mathbb{N}$, let

$$A_i = \{\ s \in \mathbb{N}\ :\ e^{f(s)+1} \geqslant a_i\ \} \quad \text{and} \quad B_j = \{\ t \in \mathbb{N}\ :\ e^{f(t)+1} \geqslant b_j\ \}$$

and

$$\tilde{i} = \min A_i \quad \text{and} \quad \tilde{j} = \min B_j.$$

Since $i \in A_i$ and $j \in B_j$, we have $\tilde{i} \leqslant i$ and $\tilde{j} \leqslant j$. From this facts and (3.8), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\frac{|S_{m,n} - E(S_{m,n})|}{c_{m,n}} \geqslant \epsilon\right\} \\
&\qquad \leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=\tilde{i}}^{\infty}\sum_{n=\tilde{j}}^{\infty}\frac{1}{e^{(pf(m)+qg(n))}} \\
&\qquad = C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=\tilde{i}}^{\infty}\frac{1}{e^{pf(m)}}\sum_{n=\tilde{j}}^{\infty}\frac{1}{e^{qg(n)}}
\end{aligned}$$

To obtain the next inequality, we have to prove that for each $m \in \mathbb{N}$,

$$a_m \geqslant e \implies 2\lfloor \ln(a_{m+1})\rfloor \geqslant \lfloor \ln(a_m)\rfloor + 1. \qquad (3.9)$$

Let $m \in \mathbb{N}$. Assume that

$$2\lfloor \ln(a_{m+1})\rfloor < \lfloor \ln(a_m)\rfloor + 1.$$

By increasing property of logarithm function,

$$2\lfloor \ln(a_{m+1}) \rfloor < \lfloor \ln(a_m) \rfloor + 1 \leqslant \lfloor \ln(a_{m+1}) \rfloor + 1$$

which implies that $\lfloor \ln(a_{m+1}) \rfloor < 1$. So $\ln(a_{m+1}) < 1$ and hence $a_m < a_{m+1} < e$. Similarly, we have (3.9) holds for $(b_n)_{n\in\mathbb{N}}$. Therefore

$$e^{\lfloor \ln(a_{m+1}) \rfloor} \geqslant e^{\frac{1}{2}(\lfloor \ln(a_m) \rfloor + 1)} \text{ and } e^{\lfloor \ln(b_{n+1}) \rfloor} \geqslant e^{\frac{1}{2}(\lfloor \ln(b_n) \rfloor + 1)}.$$

i.e. for every $m \in \mathbb{N}$,

$$e^{f(m+1)} \geqslant e^{\frac{1}{2}(f(m)+1)} \text{ and } e^{g(m+1)} \geqslant e^{\frac{1}{2}(g(m)+1)}.$$

Now, the inequality becomes

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\frac{|S_{m,n} - E(S_{m,n})|}{c_{m,n}} \geqslant \epsilon\right\}
&\leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=\tilde{i}}^{\infty}\frac{1}{e^{pf(m)}}\sum_{n=\tilde{j}}^{\infty}\frac{1}{e^{qg(n)}} \\
&\leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\sum_{m=f(\tilde{i})}^{\infty}\frac{1}{e^{\frac{1}{2}pm}}\sum_{n=g(\tilde{j})}^{\infty}\frac{1}{e^{\frac{1}{2}qn}}
\end{aligned}$$

where $\sum_{m=f(\tilde{i})}^{\infty}\frac{1}{e^{\frac{1}{2}pm}}$ and $\sum_{n=g(\tilde{j})}^{\infty}\frac{1}{e^{\frac{1}{2}qn}}$ are geometric series whose common ratio $\left(\frac{1}{e}\right)^{\frac{p}{2}}$ and $\left(\frac{1}{e}\right)^{\frac{q}{2}}$ respectively which are both less than 1 for all $p, q > 0$. This guarantees the convergences of series. Hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\frac{|S_{m,n} - E(S_{m,n})|}{c_{m,n}} \geqslant \epsilon\right\} \leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{i,j}\left(\frac{1}{e^{\frac{1}{2}pf(\tilde{i})}}\right)\left(\frac{1}{e^{\frac{1}{2}qg(\tilde{j})}}\right). \tag{3.10}$$

