CHAPTER II

PRELIMINARIES

Let us begin this thesis by recalling several definitions.

Definition 1 : A ring R is said to satisfy the descending chain condition for ideals if, given descending chain of ideals of R,

$$I_1 \supsetneq I_2 \supseteq I_3 \supsetneq \dots \supsetneq I_n \supseteq \dots,$$

there exists an integer n such that $I_n = I_{n+1} = I_{n+2} = \dots$

This definition is equivalent to the next definition.

Definition 2 : The minimum condition (for ideals) is said to hold in a ring R if every nonempty set of ideals of R, partially ordered by inclusion, has at least one minimal element.

Definition 3 : An algebra A over a field K is a ring A with an identity element which is at the same time a vector space over K. Moreover the scalar multiplication in the vector space and the ring multiplication are required to satisfy the axiom

$$\alpha(ab) = (\alpha a)b = \alpha(ab), \quad \alpha \in K, \quad a,b \in A .$$

Observe that these definitions yield the following result.

Proposition 4 : Every algabra A finite dimensional over K is a ring with minimum condition.

<u>Proof</u> : Ler $k \in K$ and $b \in A$, then Definition 3 implies that

$$kb = k(1.b) = (k1)b,$$

and

(1) $$(k1)b = k(1b) = k(b1) = b(k1) .$$

These imply that the set of elements

$$K = \{ k1 \mid k \in K \}$$

is contained in the center of A, and is a field isomorphic to K . We shall always identify K and K and regard K as embedded in A. Then (1) shows that every left, right, or two-sided ideal in the ring A is also a K subspace of the vector space A. Since the subspaces of a finite-dimensional vector space satisfy the descending chain condition, if follows that A is a ring with minimum condition.

<u>Q.E.D.</u>

<u>Definition 5</u> : Let R be a ring with identity. By a <u>left module over R</u> (or a left R-module), we mean a commutative group M (written additively) together with an operation of multiplication which associates with each $r \in R$ and $a \in M$ a unique element $ra \in M$ such that the following conditions are satisfied :

(1) $(r+s)a = ra + sa$

(2) $(rs)a = r(sa)$

(3) $r(a+b) = ra + rb$

(4) $1a = a$

for all $r, s \in R$ and $a, b \in M$. The parallel notion of a <u>right R-module</u> can be defined symmetrically.

<u>Definition 6</u> : An abelian group M is an <u>(R,R)-bimodule</u> over the ring R if M is both a left R-module and a right R-module, and if we have

$$(sm)r = s(mr)$$

for all $r, s \in R$ and $m \in M$.

<u>Definition 7</u> : A <u>homomorphism</u> of a ring R into a ring R is a mapping $f : R \longrightarrow R$ such that

$$f(x+y) = f(x) + f(y),$$

$$f(xy) = f(x)f(y), \qquad x, y \in R.$$

Let M,N be commutative groups, written additively and let Hom(M,N) denote the set of all homomorphisms of M into N. If we define the sum of two homomorphism f and g by

$$(f+g)m = f(m) + g(m), \qquad m \in M.$$

Then Hom(M,N) becomes a commutative group.

The additive group Hom(M,M) becomes a ring with an identity element if we define multiplication of homomorphisms by composition, namely

$$(fg)(m) = f(g(m)), \qquad m \in M.$$

Now, let M and N be a vector spaces over a field K . Then $Hom_K(M,N)$ denotes the subgroup of Hom(M,N) consisting of all mappings $f \in Hom(M,N)$ such that

$$f(\alpha m) = \alpha f(m) , \qquad \alpha \in K, \ m \in M$$

The mappings in $Hom_K(M,N)$ are called K-homomorphisms.

Definition 8 : An element x in a ring R is nilpotent if there exists a positive integer m such that $x^m = 0$. An element x in R is idempotent if $x^2 = x \neq 0$. A left, right or two-sided ideal I of R is nilpotent if there exists a positive integer m such that $I^m = \{0\}$.

Definition 9 : Given a finite number of ideals $I_1, I_2, \ldots, I_n$ of a ring R, we define sum in the natural way :

$$I_1 + I_2 + \ldots + I_n = \{a_1 + a_2 + \ldots + a_n \mid a_i \in I_i\}$$

Then $I_1 + I_2 + \ldots + I_n$ is likewise an ideal of R and is the smallest ideal of R which contains every I_i.

Definition 10 : Let R be a ring with minimum condition. The left ideal which is the sum of all nilpotent left ideals of R is called the radical of R and is denoted by rad R. We say that R is semisimple if rad. R = 0.

We have the following theorem of which the proof can be found in reference [2].

Theorem 11 : The sum of all nilpotent left ideals in a ring R with minimum condition is a two-sided nilpotent ideal N. The ideal N contains every nilpotent right ideal of R, and the factor ring $R/_N$ has no nilpotent ideals except 0.

Definition 12 : Let A be a finite-dimensional associative algebra over a field K . Then A is said to be a separable algebra (over K) if A is semisimple.

The proof of the following theorem can be found in [2] .

Theorem 13 : Let A be a finite-dimensional associative algebra over a field K . Then A is a separable algebra over K if and only if for some K-basis $\{a_1, a_2, \ldots, a_n\}$ of A, there exist elements $a_1, a_2, \ldots, a_n$ in A such that

(1) $$\sum_{i=1}^{n} a_i' a_i = 1 ,$$

and

(2) for $a \in A$,

$$a_i a = \sum_{j=1}^{n} k_{ij}(a) a_j , \qquad k_{ij}(a) \in K .$$

implying that

$$a a_i' = \sum_{j=1}^{n} a_j' k_{ji}(a)$$

Notation 14 : $R_0[x] = \{a_1 x + a_2 x^2 + \ldots + a_n x^n \mid a_k \in R ; n > 0\}$.

Notation 15 : For an algebra B over a field K we denote (B : K) to be the dimension of B over K.

Definition 16 : A ring $R \neq 0$ is called simple if R has no 2-sided ideals other than 0 and R.

Definition 17 : A ring R is called a division ring if its nonzero elements form a group (not necessary commutative) with respect to multiplication.