CHAPTER II

PRELIMINARY

This chapter will give some definitions and theorems which will be needed in our investigation.

The materials of this chapter are drawn from reference [3].

1. Generalized semi-metric space

2.1.1 Definition. Let E be a non-empty set. A generalized semi-metric on E is a function

$$d : E \times E \longrightarrow \mathbb{R}(\geqslant 0)^{*} = \mathbb{R}(\geqslant 0) \cup \{\infty\}$$

satisfying

1) $d(x, y) = d(y, x)$;

2) $d(x, x) = 0$;

3) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in E$.

A generalized metric on E is a generalized semi-metric on E such that

$$d(x, y) = 0 \text{ implies } x = y$$

for any $x, y \in E$. i.e. if d is a generalized metric 2) becomes

$$d(x, y) = 0 \text{ if and only if } x = y.$$

A generalized semi-metric (respectively generalized metric) space is a set E together with a generalized semi-metric d (respectively generalized metric) on E and denoted by (E, d). Note that a generalized semi-metric (respectively generalized metric) space is a semi-metric (respectively metric) space if all values of d are in $\mathbb{R}$ (≥ 0). And we can see easily that a metric space is a special case of a generalized semi-metric space.

2.1.2 Definition. Let (E,d) be a generalized semi-metric space.

An open ball with center at $x \in E$ and radius $r > 0$ is the subset $S(x, r)$ of E, defined by

$$S(x, r) = \{ y \in E \,/\, d(x, y) < r \}.$$

A closed ball with center at $x \in E$ and radius $r > 0$ is the subset $S[x, r]$ of E, defined by

$$S[x, r] = \{ y \in E \,/\, d(x, y) \leq r \}$$

2.1.3 Definition. A subset G of a generalized semi-metric space (E, d) is called an open set if, given any $x \in G$ there exists $r > 0$ such that $S(x, r) \subset G$.

2.1.4 Definition. A subset F of a generalized semi-metric space (E,d) is closed if its complement is open.

2.1.5 Definition. A point x of a generalized semi-metric space (E, d) is called a cluster point of $A \subset E$ if, for every $r > 0$ $S(x,r) \cap A \neq \emptyset$.

2.1.6. Definition. Let (E, d) be a generalized semi-metric space. The closure of $A \subset E$ is the subset $\bar{A}$ of E, defined by

$$\bar{A} = \{ x \in A \,/\, x \text{ is a cluster point of } A \}.$$

2.1.7 Definition. A sequence $\{x_n\}$ in a generalized semi-metric space (E, d) is said to d-converge to a point $x \in E$ if, given any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. The point x is called a limit of the sequence $\{x_n\}$.

Clearly, $\{x_n\}$ d-converges to x iff $d(x_n, x)$ converges to 0.

2.1.8 Definition. Let f be a mapping from a generalized semi-metric space (E_1, d_1) into a generalized semi-metric space (E_2, d_2). A function f is said to be continuous at a point $x_o \in E_1$ if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in E_1$, $d_2(f(x), f(x_o)) < \varepsilon$ whenever $d_1(x, x_o) < \delta$.

The mapping f is said to be continuous on E if it is continuous at every point on E.

2.1.9 Proposition. Let f be a mapping from a generalized semi-metric space (E_1, d_1) into a generalized semi-metric space (E_2, d_2). Then f is continuous if and only if a sequence $\{f(x_n)\}$ of E_2 d_2-converges to $f(x)$ whenever the sequence $\{x_n\}$ of E_1 d_1-converges to $x \in E_1$.

Proof. Assume f is continuous. Given any $\varepsilon > 0$ and $x \in E_1$, there exists $\delta > 0$ such that for all $y \in E_1$

$$d_1(x, y) < \delta \quad \text{implies} \quad d_2(f(x), f(y)) < \varepsilon .$$

If $\{x_n\}$ d_1-converges to x then there exists a positive integer N such that for all $n \geq N$ $d(x_n, x) < \delta$. So that $d_2(f(x_n), f(x)) < \varepsilon$. Therefore the sequence $\{f(x_n)\}$ d_2-converges to f(x).

