CHAPTER III

THE WEDDERBURN-MALCEV THEOREM

The material of this chapter is drawn from reference [2].

Throughout this chapter A denotes a finite-dimensional associative algebra over a field K and M a left A-module which is also assumed to be finite dimension over K.

<u>Definition 3.1</u> : Let A and B be two algebras. By an <u>algebra homomorphism</u> from A into B is meant a function $f:A \rightarrow B$ such that

$f(a+b) = f(a) + f(b)$, $f(ab), = f(a)\ f(b)$, $a,b \in A$.

<u>Definition 3.2</u> : A left A-module W is called an <u>extension</u> of M if there exists an A-homomorphism Φ of W onto M. The kernel of Φ is called the kernel of the extension.

Let N be the kernel of an extension $\Phi : W \rightarrow M$. Then the extension is called a <u>split extension</u> if N is an A-direct summand of W, that is,

$$W = N \oplus M'$$

for some A-submodule M' of W.

With this definition, we have the following result.

<u>Lemma 3.3</u> : Let $\Phi : W \rightarrow M$ be an extension of M with kernel N. The extension is a split extension if and only if there exists an A-homomorphism Ψ of M into W such that $\Phi\Psi = 1$.

<u>Proof</u> : First, suppose that W is a split extension of M, then $W = N \oplus M'$ for some A-submodule M' of W. Since Φ is an A-homomorphism of W onto M, then Φ maps M' isomorphically upon M.

Therefore, there exists $\Psi : M \to M'$ such that $\Phi\Psi = I$; for each $m \in M$, $\Psi(m)$ is uniquely determined by the condition that $\Phi(\Psi(m)) = m$.

To prove that Ψ is an A-homomorphism. Since Φ is an isomorphism of M' onto M, then for each m, $n \in M$ there exist m', n' in M' such that

$$\Phi(m') = m,$$
$$\Phi(n') = n.$$

Therefore,

$$\begin{aligned}\Psi(m+n) &= \Psi(\Phi(m') + \Phi(n')) \\ &= \Psi(\Phi(m' + n')) \\ &= m' + n' \\ &= \Psi(m) + \Psi(n)\end{aligned}$$

For a A, we have

$$\begin{aligned}\Psi(am) &= \Psi(a\Phi(m')) \\ &= \Psi(\Phi(am')) \\ &= am' \\ &= a\Psi(m).\end{aligned}$$

Therefore, Ψ is an A-homomorphism with the required property.

Conversely,,let Ψ : $M \longrightarrow W$ be an A-homomorphism such that $\Phi\Psi = 1$. Let $M' = \Psi(M)$, then Φ maps M' isomorphically upon M. For each $a \in W$, there exists $m' \in M'$ such that

$$\Phi(a) = \Phi(m).$$

Hence ,

$$\Phi(a-m) = 0.$$

Since N is the kernel of Φ, then $a-m'$ is in N and $M' \cap N = \{0\}$.

Therefore,

$$W = N \oplus M$$

This completes the proof of the lemma.

Q.E.D.

Remark : Let $\Phi : W \longrightarrow M$ be an arbitrary extension with kernel N, and let $T = \mathrm{Hom}_K(M,N)$. If we define operations in T by

$$(za)m = z(am),$$

and

$$(az)m = a(zm), \qquad m \in M, \ a \in A, \ z \in T$$

Then T becomes an (an(A,A)-bimodule. Let Ψ be a K-homomorphism of M into W such that $\Phi\Psi = 1$; the existence of Ψ is clear since there exist K-subspaces of W complementary to N. Define $f \in \mathrm{Hom}_K(A,T)$ by

$$f(a)m = \Psi(am) - a\Psi(m), \qquad a \in A, \ m \in M$$

The function f measures the extent to which Ψ misses being an A-homomorphism.