Since $\tilde{i} \in A_i$ and $\tilde{j} \in B_j$, we have

$$\frac{1}{e^{f(\tilde{i})}} < e \cdot \frac{1}{a_i} \text{ and } \frac{1}{e^{g(\tilde{j})}} < e \cdot \frac{1}{b_j}.$$

From this facts and (3.10) together with our assumption (ii), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P\left\{\frac{|S_{m,n} - E(S_{m,n})|}{c_{m,n}} \geqslant \epsilon\right\} \leqslant C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\rho_{i,j}}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty.$$

By Theorem 3.1, we have

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{\text{a.s.}} 0 \text{ as } m \times n \to \infty$$

as desired. □

Theorem 3.3. *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be increasing sequences of positive numbers such that $a_m, b_n \geqslant e$ and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist positive real numbers p, q such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{Var(X_{i,j})}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty,$$

then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ such that $c_{m,n} \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}$ for every $m, n \in \mathbb{N}$,

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Proof. By the virtue of ND property, it follows from Proposition 3.1 that

$$\begin{aligned} Var(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{i,j}) &= \sum_{i=1}^{m}\sum_{j=1}^{n}Var(X_{i,j}) + \sum_{\substack{i,j,k,l \\ (i,j)\neq(k,l)}} Cov(X_{i,j}, X_{k,l}) \\ &\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n}Var(X_{ij}). \end{aligned}$$

Then substituting $VarX_{ij}$ for ρ_{ij} for all $i, j \in \mathbb{N}$ in Theorem 3.2 verifies the proof. □

Corollary 3.4. *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be increasing sequences of positive numbers such that $a_m, b_n \geqslant e$ and*

$$a_m \to \infty \text{ as } m \to \infty \quad \text{and} \quad b_n \to \infty \text{ as } n \to \infty.$$

Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist $p, q \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{Var(X_{i,j})}{a_i^{\frac{p}{2}} \times b_j^{\frac{q}{2}}} < \infty, \tag{3.11}$$

then for any natural number $k \geqslant p + q$,

$$\frac{S_{m,n} - E(S_{m,n})}{(a_m + b_n)^k} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Proof. Let $p, q \in \mathbb{N}$ satisfying (3.11) and $k \in \mathbb{N}$ such that $k \geqslant p + q$. Choose $c_{m,n} = (a_m + b_n)^k$, then

$$c_{m,n} \geqslant (a_m + b_n)^{p+q} \geqslant a_m^p \times b_n^q \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}.$$

It follows from Theorem 3.3 that

$$\frac{S_{m,n} - E(S_{m,n})}{(a_m + b_n)^k} \xrightarrow{\text{a.s.}} 0 \text{ as } m \times n \to \infty.$$

□

Corollary 3.5. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ be a 2-dimensional array of pairwise ND random variables with finite variances. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{(i \times j)^2} < \infty,$$

then

$$\frac{S_{m,n} - E(S_{m,n})}{81(m \times n)^2} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Proof. For each $m, n \in \mathbb{N}$, put $a_m = 3m, b_n = 3n$ and let $c_{m,n} = a_m^2 \times b_n^2$. Then by choosing $p = q = 4$, we have $c_{m,n} \geqslant a_m^{\frac{p}{2}} \times b_n^{\frac{q}{2}}$. The result follows from Theorem 3.3. □

Theorem 3.6. *Let $(X_{i,j})_{i,j\in\mathbb{N}}$ and $(Y_{i,j})_{i,j\in\mathbb{N}}$ be 2-dimensional arrays of random variables on a probability space $(\Omega, \mathcal{F}, P)$. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X_{i,j} \neq Y_{i,j}\} < \infty \tag{3.12}$$

then

$$\frac{1}{m \times n}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j} - Y_{i,j}) \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.$$

Proof. Assume (3.12) holds. Let $\Omega_0 = \bigcap_{k=1}^{\infty} \bigcup_{\substack{i,j \\ i\times j \geqslant k}} \{X_{i,j} \neq Y_{i,j}\}$.