To prove the converse, assume that $\{f(x_n)\}$ d_2-converges to $f(x_o)$ whenever $\{x_n\}$ d_1-converges to x_o. Suppose f is not continuous at a point x_o . Therefore there exists an $\varepsilon > 0$ such that for each $\delta > 0$ there is $x' \in E_1$ such that $d_1(x', x_o) < \delta$ and $d_2(f(x'), f(x_o)) \geq \varepsilon$. So that for each positive integer n, we can choose x_n such that $d_1(x_n, x_o) < \frac{1}{n}$ and $d_2(f(x_n), f(x_o)) \geq \varepsilon$. Clearly, $\{x_n\}$ d_1-converges to x_o, but $\{f(x_n)\}$ does not d_2-converge to $f(x_o)$. This contradicts our assumption. The proof is complete.

2.1.10 Proposition. Let (E, d) be a generalized semi-metric space and $A \subset E$. Then

$$d(x, A) = 0 \text{ if and only if } x \in \bar{A}$$

where

$$d(x, A) = \inf \{ d(x, a) \ / \ a \in A \}.$$

Proof. Assume $d(x, A) = 0$, suppose $x \notin \bar{A}$, then there exist $r_x > 0$ such that $S(x, r_x) \cap A = \emptyset$. For any $y \in A$, we have $y \notin S(x, r_x)$, so that $d(x, y) \geq r_x > 0$. Therefore

$d(x, A) = \inf \{ d(x,y) / y \in A \} \geq r_x > 0$. Contradict the assumption $d(x, A) = 0$. Therefore $x \in \bar{A}$.

To show that converse, assume $x \in \bar{A}$. Suppose $d(x,A) = r > 0$. Since $d(x, y) \geq d(x.A) = r$ for all $y \in A$, i.e. there exists $r > 0$ such that for all $y \in A$, $y \notin S(x, r)$ and hence $A \cap S(x,r) = \emptyset$. Therefore $x \notin \bar{A}$. Contradict the assumption $x \in \bar{A}$. The proof is complete.

2.1.11 Theorem. Let (E,d) be a generalized semi-metric space. Define a relation R on E as follows :

$$(x, y) \in R \text{ if and only if } d(x, y) < +\infty .$$

Then R is an equivalence relation[1)] on E and E is decomposed into (disjoint) equivalence classes.

We shall call this decomposition of E the canonical decomposition.

Proof. 1) For all $x \in E$, $d(x,x) < +\infty$, we have $(x,x) \in R$;

2) Since $d(x,y) = d(y,x)$, if $(x,y) \in R$, then $(y, x) \in R$;

3) If $(x,y) \in R$ and $(y, z) \in R$, then $d(x, y) < +\infty$ and $d(y,z) < +\infty$. Since $d(x,z) \leq d(x,y) + d(y,z)$, we have $d(x,z) < +\infty$. Therefore $(x,z) \in R$. The proof is complete.

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1) See appendix

2.1.12 Definition. A sequence $\{x_n\}$ in a generalized semi-metric space (E, d) is called a d-Cauchy sequence if, given any $\varepsilon > 0$, there is an integer N such that $d(x_n, x_{n'}) < \varepsilon$ whenever $n, n' \geqslant N$.

2.1.13 Definition. A generalized semi-metric spece (E, d) is said to be d-complete if every d-Cauchy sequence in E is d-convergent to an element in E.