We can see that f(a)m is an element of N.

because for all $a \in A$ and $m \in M$ we have

$$\Phi(f(a)m) = \Phi[\Psi(am) - a\Psi(m)]$$

$$= \Phi\Psi(am) - \Phi(a\Psi(m))$$

Since Φ is an A-homomorphism and $\Phi\Psi = 1$, we obtain

$$\Phi(f(a)m) = am - am = 0.$$

To check that f is a K-homomorphism,

$$f(\alpha a)m = \Psi(\alpha am) - (\alpha a)\Psi(m), \qquad \alpha \in K$$

$$= \alpha\Psi(am) - \alpha a\Psi(m)$$

$$= \alpha f(a)m, \qquad \alpha \in K$$

Furthermore, we have

$$(af(b) + f(a)b)m = af(b)m + f(a)bm$$

$$= a[\Psi(bm) - b\Psi(m)] + [\Psi(abm) - a\Psi(bm)]$$

$$= a\Psi(bm) - ab\Psi(m + \Psi(abm) - a\Psi(bm)$$

$$= \Psi(abm) - ab\Psi(m)$$

$$= f(ab)m$$

That is, $\quad f(ab) = af(b) + f(a)b$

This remarks lead to the following definition.

<u>Definition 3.3</u> : Let T be an arbitrary (A,A)-bimodule.
A K-homomorphism $f : A \longrightarrow T$ is called a <u>generalized derivation</u> if $f(ab) = af(b) + f(a)b$ for all a,b in A.

For any fixed $t \in T$, if we define $f : A \longrightarrow T$ such that

$$f(a) = at - ta, \qquad a \in A$$

Then, we can see that f is a generalized derivation because

$$\begin{aligned} af(b) + f(a)b &= a(bt-tb)+(at-ta)b \\ &= abt - atb + atb - tab \\ &= abt - tab \\ &= f(ab) \end{aligned}$$

This generalized derivation is called an <u>inner generalized derivation.</u>

The connection between these ideas and extensions of modules is given by the following lemma.

<u>Lemma 3.4</u> : Let W be an A-module and $\Phi : W \longrightarrow M$ be an extension of a left A-module M with kernel N, let $\Psi : M \longrightarrow W$ be a K-homomorphism such that $\Phi\Psi = 1$. The extension $\Phi : W \longrightarrow M$ is a split extension iff the generalized derivation $f : A \longrightarrow T = Hom_K(M,N)$ given by

$$f(a)m = \Psi(am) - a\Psi(m), \quad a \in A, \quad m \in M$$

is an inner generalized derivation.

<u>Proof</u> : First, suppose that f is an inner generalized derivation. Then $\exists$ an element $t \in T$ such that

(1) $$f(a) = at - ta, \qquad a \in A.$$

Let Ψ' be a K-homomorphism of $M \longrightarrow W$ such that

$$\Psi'(m) = \Psi(m) + t(m), \qquad m \in M.$$

Then $\Phi\Psi'(m) = \Phi\Psi(m) + \Phi\, t(m)$

Since $t \in T$, then $t(m) \in N$ which is a kernel of Φ; $\Phi t = 0$

and since $\Phi\Psi = 1$, we have

$$\Phi\Psi'(m) = m \quad , \quad m \in M$$

That is $\Phi\Psi' = 1$.

Moreover, for all $a \in A$ we have

$$\Psi'(am) = \Psi(am) + t(am).$$

Since $f(a)m = \Psi(am) - a\Psi(m)$, then

$$\Psi'(am) = f(a)m + a\Psi(m) + t(am)$$

and since T is an (A,A)-bimodule $t(am) = (ta)m$.

Therefore, $\Psi'(am) = f(a)m + a\Psi(m) + (ta)m$.

Then from (1),

$$\begin{aligned}\Psi'(am) &= a\Psi(m) + (at - ta)m + (ta)m \\ &= a\Psi(m) + (at)m - (ta)m + (ta)m \\ &= a\Psi(m) + a\,t(m) \\ &= a\Psi'(m).\end{aligned}$$

Hence, Ψ' is an A-homomorphism of $M \longrightarrow W$.

Then by lemma 3.2, $\Phi : W \longrightarrow M$ is a split extension .

Conversely, suppose that $\exists$ an A-homomorphism $\Psi^* : M \longrightarrow W$ such that $\Phi\Psi^* = 1$.

