CHAPTER IV

EMBEDDING THEOREMS

Some theorems concerning the embedding of a semiring in a P.R.D. and a semiring in a semifield have been given already in Theorem 2.11, Proposition 2.13, Theorem 3.21, Proposition 3.23 and Corollary 3.24. We will develop some further embedding theorems in this chapter.

Theorem 4.1. If S is a semiring, then S can be embedded into a ring iff S has additive cancellation.

Proof : Assume that S can be embedded into a ring R. Let $x, y, z \in S$ be such that $x + y = x + z$. Hence $(- x + x) + y = (- x + x) + z$ and so $y = z$. Thus S has additive cancellation.

Conversely, assume that S has additive cancellation. Define a relation $\approx$ on $S \times S$ by $(a, b) \approx (c, d)$ iff $a + d = b + c$. Clearly $\approx$ is reflexive and symmetric. Let $(a, b), (c, d), (e, f) \in S \times S$ be such that $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Hence $a + d + c + f = b + c + d + e$, and so $(d + c) + (a + f) = (d + c) + (b + e)$. Therefore $a + f = b + e$ and $(a, b) \approx (e, f)$, so $\approx$ is transitive. Hence $\approx$ is an equivalence relation on S.

Define + and $\cdot$ on $\underset{\approx}{\underline{S \times S}}$ as follows : Let $\alpha, \beta \in \underset{\approx}{\underline{S \times S}}$. Choose $(a, b) \in \alpha$ and $(c, d) \in \beta$. Then let $\alpha + \beta = [(a + c, b + d)]$ and $\alpha\beta = [(ac + bd, ad + bc)]$. To show + and $\circ$ are well-defined, let

$(a', b') \in \alpha$ and $(c', d') \in \beta$. Then we have

$a + b' = b + a'$ ________________ (1), and $c + d' = d + c'$ ________________ (2).

(1) + (2); $a + b' + c + d' = b + a' + d + c'$. Hence $(a + c) + (b' + d') =$

$(b + d) + (a' + c')$, so $(a + c, b + d) \approx (a' + c', b' + d')$ and + is well-defined.

(1) d; $bd + a'd = ad + b'd$ ________________ (3).

(1) c; $ac + b'c = bc + a'c$ ________________ (4).

b' (2); $b'd + b'c' = b'c + b'd'$ ________________ (5).

a' (2); $a'c + a'd' = a'd + a'c'$ ________________ (6).

(3) + (4) + (5) + (6);

$(ac + bd + a'd' + b'c') + (a'd + a'c + b'c + b'd) = (ad + bc + a'c' + b'd') +$

$(a'd + a'c + b'c + b'd)$. Hence $(ac + bd) + (a'd' + b'c') = (ad + bc) + (a'c' + b'd')$,

so $(ac + bd, ad + bc) \approx (a'c' + b'd', a'd' + b'c')$ and $\cdot$ is well-defined.

Claim that $(\frac{S \times S}{\approx}, +, \cdot)$ is a ring.

Clearly $\forall \alpha, \beta, \gamma \in \frac{S \times S}{\approx}$ we have that $\alpha + \beta \in \frac{S \times S}{\approx}$, $\alpha + \beta = \beta + \alpha$, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, $\alpha\beta \in \frac{S \times S}{\approx}$, $\alpha\beta = \beta\alpha$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. Choose $(a, b) \in \alpha$, $(c, d) \in \beta$ and $(e, f) \in \gamma$. Then

$$\gamma(\alpha + \beta) = [(e(a + c) + f(b + d), f(a + c) + e(b + d))]$$
$$= [(ea + fb + ec + fd, fa + eb + fc + ed)]$$
$$= [(ea + fb, eb + fa)] + [(ec + fd, ed + fc)]$$
$$= [(e, f)][(a, b)] + [(e, f)][(c, d)]$$
$$= \gamma\alpha + \gamma\beta .$$

Let $x \in S$ and $\alpha \in \frac{S \times S}{\approx}$. Choose $(a, b) \in \alpha$. Then $\alpha + [(x, x)] = [(a + x, b + x)] = [(a, b)] = \alpha$ and $\alpha + [(b, a)] = [(a + b, b + a)] = [(x, x)]$. Hence $[(x, x)]$ $\forall x \in S$ is the additive identity and the additive inverse of α is $[(b, a)]$. Therefore $\frac{S \times S}{\approx}$ is a ring.