By Proposition 3.3, we have

$$P(\Omega_0) = P\left(\bigcap_{k=1}^{\infty} \bigcup_{\substack{i,j \\ i\times j \geqslant k}} \{X_{i,j} \neq Y_{i,j}\}\right) = P(\{X_{i,j} \neq Y_{i,j}\} \text{ i.o.}) = 0$$

For every $\omega \in \Omega_0^c$, we will first show that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i\times j}(X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty, \tag{3.13}$$

$$\sum_{j=1}^{\infty} \frac{1}{i\times j}(X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty \text{ for every } i \in \mathbb{N} \tag{3.14}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i\times j}(X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty \text{ for every } j \in \mathbb{N}. \tag{3.15}$$

Then from (3.13), (3.14) and (3.15), we can apply Proposition 3.2 with $\lambda_{i,j} = i\times j$ that

$$\frac{1}{m\times n}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j}(\omega) - Y_{i,j}(\omega)) \to 0 \text{ as } \max\{m,n\} \to \infty.$$

We here note that for a double sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ and $a \in \mathbb{R}$,

$$a_{m,n} \to a \text{ as } \max\{m,n\} \to \infty \Longrightarrow a_{m,n} \to a \text{ as } m\times n \to \infty.$$

So for every $\omega \in \Omega_0^c$,

$$\frac{1}{m\times n}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j}(\omega) - Y_{i,j}(\omega)) \to 0 \text{ as } m\times n \to \infty.$$

By the fact that $P(\Omega_0^c) = 1$, we have

$$\frac{1}{m\times n}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j} - Y_{i,j}) \xrightarrow{\text{a.s.}} 0 \text{ as } m\times n \to \infty$$

as desired.

To prove (3.13), (3.14) and (3.15), let $\omega \in \Omega_0^c$. Then there exists $k_\omega \in \mathbb{N}$ such that for $i,j \in \mathbb{N}$,

$$i\times j \geqslant k_\omega \Longrightarrow X_{i,j}(\omega) = Y_{i,j}(\omega). \tag{3.16}$$

Thus for each $\omega \in \Omega_0^c$, $(X_{i,j}(\omega))_{i,j\in\mathbb{N}}$ and $(Y_{i,j}(\omega))_{i,j\in\mathbb{N}}$ are different only finitely many terms. This implies that (3.13) holds.

For fixed $i \in \mathbb{N}$, we can find a large $j_0 \in \mathbb{N}$ such that (3.16) holds for all $j \geqslant j_0$ which means that there are only finitely many different terms of $(X_{i,j}(\omega))_{i,j\in\mathbb{N}}$ and $(Y_{i,j}(\omega))_{i,j\in\mathbb{N}}$. So for fixed $i \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} \frac{1}{i \times j}(X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty.$$

Similarly, for fixed $j \in \mathbb{N}$,

$$\sum_{i=1}^{\infty} \frac{1}{i \times j}(X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty.$$

Now (3.14) and (3.15) are proved and this completes the proof. □

3.3 Examples

Example 3.1. *A box contains pq balls of p different colors and q different sizes in each color. Pick 2 balls randomly.*

Let $\widetilde{X}_{i,j}, i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ be a random variable indicating the presence of a ball of the i^{th} color and the j^{th} size such that

$$\widetilde{X}_{i,j} = \begin{cases} 1, & \textit{if the } i^{th} \textit{ color and } j^{th} \textit{ size of ball is picked,} \\ 0, & \textit{otherwise.} \end{cases}$$

For $i, j \in \mathbb{N}$, let $X_{i,j}$ be a random variable defined by

$$X_{i,j} = \begin{cases} \widetilde{X}_{i,j}, & \textit{if } 1 \leqslant i \leqslant p \textit{ and } 1 \leqslant j \leqslant q, \\ 0, & \textit{otherwise.} \end{cases}$$

Then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ such that $c_{m,n} \geqslant 81(m \times n)^2$ for every $m, n \in \mathbb{N}$, we have