Let t be defined by

(2) $$t(m) = \Psi^*(m) - \Psi(m), \qquad m \in M$$

Then

$$\Phi\, t(m) = \Phi\Psi^*(m) - \Phi\Psi(m),$$

$$= 0\ , \qquad m \in M$$

That is $\Phi t = 0$ and $t\ (m)$ is in N. Since Ψ^* and Ψ are K-homomorphism, t is a K-homomorphism. Therefore $t \in T = \mathrm{Hom}_K\ (M,N)$. Moreover, for all $a \in A$ we have

$$f(a)m = \Psi\,(am) - a\,\Psi(m), \qquad a \in A,\ m \in M$$

From (2), we get

$$f(a)m = \left[\Psi^*(am) - t(am)\right] - a\left[\Psi^*(m) - t(m)\right]$$

Since Ψ^* is an A-homomorphism, $\Psi^*(am) = a\Psi^*(m)$. Therefore

$$f(a)m = a\,\Psi^*(m) - t(am) - a\,\Psi^*(m) + a\,t(m)$$

$$= (a\,t - ta)m$$

That is $f(a) = a\,t - ta$, and the lemma is proved

Q.E.D.

Now, we come to the case of extensions of algebras which is similar to the case of the extensions of modules.

<u>Definition 3.5</u> : Let B be a finite-dimensional algebra, and $\Phi : B \to A$ a homomorphism of B onto an algebra A, with kernel N. Then B is called an <u>extension</u> of A with kernel N.

The extension $\Phi : B \to A$ is called a <u>split extension</u> if there exists an algebra-homomorphism Ψ of $A \to B$ such that $\Phi\Psi = 1$.

The proof of the following lemma is similar to the proof in the module case.

<u>Lemma 3.6</u> : Let $\Phi : B \longrightarrow A$ be an extension of A with kernel N. The extension is a split extension if and only if there exists a subalgebra A_1 of B such that

$$B = A_1 \oplus N \qquad \text{(vector space direct sum).}$$

<u>Proof</u> : If the extension is a split extension, then there exists an algebra homomorphism $\Psi : A \longrightarrow B$ such that $\Phi\Psi = 1$. Let $A' = \Psi(A)$, then Φ maps A' isomorphically onto A. Therefore, for each b in B there exists $a' \in A'$ such that

$$\Phi(b) = \Phi(a').$$

That is

$$\Phi(b-a') = 0 .$$

Since N is the kernel of Φ, $b-a'$ is in N and $A' \cap N = \{0\}$.

Therefore,

$$B = A' \oplus N \qquad \text{(vector space direct sum)}$$

Conversely, let

$$B = A' \oplus N \qquad \text{(vector space direct sum)}$$

Since Φ is a homomorphism of B onto A with kernel N and $A' \cap N = \{0\}$, Φ maps A' isomorphically onto A.

That is, for each a in A, we may let $\Psi(a)$ to be uniquely determined in A such that

$$\Phi(\Psi(a)) = a$$

Then Ψ is an algebra homomorphism with the required property, and B is a split extension.

This completes the prove of the lemma.

<u>Q.E.D.</u>

<u>Remark</u> : As in the case of modules, we can associate with an extension $\Phi : B \longrightarrow A$ a function from A to an (A,A)-bimodule. Let Ψ be a K-homomorphism of $A \longrightarrow B$ such that $\Phi\Psi = 1$ and let N be the kernel of Φ, the equation

(1) $$f(a,b) = \Psi(ab) - \Psi(a)\Psi(b), \qquad a,b \in A$$

is a bilinear function such that

$$\Phi f(a,b) = \Phi\Psi(ab) - \Phi(\Psi(a)\Psi(b))$$

Since Φ is algebra homomorphism of $B \longrightarrow A$ and $\Phi\Psi = 1$, then

$$\Phi f(a,b) = ab - ab = 0 \qquad a,b \in A.$$

Therefore $f : A \times A \longrightarrow N$. The function f measures the extent to which Ψ fails to be an algebra homomorphism.

From (1), we have

$$\Psi((ab)c) = f(ab,c) + \Psi(ab)\Psi(c)$$

Using (1) again, we get

(2) $$\Psi((ab)c) = f(ab,c) + \Psi(a)\Psi(b)\Psi(c) + f(a,b)\Psi(c)$$

where as

$$\Psi(a(bc)) = f(a,bc) + \Psi(a)\Psi(bc)$$

$$= f(a,bc) + \Psi(a)\Psi(b)\Psi(c) + \Psi(a)f(b,c)$$

Subtracting (2) with this, we obtain

$$f(ab,c) - f(a,bc) + f(a,b)\Psi(c) - \Psi(a)\, f(b,c) = 0$$

In order to define f to be a function from A to an (A,A)-bimodule we have to make N into an (A,A)-bimodule in which the operations are defined by

$$na = n\Psi(a) \quad \text{and}$$

$$an = \Psi(a)n, \qquad n \in N,\ a \in A$$

and the sufficient condition to make N into an (A,A)-bimodule is that $N^2 = \{0\}$.