Fix $x \in S$. Define $\theta : S \longrightarrow \underset{\approx}{S \times S}$ by $\theta(a) = [(a + x, x)]$ $\forall a \in S$. Let $a_1, a_2 \in S$. Then $\theta(a_1 + a_2) = [((a_1 + a_2) + x, x)] = [(a_1 + a_2 + x, x)] + [(x, x)] = [(a_1 + x + a_2 + x, x + x)] = [(a_1 + x, x)] + [(a_2 + x, x)] = \theta(a_1) + \theta(a_2)$ and $\theta(a_1a_2) = [(a_1a_2 + x, x)] = [(a_1a_2 + x^2, x^2)] = [(a_1a_2 + x^2, x^2)] + [(a_1x, a_1x)] + [(a_2x, a_2x)] + [(x^2, x^2)] = [(a_1a_2 + x^2 + a_1x + a_2x + x^2, x^2 + a_1x + a_2x + x^2)] = [(a_1(a_2 + x) + x(a_2 + x) + x^2, x(a_1 + x) + x(a_2 + x))] = [((a_1 + x)(a_2 + x) + x^2, (a_1 + x)x + x(a_2 + x))] = [(a_1 + x, x)][(a_2 + x, x)] = \theta(a_1)\theta(a_2)$. Hence θ is a homomorphism. Let $r_1, r_2 \in S$ be such that $\theta(r_1) = \theta(r_2)$. Then $[(r_1 + x, x)] = [(r_2 + x, x)]$, so $r_1 + x + x = r_2 + x + x$. Hence $r_1 = r_2$ and we have that θ is one-to-one. Therefore S can be embedded into $\underset{\approx}{S \times S}$. #

<u>Remark 4.2</u>. If S has an additive identity (0) or multiplicative identity (1), then we have a natural embedding $x \longmapsto [(x, 0)]$ or $x \longmapsto [(x + 1, 1)]$.

<u>Proposition 4.3</u>. If S is a semiring with additive cancellation, then $\underset{\approx}{S \times S}$ is the smallest ring containing S up to isomorphism.

<u>Proof</u> : Let R be a ring such that $S \subseteq R$. Define $\theta : \underset{\approx}{R \times R} \longrightarrow R$ in the following way : Let $\alpha \in \underset{\approx}{R \times R}$. Choose $(a, b) \in \alpha$ and let $\theta(\alpha) = a - b$. To show θ is well-defined, let $(a', b') \in \alpha$. Then $a + b' = b + a'$, so $a - b = a' - b'$ and θ is well-defined.

Let $\alpha, \beta \in \underset{\approx}{R \times R}$. Choose $(a, b) \in \alpha$, $(c, d) \in \beta$. Then $\theta(\alpha + \beta) = (a + c) - (b + d) = (a - b) + (c - d) = \theta(\alpha) + \theta(\beta)$ and $\theta(\alpha\beta) = (ac + bd) - (ad + bc) = (ac - bc) - (ad - bd) = (a - b)(c - d) = \theta(\alpha)\theta(\beta)$.

Hence θ is a homomorphism. Clearly θ is one-to-one and onto. Therefore $\frac{R \times R}{\approx} \cong R$.

Define $\phi : \frac{S \times S}{\approx} \longrightarrow \frac{R \times R}{\approx}$ in the following way : Let $\alpha \in \frac{S \times S}{\approx}$. Choose $(a, b) \in \alpha$ and let $\phi(\alpha) = [(a, b)]'$ where $[(a, b)]'$ is the equivalence class of (a, b) in $R \times R$. Clearly ϕ is a monomorphism. Hence $\frac{S \times S}{\approx}$ is isomorphic to a subring of $\frac{R \times R}{\approx}$. Since $R \cong \frac{R \times R}{\approx}$, we have that $\frac{S \times S}{\approx}$ is isomorphic to a subring of R and so $\frac{S \times S}{\approx}$ is the smallest ring containing S up to isomorphism. #

<u>Proposition 4.4.</u> If S is a finite semiring with additive cancellation, then S is a ring.