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \textit{ as } m \times n \to \infty.$$

In particular,

$$\frac{S_{m,n} - E(S_{m,n})}{81(m \times n)^2} \xrightarrow{a.s.} 0 \ \mathit{as} \ m \times n \to \infty.$$

Proof. We first show that $X_{i,j}$'s are pairwise ND random variables, i.e. for $i, j, k, l \in \mathbb{N}$ such that $(i, j) \neq (k, l)$ and $a, b \in \mathbb{R}$ such that

$$P(X_{i,j} > a, X_{k,l} > b) \leqslant P(X_{i,j} > a)P(X_{k,l} > b). \tag{3.17}$$

Let $i, j, k, l \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that $(i, j) \neq (k, l)$.
If $i > p$ or $j > q$, we have

$$P(X_{i,j} > a) = \begin{cases} 1, & \text{if } a < 0, \\ 0, & \text{if } a \geqslant 0 \end{cases}$$

which implies that (3.17) holds for all $k, l \in \mathbb{N}$ and $b \in \mathbb{R}$. Similarly, if $k > p$ or $l > q$, we have (3.17) holds for all $i, j \in \mathbb{N}$ and $a \in \mathbb{R}$. WLOG, we will show that (3.17) holds for all $1 \leqslant i, k \leqslant p$ and $1 \leqslant j, l \leqslant q$.
Assume that $1 \leqslant i, k \leqslant p$ and $1 \leqslant j, l \leqslant q$.
case 1 $a < 0$. We have $P(X_{i,j} > a) = 1$. So (3.17) holds.
case 2 $0 \leqslant a < 1$. We have

$$P(X_{i,j} > a) = P(X_{i,j} = 1) = \frac{pq-1}{\binom{pq}{2}} = \frac{2}{pq}.$$

case 2.1 $b < 0$. We have $P(X_{k,l} > b) = 1$ and hence (3.17) holds.
case 2.2 $0 \leqslant b < 1$. We have

$$P(X_{k,l} > b) = P(X_{k,l} = 1) = \frac{pq-1}{\binom{pq}{2}} = \frac{2}{pq}$$

and

$$P(X_{i,j} > a, X_{k,l} > b) = P(X_{i,j} = 1, X_{k,l} = 1) = \frac{1}{\binom{pq}{2}} = \frac{2}{pq(pq-1)}.$$

Since $pq \geqslant 2$, we have $2pq - 2 \geqslant pq$ and then

$$\frac{2}{pq} \geqslant \frac{1}{pq-1}.$$

Therefore

$$P(X_{i,j} > a, X_{k,l} > b) = \frac{2}{pq(pq-1)} \leqslant \left(\frac{2}{pq}\right)^2 = P(X_{i,j} > a)P(X_{k,l} > b).$$

<u>case 2.3</u> $b \geqslant 1$. We have $P(X_{k,l} > b) = 0$. So $P(X_{i,j} > a, X_{k,l} > b) = 0$ and then (3.17) holds.

<u>case 3</u> $a \geqslant 1$. We have $P(X_{i,j} > a) = 0$. So $P(X_{i,j} > a, X_{k,l} > b) = 0$ and then (3.17) holds.

By any cases, we have $X_{i,j}$'s are pairwise ND.

Next, we will find $E(X_{i,j})$ and $Var(X_{i,j})$.

$$E(X_{i,j}) = \sum_{x=0}^{1} xP(X_{i,j} = x) = P(X_{i,j} = 1) = \frac{pq-1}{\binom{pq}{2}} = \frac{2}{pq}$$

and

$$\begin{aligned} Var(X_{i,j}) &= E(X_{i,j}^2) - E^2(X_{i,j}) \\ &= \sum_{x=0}^{1} x^2 P(X_{i,j} = x) - \left(\frac{2}{pq}\right)^2 \\ &= \frac{2}{pq} - \frac{4}{(pq)^2}. \end{aligned}$$