2.1.14 Theorem. Let (E, d) be a generalized semi-metric space. $E = \cup \{ E_\alpha / \alpha \in \mathcal{A} \}$ be the canonical decomposition and for each $\alpha \in \mathcal{A}$ $d_\alpha = d|_{E_\alpha \times E_\alpha}$, the restriction of d to $E_\alpha \times E_\alpha$. Then

a) for each $\alpha \in \mathcal{A}$, (E_α, d_α) is a semi-metric space;

b) for each $\alpha, \beta \in \mathcal{A}$, with $\alpha \neq \beta$

$$d(x, y) = +\infty$$

for any $x \in E_\alpha$ and $y \in E_\beta$;

c) (E, d) is a complete generalized semi-metric space if and only if for each $\alpha \in \mathcal{A}$, (E_α, d_α) is a complete semi-metric space.

Proof. a) Clearly, d_α is a semi-metric.

b) Suppose for some $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq \beta$, there exists $x \in E_\alpha$ and $y \in E_\beta$ such that $d(x, y) < +\infty$. Therefore x and y are in the some class. Contradict the assumption that x and y are not in the same class. The proof is complete.

c) Assume (E, d) is a complete generalized semi-metric space. For each $\alpha \in \mathcal{A}$, let $\{x_n\}$ be a d_α-Cauchy sequence in E_α. Then $\{x_n\}$ is also d-Cauchy in E. Since E is complete, $\{x_n\}$ d-converges to a point $x \in E$. Since $\{x_n\}$ d-converges to x, hence $d(x_n, x) < +\infty$ for sufficiently large n. It follows that $x \in E_\alpha$. Therefore (E_α, d_α) is a complete semi-metric space.

To prove the converse, suppose that for each $\alpha \in \mathcal{A}$, (E_α, d_α) is a complete semi-metric space. Let $\{x_n\}$ be a d-Cauchy sequence in E. Then there exists a positive integer N such that $d(x_m, x_n) < +\infty$ for $m, n \geq N$ so that there exists an $\alpha \in \mathcal{A}$ such that $x_n \in E_\alpha$ for $n \geq N$. Since E_α is a complete semi-metric space, the sequence $\{x_n \mid n \geq N\}$ d_α-converges to $x \in E_\alpha \subset E$. Therefore (E, d) is a complete generalized semi-metric space.

2. Metric space and the complete metric space

2.2.1 Theorem. The set of all real number X with a function d defined as follows :

$$d(x, y) = |x - y|$$

where x, y are any real numbers, is a metric space. We shall denote this metric space by (R^1, d) or simply R^1.

Proof. The nonnegative function d satisfies property 1) and 2) in definition 2.1.1. Moreover, for any x, y, z in X, d satisfies property 3) by setting $a = x - z$, $b = z - y$ in inequality

$$|a + b| \leq |a| + |b|$$

for any real numbers a, b so that

$$|x - y| \leq |x - z| + |z - y| .$$

Therefore (X, d) is a metric space.

2.2.2 Lemma. If f and g are any bounded real valued functions defined on a set X then

$$\sup_{x \in X} |f(x) + g(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| .$$

Proof. Since f and g are bounded real valued functions, f + g is also bounded real valued function. Let

$$a = \sup_{x \in X} |f(x)| , \quad b = \sup_{x \in X} |g(x)| .$$

For any $x \in X$, $|f(x)| \leq a$ and $|g(x)| \leq b$ so that

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq a + b .$$

Therefore

$$\sup_{x \in X} |f(x) + g(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)|$$

The proof is complete.

2.2.3 Theorem. The set of all continuous functions defined on the closed interval $[a, b]$, with a function d given by

$$d(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)| ,$$

is a metric space. We shall denote this metric space by $C_{[a,b]}$.

Proof. Let f, g, h be any three functions in $C_{[a,b]}$. Since

$$d(f,g) = \sup_{a \leq x \leq b} \left| f(x) - g(x) \right|$$

$$\leq \sup_{a \leq x \leq b} \left\{ \left| f(x) - h(x) \right| + \left| h(x) - g(x) \right| \right\}$$

By lemma 2.2.2 , we have

$$d(f,g) \leq \sup_{a \leq x \leq b} \left| f(x) - h(x) \right| + \sup_{a \leq x \leq b} \left| h(x) - g(x) \right|$$

$$= d(f, h) + d(h, g).$$

Moreover, d satisfies the properties 1) and 2) in definition 2.1.1 obviously. Our proof is complete.