<u>Proof</u> : Let $x \in S$. Define $f_x : S \rightarrow S$ by $f_x(y) = y + x$ $\forall y \in S$. let $y_1, y_2 \in S$ be such that $f_x(y_1) = f_x(y_2)$. Then $y_1 + x = y_2 + x$, so $y_1 = y_2$ and hence f_x is one-to-one. Since S is finite, f_x is onto. $\exists\, e \in S$ such that $f_x(e) = x$, so $x = e + x = x + e$. Let $z \in S$. Then $\exists\, u \in S$ such that $f_x(u) = z$, so $u + x = x + u = z$. Therefore $z + e = e + z = e + (x + u) = (e + x) + u = x + u = z$, so e is the additive identity. Let $v \in S$. $\exists\, -v \in S$ such that $f_v(-v) = e$. Hence $-v + v = e$ and $-v$ is the additive inverse of v. Therefore we have that S is a ring. #

<u>Remark 4.5.</u> Proposition 4.4 is not true for infinite semirings since $\mathbb{N}$ is an additively cancellative semiring which is not a ring.

<u>Proposition 4.6.</u> If S is a finite semiring of order > 1 with multiplicative zero having 0-multiplicative cancellation, then S is a semifield.

<u>Proof</u> : Let $x \in S - \{0\}$. Define $f_x : S - \{0\} \rightarrow S$ by $f_x(y) = xy$

$\forall y \in S - \{0\}$. If $\exists\, y \in S - \{0\}$ such that $f_x(y) = 0$, then $xy = 0 = x0$. Since $x \neq 0$, $y = 0$, a contradiction. Thus $f_x : S - \{0\} \twoheadrightarrow S - \{0\}$. The same proof given in Corollary 2.7, gives us that $(S - \{0\}, \cdot)$ is an abelian group and so S is a semifield. #

Remark 4.7. Proposition 4.6 is not true for infinite semirings since $\mathbb{N} \cup \{0\}$ with the usual addition and multiplication has 0-multiplicative cancellation but is not a semifield.

Proposition 4.8. If S is a semiring with multiplicative zero (0) such that 0 is also the additive identity and S satisfies the property that $\forall x_1, x_2, y_1, y_2\ (x_1y_1 + x_2y_2 = x_1y_2 + x_2y_1 \Rightarrow x_1 = x_2 \vee y_1 = y_2)$, then S has 0-multiplicative cancellation.

Proof : Let $x,y,z \in S$ be such that $xy = xz$. Hence $xy + 0z = xz + 0y$ and so $x = 0$ or $y = z$. #

We shall call the above property, property (*).

From Proposition 4.8, we know that property (*) together with the condition that the multiplicative zero and the additive identity coincide is a specialization of 0-multiplicative cancellation in a semiring with multiplicative zero (0).

Note that $\mathbb{N} \cup \{0\}$ with the usual multiplication and + defined by $x + y = \min.\{x, y\}\ \forall x,y \in \mathbb{N} \cup \{0\}$ is 0-multiplicatively cancellative but does not have an additive identity and does not satisfy property (*) since $0\cdot 5 + 2\cdot 1 = 0\cdot 1 + 2\cdot 5$ but $0 \neq 2$ and $5 \neq 1$.

Theorem 4.9. If S is a semiring, then S can be embedded into an integral domain iff S has additive cancellation and satisfies property (*).

Proof : Assume that S can be embedded into an integral domain R. Obviously S has additive cancellation. Let $x_1, x_2, y_1, y_2 \in S$ be such that $x_1y_1 + x_2y_2 = x_1y_2 + x_2y_1$. Suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. Hence $x_1 - x_2 \neq 0$ and $y_1 - y_2 \neq 0$. Since $x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1 = 0$, $(y_1 - y_2)(x_1 - x_2) = 0$. Thus $y_1 - y_2 = 0$ or $x_1 - x_2 = 0$ which implies that $y_1 = y_2$ or $x_1 = x_2$, a contradiction. Therefore $x_1 = x_2$ or $y_1 = y_2$.

