CHAPTER VI

FIXED POINT THEOREM OF THE ALTERNATIVE FOR CONTRACTION ON A GENERALIZED SEMI-METRIC SPACE

The materials of this chapter are drawn from reference [5].

6.1.1 Theorem. Let (E, d) be a generalized semi-metric space with the canonical decomposition $E = \cup\{E_\alpha / \alpha \in \mathcal{A}\}$. Let $T : E \longrightarrow E$ be a contraction. If there exists an $x_o \in E$ such that $d(x_o, T(x_o)) < +\infty$. Then for some $\alpha_o \in \mathcal{A}$, the restriction $T/_{E_{\alpha_o}}$, which will be denoted by T_{α_o}, is a contraction.

Proof. Assume that there exists $x_o \in E$ such that $d(x_o, T(x_o)) < +\infty$. Therefore x_o, $T(x_o)$ belong to the same E_{α_o} for some $\alpha_o \in \mathcal{A}$. Since T is contraction, hence there exists $q \in [0, 1)$ such that for any $y \in E_{\alpha_o}$

$$d(T(x_o), T(y)) \leq q\, d(x_o, y).$$

Since $x_o, y \in E_{\alpha_o}$, hence $d(x_o, y) < +\infty$. So that $d(T(x_o), T(y)) < +\infty$, i.e. $T(y) \in E_{\alpha_o}$. Therefore the restriction T_{α_o} is contraction. The proof is complete.

6.1.2. Corollary. Let (E, d) be a complete generalized semi-metric space with the canonical decomposition $E = \cup\{E_\alpha / \alpha \in \mathcal{A}\}$. Let $T : E \longrightarrow E$ be a continuous map such that

(i) $T/_{E_\alpha} : E_\alpha \longrightarrow E_\alpha$ for each $\alpha \in \mathcal{A}$,

(ii) T^p is contraction for some positive integer p.

Then either

(A) for every $x \in E$, $d(x, T^p(x)) = +\infty$

or (B) there is an $x \in E$ such that $d(x, T(x)) = 0$.

<u>Proof</u>. If (A) does not hold then there is an $x_o \in E$ such that $d(x_o, T^p(x_o)) < +\infty$. By theorem 6.1.1, there exists $\alpha_o \in \mathcal{A}$ such that the restriction $T^p_{\alpha_o}$ is contraction. By Banach's contraction theorem, there exists an $x \in E_{\alpha_o} \subset E$ such that $d(T(x), x) = 0$. The proof is complete.

6.1.3 <u>Remark</u>. Without condition (i), our proof of corollary 6.1.2 is not valid. To justify this, we give the following counter example.

Let $E = E_1 \cup E_2$, where $E_1 = [0, 1]$ and $E_2 = [2, 3]$ with a generalized metric d, given by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in E_1 \text{ or } x, y \in E_2, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $E = E_1 \cup E_2$ is the canonical decomposition of E. Let $T : E \longrightarrow E$ be defined by

$$T(x) = \begin{cases} \frac{1}{2}x + 2 & x \in E_1, \\ \frac{1}{2}(x - 2) & x \in E_2. \end{cases}$$

Then $T^2(x)$ is a contraction, but $T/_{E_1} : E_1 \longrightarrow E_2$, $T/_{E_2} : E_2 \longrightarrow E_1$ and neither (A) nor (B) hold.

Proof. For any x, y $\in$ E

$$T^2(x) = \begin{cases} \frac{1}{4} x & x \in E_1, \\ \frac{1}{4}(x-2)+2 & x \in E_2. \end{cases}$$

and

$$d(T^2(x), T^2(y)) = \begin{cases} \frac{1}{4}|x-y| & x, y \in E_1 \text{ or } x, y \in E_2, \\ +\infty & \text{otherwise} \end{cases}$$

Hence $d(T^2(x), T^2(y)) \leq \frac{1}{4} d(x, y)$ for all x, y $\in$ E. Therefore T^2 is a contraction map. By the definition of T, we see that $T/_{E_1} : E_1 \longrightarrow E_2$, $T/_{E_2} : E_2 \longrightarrow E_1$ and hence $d(x, T(x)) = +\infty$ for all x. Therefore observe that $d(x, T^2(x)) < +\infty$ for all x and $d(x, T(x)) \neq 0$ for any x. The proof is complete,

6.1.4 Corollary. Let (E, d), T and $T/_{E_\alpha}$ for each $\alpha \in \mathcal{A}$ be as in corollary 6.1.2. For any $x_0 \in E$, let

$$x_n = T^p(x_{n-1}) \quad \text{for all positive integers } n.$$

Then either

(A) $d(x_{n+1}, x_n) = +\infty$ for all positive integers n

or (B) $\{x_n\}$ d-converges to some point $x \in E$ and $d(T(x), x) = 0$. Furthermore, if there exists $y \in E$ such that $d(T(y), y) = 0$, then $d(x, y) = 0$.