2.2.4 Lemma. Let f_i, g_i for i = 1, 2,..., n be any bounded real valued functions defined on a set X. Then for any $x \in X$ and i = 1, 2,..., n

$$\sup_{x,i} \left| f_i(x) + g_i(x) \right| \leq \sup_{x,i} \left| f_i(x) \right| + \sup_{x,i} \left| g_i(x) \right|$$

Proof. Since f_i, g_i are bounded real valued functions for i = 1, 2,..., n, hence $f_i + g_i$ is also bounded real valued function for each i = 1, 2, ..., n. Let

$$a = \sup_{x,i} \left| f_i(x) \right| , \quad b = \sup_{x,i} \left| g_i(x) \right|$$

so that for any x and i

$$\left| f_i(x) \right| \leq a \quad , \quad \left| g_i(x) \right| \leq b$$

and hence

$$\left| f_i(x) + g_i(x) \right| \leqslant \left| f_i(x) \right| + \left| g_i(x) \right| \leqslant a + b .$$

Therefore

$$\sup_{x,i} \left| f_i(x) + g_i(x) \right| \leqslant \sup_{x,i} \left| f_i(x) \right| + \sup_{x,i} \left| g_i(x) \right| .$$

The proof is complete.

2.2.5 Theorem. Let $C^n_{[a,b]}$ be a space of n-tuples $f = (f_1, f_2, \dots, f_n)$ of continuous function $f_1, f_2, \dots, f_n$ defined on the closed interval $[a, b]$ with a function d given by

$$d(f, g) = \sup_{x,i} \left| f_i(x) - g_i(x) \right| .$$

Then $C^n_{[a,b]}$ is a metric space.

Proof. Let f, g, h be any three functions in $C^n_{[a,b]}$ so that

$$d(f, g) = \sup_{x,i} \left| f_i(x) - g_i(x) \right|$$

$$\leqslant \sup_{x,i} \left\{ \left| f_i(x) - h_i(x) \right| + \left| h_i(x) - g_i(x) \right| \right\}$$

By lemma 2.2.4 , we have

$$d(f,g) \leqslant \sup_{x,i} \left| f_i(x) - h_i(x) \right| + \sup_{x,i} \left| h_i(x) - g_i(x) \right|$$

$$= d(f, h) + d(h, g)$$

and hence d satisfies property 3) of definition 2.1.1. Moreover, d satisfies properties 1) and 2) of definition 2.1.1 obviously. The proof is complete.

2.2.6 Lemma. Cauchy sequence in $\mathbb{R}^1$ is bounded.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^1$. Let $\varepsilon = 1$ there exists N such that

$$|x_n - x_m| < 1$$

for all m, n $\geq$ N. Since

$$x_n = x_n - x_N + x_N$$

$$|x_n| \leq |x_n - x_N| + |x_N|$$

$$\leq |x_N| + 1$$

for all $n \geq N$. Let

$$M = \max(|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1),$$

therefore $|x_n| \leq M$ for all n. The proof is complete.