Conversely, assume that S has additive cancellation and satisfies property (*). By theorem 4.1, S can be embedded into a ring $\underline{S \times S}_{\approx}$ with $[(x, x)]$ as its additive identity $\forall x \in S$. Let $x \in S$ and let $\alpha, \beta \in \underline{S \times S}_{\approx}$ be such that $\alpha\beta = [(x, x)]$. Choose $(a, b) \in \alpha$ and $(c, d) \in \beta$. Then $[(ac + bd, ad + bc)] = [(x, x)]$. Hence $(ac + bd) + x = (ad + bc) + x$, so $ac + bd = bc + ad$. By (*), $a = b$ or $c = d$ which implies that $\alpha = [(x, x)]$ or $\beta = [(x, x)]$. Therefore $\underline{S \times S}_{\approx}$ is an integral domain. #

Theorem 4.10. If S is an additively cancellative semiring of order > 1 with multiplicative zero (0) such that 0 is also the additive identity and S satisfies property (*), then $\underline{S \times (S - \{0\})}_{\sim}$, the smallest semifield containing S, also has additive cancellation and satisfies the property that $\forall \alpha, \beta \in \underline{S \times (S - \{0\})}_{\sim}$ $(1 + \alpha\beta = \alpha + \beta \Rightarrow \alpha = 1 \vee \beta = 1)$.

Proof : From Proposition 4.8, S has 0-multiplicative cancellation. By Proposition 3.23, $\underline{S \times (S - \{0\})}_{\sim}$ is the smallest semifield containing S. Let $\alpha, \beta, \gamma \in \underline{S \times (S - \{0\})}_{\sim}$ be such that $\alpha + \beta = \alpha + \gamma$. Choose $(a, b) \in \alpha$,

$(c, d) \in \beta$ and $(e, f) \in \gamma$. Then $[(ad + bc, bd)] = [(af + be, bf)]$, so $(ad + bc)bf = (af + be)bd$. Since $b \neq 0$, $(ad + bc)f = (af + be)d$. Hence $adf + bcf = adf + bde$ so $bcf = bde$ and $cf = de$. Hence $\beta = \gamma$ and so $\underset{\sim}{S \times (S - \{0\})}$ is additively cancellative.

Let $\alpha, \beta \in \underset{\sim}{S \times (S - \{0\})}$ be such that $1 + \alpha\beta = \alpha + \beta$. Choose $(a, b) \in \alpha$ and $(c, d) \in \beta$. Then $[(bd + ac, bd)] = [(ad + bc, bd)]$, so $(bd + ac)\, bd = (ad + bc)\, bd$. Hence $bd + ac = ad + bc$ and by property (*), $a = b$ or $c = d$. Thus $\alpha = 1$ or $\beta = 1$. #

We shall call the property in Theorem 4.10, property (**).

<u>Theorem 4.11.</u> If K is a semifield, then K can be embedded into a field iff K has additive cancellation and satisfies property (**).

<u>Proof</u> : Assume that K can be embedded into a field. Obviously K has additive cancellation. Let $x, y \in K$ be such that $1 + xy = x + y$. Suppose $x \neq 1$ and $y \neq 1$, then $x - 1 \neq 0$ and $y - 1 \neq 0$. Since $1 + xy - x - y = 0$, we get that $(1 - x)(1 - y) = 0$. Hence $(1 - x)^{-1}(1 - x)(1 - y) = (1 - x)^{-1}0 = 0$, so $1 - y = 0$. Thus $1 = y$ which is a contradiction. Therefore $x = 1$ or $y = 1$ and K satisfies property (**).

Conversely, assume that K has additive cancellation and satisfies property (**). Since K is a semiring, by Theorem 4.1 $\underset{\approx}{K \times K}$ is a ring with $[(x, x)]$ as its additive identity $\forall x \in K$. Fix $x \in K$. Let $\alpha, \beta \in \underset{\approx}{K \times K} - \{[(x, x)]\}$. Choose $(a, b) \in \alpha$ and $(c, d) \in \beta$.
Then $\alpha\beta = [(ac + bd, ad + bc)]$. Suppose $\alpha\beta = [(x, x)]$, then $ac + bd + x = ad + bc + x$. Hence $ac + bd = ad + bc$. Suppose that $a = 0$ or $c = 0$, say

$a = 0$. Then $b \neq 0$ and $ac = ad = 0$. Hence $0 + bd = 0 + bc$, so $bd = bc$. Since $b \neq 0$, $d = c$. Thus $\beta = [(x, x)]$, a contradiction. Therefore $a \neq 0$ and $c \neq 0$, so $1 + c^{-1}da^{-1}b = c^{-1}d + a^{-1}b$. By property (**), $c^{-1}d = 1$ or $a^{-1}b = 1$, so $c = d$ or $a = b$. Thus $\alpha = [(x, x)]$ or $\beta = [(x, x)]$, a contradiction. Therefore $\alpha\beta \neq [(x, x)]$ and we have that $\underset{\approx}{\underline{K \times K}}$ is an integral domain containing K. Since $\underset{\approx}{\underline{K \times K}}$ can be embedded into its quotient field, K also can be embedded into a field. #