Proof. If (A) does not hold, then there exists n_0 such that $d(x_{n_0+1}, x_{n_0}) = d(T^p(x_{n_0}), x_{n_0}) < +\infty$. By theorem 6.1.1, there exists $\alpha_0 \in \mathcal{A}$ such that $x_{n_0+1}, x_{n_0} \in E_{\alpha_0}$ and the restriction $T^p_{\alpha_0}$ is contraction. So that there exists $q \in [0, 1)$ such that

$$d(x_{n_o+2}, x_{n_o+1}) = d(T^p(x_{n_o+1}), T^p(x_{n_o})) \leqslant q\, d(x_{n_o+1}, x_{n_o}) < +\infty$$

and hence $x_{n_o+2} \in E_{\alpha_o}$. Therefore $x_n \in E_{\alpha_o}$ for all $n \geqslant n_o$. By corollary 3.1.6, $\{x_n\}$ d-converges to a point $x \in E_{\alpha_o} \subset E$ and $d(T(x), x) = 0$. By Banach's contraction theorem, if there exists $y \in E_{\alpha_o} \subset E$ such that $d(T(y), y) = 0$, then $d(x, y) = 0$. Our proof is complete.

6.1.5 Corollary. Let (E, d) be a complete generalized metric space with the canonical decomposition $E = \cup\{E_\alpha / \alpha \in \mathcal{A}\}$. Let $T : E \longrightarrow E$ be a continuous map such that

(i) $T/_{E_\alpha} : E_\alpha \longrightarrow E_\alpha$ for each $\alpha \in \mathcal{A}$,

(ii) T^p is contraction for some positive integer p.

Then either

(A) for every $x \in E$, $d(x, T^p(x)) = +\infty$,

or (B) T has a fixed point.

Proof. The proof is similar to the proof of corollary 6.1.2, except we use the result of (2) instead of (1) of Banach's contraction theorem.

6.1.6 Corollary. Let (E, d), T and $T/_{E_\alpha}$ for each $\alpha \in \mathcal{A}$ be as in corollary 6.1.5. For any $x_o \in E$, let

$$x_n = T^p(x_{n-1}) \text{ for all positive integers } n.$$

Then either

(A) $d(x_{n+1}, x_n) = +\infty$ for all positive integers n

or (B) $\{x_n\}$ d-converges to the unique fixed point of T.

Proof. The proof is similar to the proof of corollary 6.1.4. But we use the result of (2) instead of (1) of Banach's contraction theorem.

APPENDIX

Partition of sets

1.1 Definition. A relation R in X is an equivalence relation if it satisfies the following conditions :

(1) $(x, x) \in R$ for all $x \in X$, i.e. R is reflexive ;

(2) $(y, x) \in R$ whenever $(x, y) \in R$, i.e. R is symmetric;

(3) $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, i.e. R is transitive.

1.2 Definition. Let R be an equivalence relation in X, the set

$$R_x = \{ y \,/\, (x, y) \in R \}$$

is called the equivalence class in X containing x.

1.3 Definition. Let X be a set. A partition of X is a class of non-empty disjoint subset C_i of X such that $\cup C_i = X$.

1.4 Theorem. Let R be an equivalence relation in $X \neq \emptyset$, then the set of all equivalence class forms a partition of X, called the partition induced by R.

Proof. Since R is an equivalence relation, hence $(x, x) \in R$ for all $x \in X$. So that $x \in R_x$ and therefore $R_x \neq \emptyset$. Let z belong to both R_x and R_y, so that $(x, z) \in R$ and $(y, z) \in R$. By the symmetry and transitivity of R, we have

$$(y, x) \in R.$$

If now $a \in R_x$, then by transitivity of R, $(y, a) \in R$. Hence $a \in R_y$, i.e. $R_x \subset R_y$. The same argument show that $a \in R_y$ implies $a \in R_x$, i.e. $R_y \subset R_x$. Hence $R_x = R_y$. Therefore distinct members of a set of all equivalence classes are disjoint. Since for any $x \in X$, $x \in R_x \subset X$, hence $X = \bigcup_{x \in X} \{x\} \subset \bigcup_{x \in X} R_x \subset X$.

Therefore $X = \bigcup_{x \in X} R_x$. The proof is complete.