For in this case,

$$(na)b - n(ab) = (n\Psi(a))\Psi(b) - n\Psi(ab)$$

$$= - nf(a,b)$$

Since f(a,b) is in N and $N^2 = \{0\}$,

$$(na)b - n(ab) = 0 .$$

Now, we come to the next definition which is motivated by the module case.

<u>Definition 3.7</u> : Let A be an algebra over K and N an (A,A)-bimodule. A bilinear function $f : A \times A \longrightarrow N$ is called a <u>factor set</u> provided that

$$f(ab,c) - f(a,bc) + f(a,b)c - af(b,c) = 0 \text{ for all } a,\ b,\ c \in A.$$

The factor set f is called a <u>split factor set</u> if $\exists$ a linear transformation $F : A \longrightarrow N$ such that for all a and b

$$f(a,b) = aF(b) - F(ab) + F(a)b.$$

<u>Lemma 3.8</u> : Let $\Phi : B \to A$ be an extension whose kernel N has the property that $N^2 = \{0\}$, and let f be the factor set defined by

(1) $$f(a,b) = \psi(ab) \quad (a) \quad (b)$$

relative to a K-homomorphism $\psi : A \to B$ such that $\Phi\psi = 1$. Then f is a split factor set if and only if the extension is a split extension.

<u>Proof</u> : First, suppose that f is a split factor set, then there exists a linear transformation $F: A \to N$ such that for all a,b in A

(2) $$f(a,b) = aF(b) - F(ab) + F(a)b.$$

Define the linear map $\psi' : A \to B$ by

$$\psi'(a) = \psi(a) + F(a).$$

Then,

$$\Phi\psi'(a) = \Phi\psi(a) + \Phi F(a)$$

Since F(a) is in N, then $\Phi F(a) = 0$. Therefore $\Phi\psi'(a) = a$ implies that $\Phi\psi' = 1$

Furthermore, for all a, b in A, we have

$$\psi'(ab) = \psi(ab) + F(ab)$$

Using (1), we get

$$\psi'(ab) = f(a,b) + \psi(a)\psi(b) + F(ab).$$

Then by using (2)

$$\psi'(ab) = aF(b) - F(ab) + F(a)b + \psi(a)\psi(b) + F(ab)$$
$$= aF(b) + F(a)b + \psi(a)\psi(b)$$

Since $F(a), F(b) \in N$

$$F(a)b = F(a)\psi(b) \quad \text{and}$$

$$aF(b) = \psi(a)F(b)$$

and $\quad F(a)F(b) = 0$, for $N^2 = \{0\}$.

Therefore, $\psi'(ab) = \psi(a)F(b)+F(a)\psi(b)+\psi(a)\psi(b)+F(a)F(b)$

$$= [\psi(a)+F(a)][\psi(b)+F(b)]$$

$$= \psi'(a)\psi'(b).$$

That is ψ' is an algebra homomorphism such that $\phi\psi' = 1$.

Thus, the extension is a split extension

Conversely, suppose that the extension is a split extension. Then $\exists$ an algebra homomorphism $\psi' : A \longrightarrow B$ such that $\phi\psi' = 1$. Define a linear transformation F by

(3) $$F(a) = \psi'(a) - \psi(a), \qquad a \in A$$

Then, $$\phi F(a) = \phi\psi'(a) - \phi\psi(a)$$

$$= a - a$$

$$= 0 \qquad \text{for all a in A.}$$

This implies that $\phi F = 0$ and F(a) is in N for all a in A : $F : A \longrightarrow N$.

Moreover, for all a, b in A we have from (1) that

$$f(a,b) = \psi(ab) - \psi(a)\psi(b)$$

Using (3), we get

$$f(a,b) = [\psi'(ab)-F(ab)] - [\psi'(a)-F(a)][\psi'(b)-F(b)]$$

$$= \psi'(ab)-F(ab)-\psi'(a)\psi'(b)+F(a)\psi'(b)+\psi'(a)F(b)-F(a)F(b)$$

Since $N^2 = 0$, $F(a)F(b) = 0$ and since Ψ' is an algebra homomorphism,

$\Psi'(ab) = \Psi'(a)\Psi'(b)$.