<u>Remark 4.12.</u> (i) $\mathbb{Q}^+ \cup \{0\}$ with the usual multiplication and + defined by $x + y = \max.\{x, y\}\ \forall x, y\ \mathbb{Q}^+ \cup \{0\}$ is the semifield that satisfies property (**) but is not additively cancellative.

(ii) The semifield of infinity type cannot embed into a field since additive cancellation is the necessary condition for embedding semifield into a field. Let K be a semifield of zero type. By theorem 3.25, the prime semifield of K is either isomorphic to $\mathbb{Q}^+ \cup \{0\}$ with the usual addition and multiplication or $\mathbb{Z}_p$ where p is a prime number or is the semifield in table 3, page 25. Moreover, if the prime semifield of K is $\mathbb{Z}_p$ for some prime number p, then K itself is a field. Consider the semifield in table 3, page 25. Since $0 + 1 = 1 + 1$ but $0 \neq 1$, this semifield is not additively cancellative. Therefore we conclude that a necessary condition that a semifield which is not a field is embeddable in a field is that its prime semifield is $\mathbb{Q}^+ \cup \{0\}$ with the usual addition and multiplication.

<u>Proposition 4.13.</u> If K is a semifield with additive cancellation satisfying property (**), then the quotient field F of $\frac{K \times K}{\approx}$ is the smallest field containing K up to isomorphism.

<u>Proof</u> : Let F' be a field containing K. Since F' is a field, as we already showed in the proof in Proposition 4.3, the map $\theta : \frac{F' \times F'}{\approx} \twoheadrightarrow F'$ defined by $\theta([(x, y)]) = x - y \ \forall x, y \in F'$ gives us an isomorphism, i.e. $\frac{F' \times F'}{\approx} \cong F'$. Let $\alpha \in \frac{K \times K}{\approx}$. Choose $(a, b) \in \alpha$ and define $\phi : \frac{K \times K}{\approx} \to \frac{F' \times F'}{\approx}$ by $\phi(\alpha) = [(a, b)]'$ where $[(a, b)]'$ is the equivalence class of (a, b) in $F' \times F'$. We have that ϕ is a monomorphism. Hence up to isomorphism $\frac{K \times K}{\approx} \subseteq \frac{F' \times F'}{\approx}$, and so up to isomorphism we have that $F \subseteq \frac{F' \times F'}{\approx} \cong F'$. Therefore F is the smallest field containing K up to isomorphism. #

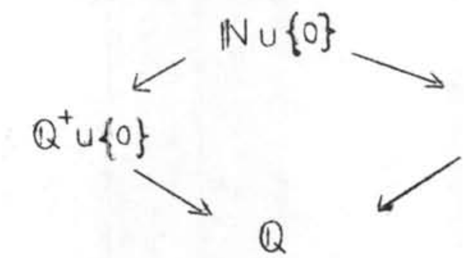
<u>Proposition 4.14.</u> A finite semifield S with additive cancellation is a field.

<u>Proof</u> : It follows from the proof in Proposition 4.4 that (S, +) is an abelian group and so S is a field. #

Let S be an additively cancellative semiring of order > 1 with multiplicative zero (0) such that 0 is also the additive identity and S satisfies property (*). By theorem 4.10, theorem 4.11 and Proposition 4.13, S can be embedded into F_1 the smallest field containing S. And also by Theorem 4.9 and Proposition 4.3, S can be embedded into an integral domain $\frac{S \times S}{\approx}$ which is the smallest ring containing S. Let F_2 be the quotient field of $\frac{S \times S}{\approx}$. We claim that F_2 is the smallest field containing