2.2.7 Proposition. The metric space $(\mathbb{R}^1, d)$ is complete

Proof. Let $\{x_n\}$ be a d-Cauchy sequence of points in $\mathbb{R}^1$. Given any $\varepsilon > 0$, there exists an N such that

$$|x_n - x_m| < \frac{\varepsilon}{2}$$

for all m, n $\geq$ N. Let

$$A_{N+k} = \{x_n \,/\, n \geq N + k, \text{ for all nonnegative integers } k\}$$

so that $A_N \supset A_{N+1} \supset \ldots$. By lemma 2.2.6, A_{N+k} is bounded above by a real number M. So that each A_{N+k} has the least upper bound. Let $\alpha_k = \sup A_{N+k}$, then $\alpha_0 \geq \alpha_1 \geq \ldots\ldots$ Let

$$B = \{\alpha_0, \alpha_1, \ldots\},$$

B has a lower bound namely $-M$, since $-M \leqslant x_n \leqslant M$ for all n, So that B has the greatest lower bound. Let

$$x = \inf B$$

then there exists α_{k_o} such that

$$x \leqslant \alpha_{k_o} < x + \frac{\varepsilon}{2} \qquad \ldots\ldots\ldots (1)$$

where $\alpha_{k_o} = \sup A_{N+k_o}$. Therefore there exists $x_m \in A_{N+k_o}$ such that

$$\alpha_{k_o} - \frac{\varepsilon}{2} < x_m \leqslant \alpha_{k_o} \qquad \ldots\ldots\ldots (2)$$

where $m \geqslant N + k_o$.

By equation (1) and (2)

$$x - \frac{\varepsilon}{2} \leqslant \alpha_{k_o} - \frac{\varepsilon}{2} < x_m \leqslant \alpha_{k_o} < x + \frac{\varepsilon}{2}$$

$$x - \frac{\varepsilon}{2} < x_m < x + \frac{\varepsilon}{2}$$

and hence $|x_m - x| < \frac{\varepsilon}{2}$.

Now, for any $n \geqslant m$ we have

$$|x_n - x| \leqslant |x_n - x_m| + |x_m - x|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Therefore $\{x_n\}$ converges to x. The proof is complete.

2.2.8 Definition. Let $\{f_n\}$ be a sequence of functions from an arbitrary set X into a metric space (Y, d). Then $\{f_n\}$ is said to d-converge uniformly to a function $f : X \longrightarrow Y$ if, for every $\varepsilon > 0$, there exists a positive integer N such that $n \geqslant N$ implies $| f_n(x) - f(x) | < \varepsilon$ for all $x \in X$.

2.2.9 Lemma. Let $\{ f_n \}$ be a sequence of continuous functions on $[a, b]$ which converges uniformly to f, then f is continuous on $[a, b]$.

Proof. Given any $\varepsilon > 0$, there exists a positive integer N such that for all $n \geqslant N$ $| f_n(x) - f(x) | < \frac{\varepsilon}{3}$ for all $x \in [a, b]$. Let x_o be any element in $[a, b]$ so that f_N is continuous at x_o, i.e. there exists $\delta > 0$ such that

$$| x - x_o | < \delta \quad \text{implies} \quad | f_N(x) - f_N(x_o) | < \frac{\varepsilon}{3} .$$

Therefore we have

$$| f(x) - f(x_o) | \leqslant | f(x) - f_N(x) | + | f_N(x) - f_N(x_o) | + | f_N(x_o) - f(x_o) |$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

whenever $| x - x_o | < \delta$. This completes the proof.

2.2.10 Proposition. The metric space $C_{[a,b]}$, where d is defined in theorem 2.2.3, is complete.

Proof. Let $\{ f_n \}$ be any d-Cauchy sequence in a metric space $C_{[a,b]}$. Then, given any $\varepsilon > 0$, there exists a positive integer N such that

$$d(f_n, f_{n'}) = \sup_{a \leqslant x \leqslant b} | f_n(x) - f_{n'}(x) | < \frac{\varepsilon}{2}$$

for all n, $n' \geq N$. Therefore we have

$$(*) \qquad | f_n(x) - f_{n'}(x) | < \frac{\varepsilon}{2}$$

for all $n, n' \geq N$ and $x \in [a, b]$. For each x fixed in $[a, b]$, the sequence $\{ f_n(x) \}$ forms a Cauchy sequence in $\mathbb{R}^1$. Since $\mathbb{R}^1$ is complete, $\{ f_n(x) \}$ converges to an element in $\mathbb{R}^1$. Let

$$c_x = \lim_{n \to \infty} f_n(x)$$

Now we define a function f on $[a, b]$ such that

$$f(x) = c_x$$

for all $x \in [a, b]$. Then we have

$$f(x) = \lim_{n \to \infty} f_n(x).$$

for each $x \in [a, b]$.