Since

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Var(X_{i,j})}{(i \times j)^2} = \left(\frac{2}{pq} - \frac{4}{(pq)^2}\right) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{(i \times j)^2},$$

we have

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{(i \times j)^2} &= \lim_{m,n\to\infty} \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Var(X_{i,j})}{(i \times j)^2}\right) \\ &= \lim_{m,n\to\infty} \left(\left(\frac{2}{pq} - \frac{4}{(pq)^2}\right) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{(i \times j)^2}\right) \\ &= \left(\frac{2}{pq} - \frac{4}{(pq)^2}\right) \lim_{m,n\to\infty} \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{(i \times j)^2}\right) \\ &= \left(\frac{2}{pq} - \frac{4}{(pq)^2}\right) \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(i \times j)^2} \\ &< \infty \end{aligned}$$

because $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(i\times j)^2}<\infty$. By applying Corollary 3.5, for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ such that $c_{m,n}\geqslant 81(m\times n)^2$ for every $m,n\in\mathbb{N}$, we have

$$\frac{S_{m,n}-E(S_{m,n})}{c_{m,n}}\xrightarrow{\text{a.s.}}0 \text{ as } m\times n\to\infty.$$

In particular,

$$\frac{S_{m,n}-E(S_{m,n})}{81(m\times n)^2}\xrightarrow{\text{a.s.}}0 \text{ as } m\times n\to\infty.$$

□

Example 3.2. *An urn contains pq candies of p different brands and each brand having q distinct flavours. Pick 2 candies randomly.*

Let $\widetilde{X}_{i,j}, i=1,2,\ldots,p$ and $j=1,2,\ldots,q$ be a random variable indicating the presence of a candy of the i^{th} brand and the j^{th} flavour such that

$$\widetilde{X}_{i,j}=\begin{cases}2ij, & \textit{if the } i^{th} \textit{ brand and } j^{th} \textit{ flavour of candy is chosen,}\\ 0, & \textit{otherwise.}\end{cases}$$

For $i,j\in\mathbb{N}$, a random variable $X_{i,j}$ be defined such that

$$X_{i,j}=\begin{cases}\widetilde{X}_{i,j}, & \textit{if } 1\leqslant i\leqslant p \textit{ and } 1\leqslant j\leqslant q,\\ 0, & \textit{otherwise.}\end{cases}$$

Then for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ such that $c_{m,n}\geqslant(3m)^{\frac{p_0}{2}}\times(3n)^{\frac{q_0}{2}}$ for any $p_0,q_0>3$, we have

$$\frac{S_{m,n}-E(S_{m,n})}{c_{m,n}}\xrightarrow{a.s.}0 \textit{ as } m\times n\to\infty.$$

In particular,

$$\frac{S_{m,n}-E(S_{m,n})}{81(m\times n)^2}\xrightarrow{a.s.}0 \textit{ as } m\times n\to\infty.$$

Proof. We first show that $X_{i,j}$'s are pairwise ND random variables. Let $i,j,k,l\in\mathbb{N}$ and $a,b\in\mathbb{R}$ such that $(i,j)\neq(k,l)$.

If $i>p$ or $j>q$, we have

$$P(X_{i,j} > a) = \begin{cases} 1, & \text{if } a < 0, \\ 0, & \text{if } a \geqslant 0 \end{cases}$$

which implies that (3.17) holds for all $k, l \in \mathbb{N}$ and $b \in \mathbb{R}$. Similarly, if $k > p$ or $l > q$ we have (3.17) holds for all $i, j \in \mathbb{N}$ and $a \in \mathbb{R}$. WLOG, we will show that (3.17) holds for all $1 \leqslant i, k \leqslant p$ and $1 \leqslant j, l \leqslant q$.

Assume that $1 \leqslant i, k \leqslant p$ and $1 \leqslant j, l \leqslant q$.

<u>case 1</u> $a < 0$. We have $P(X_{i,j} > a) = 1$. So (3.17) holds.

<u>case 2</u> $0 \leqslant a < 2ij$. We have

$$P(X_{i,j} > a) = P(X_{i,j} = 2ij) = \frac{pq-1}{\binom{pq}{2}} = \frac{2}{pq}.$$

<u>case 2.1</u> $b < 0$. We have $P(X_{k,l} > b) = 1$ and hence (3.17) holds.