S. Let F_2' be a field such that $S \subseteq F_2'$. Since F_2' is a field, the map $\phi : \frac{F_2' \times F_2'}{\approx} \longrightarrow F_2'$ defined by $\phi\,(\,[(x, y)]\,) = x - y$ is an isomorphism. Let $\alpha \in \frac{S \times S}{\approx}$. Choose $(a, b) \in \alpha$ and define $\phi : \frac{S \times S}{\approx} \longrightarrow \frac{F_2' \times F_2'}{\approx}$ by $\phi\,(\alpha) = [(a, b)]'$ where $[(a, b)]'$ is the equivalence class of (a, b) in $F_2' \times F_2'$. We have that ϕ is a monomorphism and so up to isomorphism $\frac{S \times S}{\approx}$ is a subring of F_2'. Since F_2 is the smallest field containing $\frac{S \times S}{\approx}$, up to isomorphism $F_2 \subseteq F_2'$. Therefore F_2 is the smallest field containing S up to isomorphism. From this we get that $F_1 \cong F_2$ and we conclude that if S is a semiring with multiplicative zero (0) such that 0 is also the additive identity and S satisfies property (*), then we have two ways to embed S into a field and each way gives the same smallest field containing S up to isomorphism.

As a special case of this we can embed $\mathbb{N} \cup \{0\}$ with the usual addition and multiplication into the field of rational numbers $\mathbb{Q}$, in at least two different ways, as show in diagram :



We now consider the embedding theorem of a P.R.D. into a field.

<u>Theorem 4.15.</u> If D is a P.R.D., then D can be embedded into a field iff D is additively cancellative and satisfies property (**).

<u>Proof</u> : As the proof in Theorem 4.10, we get that if D can be

embedded into a field then D is additively cancellative and satisfies property (**).

Assume that D has additive cancellation and satisfies property (**). By Theorem 4.1, we have that $\frac{D \times D}{\approx}$ is a ring with $[(x, x)]$ as its additive identity $\forall x \in D$. Fix $x \in D$. Let $\alpha, \beta \in \frac{D \times D}{\approx} - \{[(x, x)]\}$. Choose $(a, b) \in \alpha$ and $(c, d) \in \beta$. Then $\alpha\beta = [(ac + bd, ad + bc)]$. Suppose $\alpha\beta = [(x, x)]$, then $ac + bd + x = ad + bc + x$. Hence $ac + bd = ad + bc$ and so $1 + c^{-1}da^{-1}d = c^{-1}d + a^{-1}b$. Therefore $c^{-1}d = 1$ or $a^{-1}b = 1$ so $c = d$ or $a = b$. Thus $\alpha = [(x, x)]$ or $\beta = [(x, x)]$, a contradiction. Hence $\alpha\beta \neq [(x, x)]$ and so $\frac{D \times D}{\approx}$ is an integral domain which as is wellknown can be embedded into a field. Thus D can be embedded into a field. #

Remark 4.16. (i) $\mathbb{Q}^+$ with the usual multiplication and + defined by $x + y = \min.\{x, y\}$ $\forall x, y \in \mathbb{Q}^+$ is a P.R.D. that satisfies property (**) but is not additively cancellative.

(ii) The quotient field of $\frac{D \times D}{\approx}$ in Theorem 4.15 is the smallest field containing D.

Let S be a semiring with additive and multiplicative cancellation satisfying property (*). By Theorem 2.11 and Proposition 2.13, $\frac{S \times S}{\sim}$ is the smallest P.R.D. containing S and we have that $\frac{S \times S}{\sim}$ is additively cancellative and satisfies property (**). By Theorem 4.15, $\exists F_1$ the smallest field containing $\frac{S \times S}{\sim}$. Hence F_1 is the smallest field containing S. By Theorem 4.9, $\frac{S \times S}{\approx}$ is the smallest integral domain containing S. Let F_2 be the quotient field of $\frac{S \times S}{\approx}$. Since up to isomorphism $S \subseteq F_2$, up to isomorphism $F_2 \subseteq F_1$. Since $\frac{S \times S}{\sim} \subseteq \frac{F_2 \times F_2}{\sim} \cong F_2$, up to isomorphism $F_1 \subseteq F_2$. Thus $F_1 \cong F_2$. From this we get that if S is a semiring with additive and multiplicative cancellation satisfying (*), then we have two ways to embed S into a field and each way gives the same smallest field containing S up to isomorphism.

As a special case of this we can embed $\mathbb{N}$ with the usual addition and multiplication into the field of rational numbers $\mathbb{Q}$ in at least two different ways, as show in diagram :