Therefore,

$$f(a,b) = F(a)\Psi'(b) + \Psi'(a)F(b) - F(ab).$$

By using (3) again

$$f(a,b) = F(a)\left[F(b)+\Psi(b)\right] + \left[F(a)+\Psi(a)\right]F(b)-F(ab)$$

$$= F(a)\Psi(b)+\Psi(a)F(b)-F(ab).$$

Since the operations in N is defined by

$$na = n\Psi(a) \quad \text{and}$$

$$an = \Psi(a)n., \qquad \text{we obtain}$$

$$f(a,b) = F(a)b+aF(b)-F(ab).$$

That is f is a split factor set, and the lemma is proved

Q.E.D.

Theorem 3.9 : Let A be a separable algebra over a field K, then every generalized derivation is inner and every factor set defined on A is a split factor set.

Proof : Since A is separable, then by the definition of separable albebra in chapter I, there exists a K-basis $\{a_1,\ldots,a_n\}$ of A and a set of elements $\{a_1',\ldots a_n'\}$ of A such that

(1) $$\sum_{i=1}^{n} a_i' a_i = 1 \qquad \text{and}$$

(2) For all a in A

$$a_i a = \sum_{j=1}^{n} \lambda_{ij}(a) a_j \quad \text{where} \quad \lambda_{ij}(a) \in K$$

implies

$$a a_i' = \sum_{j=1}^{n} a_j' \lambda_{ji}(a).$$

Let T be an (A,A)-bimodule, and let $f : A \longrightarrow T$ be a generalized derivation of A. Let $t = \sum_{i=1}^{n} a_i' f(a_i)$.

Since $f(a_i)$ is in T and T is an (A,A)-bimodule, we have that $t \in T$. Then

$$at - ta = a \sum_{i=1}^{n} a_i' f(a_i) - \left[\sum_{i=1}^{n} a_i' f(a_i) \right] a$$

$$= \sum_{i=1}^{n} a a_i' f(a_i) - \sum_{i=1}^{n} a_i' f(a_i) a$$

Since f is a generalized derivation,

$$f(ab) = af(b) + f(a)b , \qquad a, b \in A$$

Therefore,

$$at-ta = \sum_{i=1}^{n} a a_i' f(a_i) - \sum_{i=1}^{n} a_i' f(a_i a) + \sum_{i=1}^{n} a_i' a_i f(a).$$

Using (1), we get

$$\text{(3)} \qquad at-ta = \sum_{i=1}^{n} a a_i' f(a_i) - \sum_{i=1}^{n} a_i' f(a_i a) + f(a).$$

Claim that $\sum_{i=1}^{n} a_i' f(a_i a) = \sum_{i=1}^{n} aa_i' f(a_i)$.

To prove this, let us consider $\sum_{i=1}^{n} a_i' f(a_i a)$,

$$\sum_{i=1}^{n} a_i' f(a_i a) = \sum_{i=1}^{n} a_i' f\left(\sum_{j=1}^{n} \lambda_{ij}(a) a_j\right).$$

Since f is a K-homomorphism and $\lambda_{ij}(a)$ is in K, we have

$$\sum_{i=1}^{n} a_i' f(a_i a) = \sum_{i=1}^{n} a_i' \sum_{j=1}^{n} \lambda_{ij}(a) f(a_j)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_i' \lambda_{ij}(a) f(a_j).$$

Using (2),

$$\sum_{i=1}^{n} a_i' f(a_i a) = \sum_{j=1}^{n} aa_j' f(a_j) = \sum_{i=1}^{n} aa_i' f(a_i).$$

Therefore, in (3) we obtain

$$at - ta = f(a).$$

That is f is an inner generalized derivation.

Secondly, let h be a factor set such that $h : A \times A \longrightarrow N$ where N is an (A,A)-bimodule. Let F be defined by

$$F(a) = \sum_{i=1}^{n} h(a, a_i') a_i .$$

Since $h(a, a_i')$ is in N which is an (A,A)-bimodule, F(a) is in N for all a in A.