<u>case 2.2</u> $0 \leqslant b < 2kl$. We have

$$P(X_{k,l} > b) = P(X_{k,l} = 2kl) = \frac{pq-1}{\binom{pq}{2}} = \frac{2}{pq}$$

and

$$P(X_{i,j} > a, X_{k,l} > b) = P(X_{i,j} = 2ij, X_{k,l} = 2kl) = \frac{1}{\binom{pq}{2}} = \frac{2}{pq(pq-1)}.$$

Since $pq \geqslant 2$, we have $2pq - 2 \geqslant pq$ and then

$$\frac{2}{pq} \geqslant \frac{1}{pq-1}.$$

Therefore

$$P(X_{i,j} > a, X_{k,l} > b) = \frac{2}{pq(pq-1)} \leqslant \left(\frac{2}{pq}\right)^2 = P(X_{i,j} > a)P(X_{k,l} > b).$$

<u>case 2.3</u> $b \geqslant 2kl$. We have $P(X_{k,l} > b) = 0$. So $P(X_{i,j} > a, X_{k,l} > b) = 0$ and then (3.17) holds.

<u>case 3</u> $a \geqslant 2ij$. We have $P(X_{i,j} > a) = 0$. So $P(X_{i,j} > a, X_{k,l} > b) = 0$ and then

(3.17) holds.

By any cases, we have $X_{i,j}$'s are pairwise ND.

Next, we will find $E(X_{i,j})$ and $Var(X_{i,j})$.

$$E(X_{i,j}) = 2ijP(X_{i,j} = 2ij) = 2ij \cdot \frac{pq-1}{\begin{pmatrix} pq \\ 2 \end{pmatrix}} = \frac{4ij}{pq}$$

and

$$\begin{aligned} Var(X_{i,j}) &= E(X_{i,j}^2) - E^2(X_{i,j}) \\ &= 4i^2j^2P(X_{i,j} = 2ij) - \left(\frac{4ij}{pq}\right)^2 \\ &= \frac{8i^2j^2}{pq} - \frac{16i^2j^2}{p^2q^2}. \end{aligned}$$

Let $p_0, q_0 > 3$. Then

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Var(X_{i,j})}{i^{p_0} \times j^{q_0}} &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{i^{p_0} \times j^{q_0}} \left(\frac{8i^2j^2}{pq} - \frac{16i^2j^2}{p^2q^2}\right) \\ &= \left(\frac{8}{pq} - \frac{16}{p^2q^2}\right) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{i^{p_0-2} \times j^{q_0-2}} \end{aligned}$$

and we have

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{Var(X_{i,j})}{i^{p_0} \times j^{q_0}} &= \lim_{m,n\to\infty} \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Var(X_{i,j})}{i^{p_0} \times j^{q_0}}\right) \\ &= \lim_{m,n\to\infty} \left(\left(\frac{8}{pq} - \frac{16}{p^2q^2}\right) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{i^{p_0-2} \times j^{q_0-2}}\right) \\ &= \left(\frac{8}{pq} - \frac{16}{p^2q^2}\right) \lim_{m,n\to\infty} \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{i^{p_0-2} \times j^{q_0-2}}\right) \\ &= \left(\frac{8}{pq} - \frac{16}{p^2q^2}\right) \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i^{p_0-2} \times j^{q_0-2}} \\ &< \infty \end{aligned}$$

because $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i^{p_0-2} \times j^{q_0-2}} < \infty$. By applying Corollary 3.5, for any double sequence $(c_{m,n})_{m,n\in\mathbb{N}}$ such that $c_{m,n} \geqslant (3m)^{\frac{p_0}{2}} \times (3n)^{\frac{q_0}{2}}$ for every $m, n \in \mathbb{N}$, we have

$$\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{\text{a.s.}} 0 \text{ as } m \times n \to \infty.$$

In particular,

$$\frac{S_{m,n} - E(S_{m,n})}{81(m \times n)^2} \xrightarrow{\text{a.s.}} 0 \text{ as } m \times n \to \infty.$$

□

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