By taking n' goes to $+\infty$ in $(*)$, we have

$$| f_n(x) - f(x) | \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geq N$ and $x \in [a, b]$, i.e. $\{ f_n \}$ d-converges uniformly to f. By lemma 2.2.9, we have $f \in C_{[a,b]}$. Therefore $\{ f_n \}$ d-converges to $f \in C_{[a,b]}$. The proof is complete.

2.2.11 <u>Proposition</u>. The metric space $C^n_{[a,b]}$, where d is defined as in theorem 2.2.5, is complete.

<u>Proof</u>. Let $\{ f^{(p)} \} = \{ (f_1^{(p)}, f_2^{(p)}, \ldots, f_n^{(p)}) \}$ be a d-Cauchy sequence of continuous function of a metric space $C^n_{[a,b]}$.

Given any $\varepsilon > 0$ there exists a positive integer N such that

$$d(f^{(p)}, f^{(q)}) = \sup_{x,i} \left| f_i^{(p)}(x) - f_i^{(q)}(x) \right| < \frac{\varepsilon}{2}$$

for all $p, q \geq N$ so that

$$(**) \qquad \left| f_i^{(p)}(x) - f_i^{(q)}(x) \right| < \frac{\varepsilon}{2}$$

for all $p, q \geq N$, $x \in [a, b]$ and $i = 1, 2, \ldots, n$. For each x and i fixed $\{ f_i^{(p)}(x) \}$ forms a Cauchy sequence in $\mathbb{R}^1$. Since $\mathbb{R}^1$ is complete, $\{ f_i^{(p)}(x) \}$ converges to an element in $\mathbb{R}^1$. Let

$$c_{i,x} = \lim_{p \to \infty} f_i^{(p)}(x)$$

for $i = 1, 2, \ldots, n$. Now we define the functions f_i on $[a, b]$ such that

$$f_i(x) = c_{i,x}$$

for all $x \in [a, b]$ and $i = 1, 2, \ldots, n$. Hence

$$f_i(x) = \lim_{p \to \omega} f_i^{(p)}(x) .$$

So that there exist positive integers N_i such that

$$\left| f_i^{(p)}(x) - f_i(x) \right| < \varepsilon$$

for all $p \geq N_i$, $x \in [a, b]$ and $i = 1,2,\ldots, n$. Let

$$N = \max \{ N_1, N_2, \ldots, N_n \} .$$

If $p \geq N$, then

$$d(f^{(p)}, f) = \sup_{x,i} \left| f_i^{(p)}(x) - f_i(x) \right| < \varepsilon .$$

Therefore $\{ f^{(p)} \}$ d-converges to f.

By taking q goes to + ∞ in (* *), we have

$$| f_i^{(p)}(x) - f_i(x) | < \varepsilon$$

for each i = 1, 2,..., n and $x \in [a, b]$, i.e. $\{ f_i^{(p)} \}$ d-converges uniformly to f_i. By lemma 2.2.9, f_i are continuous, for i = 1, 2,..., n and hence $f = (f_1, f_2, \ldots, f_n) \in C^n_{[a,b]}$. The proof is complete.

2.2.12 Proposition. If (Y, d) is a closed subspace of a complete metric space (X, d). Then (Y, d) is complete.

Proof. Let $\{ x_n \}$ be a d-Cauchy sequence in $Y \subset X$. Since (X, d) is complete and Y is closed, we have $\{ x_n \}$ d-converges to a point $x \in Y$. The proof is complete.