Therefore, F is a linear map of A into N. Then, we have

$$aF(b)-F(ab)+F(a)b = \sum_{i=1}^{n} ah(b,a_i')a_i - \sum_{i=1}^{n} h(ab,a_i')a_i + \sum_{i=1}^{n} h(a,a_i')a_i b$$

Since h is a factor set,

$$h(ab,a_i')-h(a,ba_i')+h(a,b)a_i'-ah(b,a_i') = 0 \quad \text{for all } a,b,a_i' \text{ in } A.$$

Therefore,

$$\begin{aligned} aF(b)-F(ab)+F(a)b &= \Big[\sum_{i=1}^{n} h(ab,a_i')a_i - \sum_{i=1}^{n} h(a,b\ a_i')a_i \\ &\quad + \sum_{i=1}^{n} h(a,b)a_i'\ a_i \Big] - \sum_{i=1}^{n} h(ab,a_i')a_i \\ &\quad + \sum_{i=1}^{n} h(a,a_i')a_i b \\ &= \sum_{i=1}^{n} h(a,b)a_i'\ a_i - \sum_{i=1}^{n} h(a,ba_i')a_i + \sum_{i=1}^{n} h(a,a_i')a_i b. \end{aligned}$$

Using (1), we get

$$aF(b)-F(ab)+F(a)b) = h(a,b)- \sum_{i=1}^{n} h(a,ba_i')a_i + \sum_{i=1}^{n} h(a,a_i')a_i b$$

Consider $\sum_{i=1}^{n} h(a,a_i')a_i b$, we have

$$\begin{aligned} \sum_{i=1}^{n} h(a,a_i')a_i b &= \sum_{i=1}^{n} h(a,a_i') \sum_{j=1}^{n} \lambda_{ij}(b)a_j \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_{ij}(b)h(a,a_i')a_j \end{aligned}$$

$$= \sum_{j=1}^{n} h(a, \sum_{i=1}^{n} \lambda_{ij}(b)a_i')a_j$$

$$= \sum_{j=1}^{n} h(a,ba_j')a_j.$$

Therefore,

$$aF(b)-F(ab)+F(a)b = h(a,b)$$

That is h is a split factor set and this completes the proof of the theorem.

Q.E.D.

Theorem 3.10 (Wedderburn-Malcev Theorem) :

Let B be a finite dimensional algebra with multiplication identity 1 over a field K. with radical N such that the residue class algebra $A = B/_N$ is separable. Then there exists a semi simple subalgebra S of B such that $B = S \oplus N$ (vector space direct sum) If S_1 and S_2 are subalgebras such that $B = S_i \oplus N$, $i = 1,2$, then there exists an element $n \in N$ such that

$$S_1 = (1-n)\ S_2(1-n)^{-1}$$

Proof : First, we shall prove that at least one subalgebra S exists by transfinite induction on the dimension of B, assuming that S exists for all algebras of dimension less than the dimension of B. If $N^2 = \{0\}$, then from Theorem 3.9 ; every factor set defined on A is a split factor set and from Lemma 3.8 ; the extension is a split extension. Therefore, by using Lemma 3.6 we obtain the result that there exists a subalgebra A_1 of B such that $B = A_1 \oplus N$ (vector space direct sum) and this proves the theorem.

If $N^2 \neq \{0\}$, then $(B/N^2 : K) < (B : K)$. Now, we shall prove that $N/_N2$ is a nilpotent ideal of $B/_N2$. Since N is a radical of B, N is a maximal nilpotent ideal of B, that is there exists $k > 0$ such that $N^k = \{0\}$. We have

$$(N/_N2)^k = \left\{ \sum_{\text{finite}} (a_{i1}+N^2)(a_{i2}+N^2)\dots(a_{ik}+N^2) \,\middle|\, a_{ij} \in N,\ j = 1,2,\dots k \right\}$$

$$= \left\{ \sum_{\text{finite}} (a_{i1}a_{i2}\dots a_{ik} + N^2) \,\middle|\, a_{ij} \in N,\ j = 1,\dots k \right\}$$

Since $N^k = \{0\}$, $a_{i1}a_{i2}\dots a_{ik} = 0$ for all i finite. Therefore $(N/_N2)^k = N^2$. It is clear that $N/_N2$ is an ideal of $B/_N2$. Hence $N/_N2$ is a nilpotent ideal of $B/_N2$. Claim that it is a maximal nilpotent ideal of $B/_N2$. Suppose that $J/_N2$ is a nilpotent ideal of $B/_N2$ such that

$$J/_N2 \supseteq N/_N2$$

This implies that $J \supseteq N$. Since $J/_N2$ is the nilpotent ideal of $B/_N2$, we can prove in the same way as above that J is a nilpotent ideal of B. But N is the maximal nilpotent ideal of B. Therefore $J = N$. That is $N/_N2$ is a radical of $B/_N2$ and we have

$$(B/_{N}2) \;/\; (N/_{N}2) \cong B/_{N} = A$$

which is separable. We can conclude that $N/_{N}2$ is the radical of $B/_{N}2$, $(B/_{N}2) \;/\; (N/_{N}2)$ is separable and the dimension of $B/_{N}2$ is less than the dimension of B. Therefore, we can apply our induction hypothesis that there exists a subalgebra $S_{1/N}2$ of $B/_{N}2$ such that

(1) $$B/_{N}2 = S_{1/N}2 \oplus N/_{N}2$$

Therefore,

(2) $$B = S_1 + N, \text{ where } S_1 \cap N = N^2 .$$

Since N is nilpotent, we know that $N \neq N^2$ and this implies that $S_1 \neq B$. Moreover, from (1) we have

$$S_{1/N}2 \cong (B/_{N}2) \;/\; (N/_{N}2) \cong B/_{N} = A$$

which is separable.

Again, we can apply the induction hypothesis to S_1, yielding a subalgebra S of S_1 such that

(3) $$S_1 = S + N^2 \text{ where } S \cap N^2 = 0 .$$

Combining (2) and (3), we obtain

$$B = S + N , \quad S \cap N = 0$$

and the first part of the theorem is proved.

Now, for the second part of the theorem, suppose S_1 and S_2 are subalgebras of B such that $B = S_i \oplus N$, $i = 1,2$. Then by Lemma 3.6 $\exists$ algebra homomorphisms Ψ_1 and Ψ_2 of A into B such that $\Phi\Psi_1 = 1$ and $\Phi\Psi_2 = 1$, where Φ is the natural mapping of $B \longrightarrow A$ and $S_i = \Psi_i(A)$ $i = 1,2$. Because the $\{\Psi_i\}, i = 1,2$ are algebra homomorphisms, N becomes an (A,A)-bimodule if we define

(4) $$na = n\Psi_2(a) \quad \text{and}$$
$$an = \Psi_1(a)n$$

Then consider the function $f : A \longrightarrow B$ defined by

$$f(a) = \Psi_1(a) - \Psi_2(b)$$

Since $\Phi\Psi_i = 1$, $i = 1,2$

$$\Phi f(a) = a - a = 0 \quad \text{for all a in A}$$

That is f(a) is in N for all a in A, and we have

$$f(ab) = \Psi_1(ab) - \Psi_2(ab)$$

Since Ψ_1 and Ψ_2 are algebra homomorphisms

$$\begin{aligned} f(ab) &= \Psi_1(a)\,\Psi_1(b) - \Psi_2(a)\,\Psi_2(b) \\ &= \Psi_1(a)[\Psi_1(b) - \Psi_2(b)] + [\Psi_1(a) - \Psi_2(a)]\Psi_2(b) \\ &= \Psi_1(a)\,f(a) + f(a)\,\Psi_2(b) \end{aligned}$$

But $f(a)$ is in N, by equation (4) we can write

$$f(ab) = af(b) + f(a)b, \qquad a, b \in A.$$

Therefore, f is a generalized derivation, and since A is separable, f is an inner generalized derivation by Theorem 3.9. Thus there exists n in N such that

$$f(a) = an - na$$

$$= \Psi_1(a)n - n\Psi_2(a), \qquad a \in A.$$

But $f(a) = \Psi_1(a) - \Psi_2(a)$. Therefore,

$$\Psi_1(a)(1-n) = (1-n)\Psi_2(a), \qquad a \in A.$$

Since N is the radical of B and $n \in N$, n is a nilpotent element. That is there exists $k > 0$ such that $n^k = 0$. Therefore,

$$1 - n^k = 1$$

That is

$$(1-n)(1+n+n^2+\ldots+n^{k-1}) = 1$$

Hence $(1-n)$ is invertible and

$$\Psi_1(a) = (1-n)\Psi_2(a)(1-n)^{-1} \quad \text{for all } a \in A.$$

That is

$$S_1 = (1-n)S_2(1-n)^{-1}, \qquad n \in N$$

Q.E.D.