CHAPTER IV

SEMILATTICES OF PROPER INVERSE SEMIGROUPS

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. There corresponds a semilattice Y of groups in a natural way as follows : To each $\alpha \in Y$, let σ_α denote the minimum group congruence on S_α. Set $T = \bigcup_{\alpha\in Y} S_\alpha|\sigma_\alpha$. It has been proved in [3] that under the operation $\circ$ on T defined by

$$(a\sigma_\alpha)\circ(b\sigma_\beta) = (ab)\sigma_{\alpha\beta} \qquad (\alpha,\ \beta\in Y,\ a\in S_\alpha,\ b\in S_\beta),$$

T becomes a semilattice Y of groups S_α/σ_α, and hence T is a homomorphic image of S under the homomorphism $a \longmapsto a\sigma_\alpha$ $(\alpha\in Y,\ a\in S_\alpha)$. Moreover, the two semigroups have the same maximum group homomorphic image.

In this chapter, a similar version is studied. We construct a semilattice Y of proper inverse semigroups from a given semilattice Y of inverse semigroups, with a certain condition,such that the semilattice Y of proper inverse semigroups which we construct is a homomorphic image of the given semilattice Y of inverse semigroups. Moreover, the two semigroups have isomorphic maximum group homomorphic images.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. For each $\alpha \in Y$, we denote the Green's relation $\mathcal{R}$, the minimum group congruence and the minimum proper congruence of S_α by $\mathcal{R}_\alpha$, σ_α and τ_α;

respectively. Set $\bar{S} = \bigcup_{\alpha \in Y}(S_\alpha/\tau_\alpha)$ and define an operation * on $\bar{S}$ by

$$(a\tau_\alpha)*(b\tau_\beta) \quad = \quad (ab)\tau_{\alpha\beta}$$

for all α, β in Y, $a \in S_\alpha$, $b \in S_\beta$. We show that the operation * is well-defined if the Green's relation $\mathcal{R}$ of S is a congruence. To show this, we need the following lemmas :

4.1 Lemma. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Assume that for each $\alpha \in Y$, $\mathcal{R}_\alpha$ is a congruence on S_α. Then for α, $\beta \in Y$ and $a, b \in S_\alpha$, $c \in S_\beta$, $a\mathcal{R}_\alpha b$ implies $ca\mathcal{R}_{\alpha\beta}cb$ and $ac\mathcal{R}_{\alpha\beta}bc$.

Proof : Let α, $\beta \in Y$ and $a, b \in S_\alpha$ such that $a\mathcal{R}_\alpha b$. Then $a = bx$ and $b = ay$ for some $x, y \in S_\alpha$. Let $c \in S_\beta$. Then $ca = cbx$ and $cb = cay$. Hence $ca, cb \in S_{\alpha\beta}$, and

$$ca \quad = \quad cb((cb)^{-1}cbx),$$

and
$$cb \quad = \quad ca(ca)^{-1}cay),$$

so $ca\mathcal{R}_{\alpha\beta}cb$ because $(cb)^{-1}cbx$, $(ca)^{-1}cay \in S_{\alpha\beta}$.

Since R_α is a congruence on S_α and $a\mathcal{R}_\alpha b$, we have $a^{-1}\mathcal{R}_\alpha b^{-1}$, so by the above proof, we get $c^{-1}a^{-1}\mathcal{R}_{\alpha\beta}c^{-1}b^{-1}$, that is, $(ac)^{-1}\mathcal{R}_{\alpha\beta}(bc)^{-1}$. Again, since $\mathcal{R}_{\alpha\beta}$ is a congruence on $S_{\alpha\beta}$, $ac\mathcal{R}_{\alpha\beta}bc$. Hence the lemma is proved. #

4.2 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then $\mathcal{R}$ is a congruence on S if and only if for each $\alpha \in Y$, $\mathcal{R}_\alpha$ is a congruence on S_α.

Proof : The necessary part is obvious. To show the sufficient part, let $a\mathcal{R}b$. Assume $a \in S_\alpha$ and $b \in S_\beta$. Since $a\mathcal{R}b$, there exist $x, y \in S$, say $x \in S_\gamma$, $y \in S_\lambda$ such that

$$a = bx \qquad \text{and} \qquad b = ay.$$

From a = bx, we have $S_\alpha = S_{\beta\gamma}$ so that $\alpha = \beta\gamma$ which implies $\alpha \leq \beta$ and $\alpha \leq \gamma$. Similarly, from b = ay, we have $\beta \leq \alpha$ and $\beta \leq \lambda$. Hence $\alpha = \beta$. It then follows that

$$a = b(b^{-1}bx) \qquad \text{and} \qquad b = a(a^{-1}ay)$$

and $b^{-1}bx \in S_{\beta\gamma} = S_{\alpha\gamma} = S_\alpha$, $a^{-1}ay \in S_{\alpha\lambda} = S_{\beta\gamma} = S_\beta = S_\alpha$. This proves $a, b \in S_\alpha$ and $a \mathcal{R}_\alpha b$. Then by Lemma 4.1, for any $\beta \in Y$, $c \in S_\beta$, we have $ca \mathcal{R}_{\alpha\beta} cb$ and $ac \mathcal{R}_{\alpha\beta} bc$ so that $ca \mathcal{R}\, cb$ and $ac \mathcal{R}\, bc$. #

4.3 __Lemma.__ Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If $\mathcal{R}$ is a congruence on S, then for $\alpha \in Y$, $a, b \in S_\alpha$, $a \tau_\alpha b$ implies $ac\, \tau_{\alpha\beta} bc$ and $ca\, \tau_{\alpha\beta} cb$ for all $\beta \in Y$, $c \in S_\beta$.

__proof__ : Assume that $\mathcal{R}$ is a congruence on S. Then by Corollary 4.2, for each $\alpha \in Y$, $\mathcal{R}_\alpha$ is a congruence on S_α. Hence $\tau_\alpha = \mathcal{R}_\alpha \cap \sigma_\alpha$ for all α in Y.

Let $\alpha \in Y$, $a, b \in S_\alpha$ and $a \tau_\alpha b$. Then $a \mathcal{R}_\alpha b$ and $a\, \sigma_\alpha b$. Let $\beta \in Y$ and $c \in S_\beta$. By Lemma 4.1, $ac \mathcal{R}_{\alpha\beta} bc$ and $ca \mathcal{R}_{\alpha\beta} cb$. Since $a\, \sigma_\alpha b$, ae = be and fa = fb for some $e, f \in E(S_\alpha)$ so that cae = cbe and fac = fbc. Let $e' \in E(S_\beta)$. Thus ee', $e'f \in E(S_{\alpha\beta})$,

$$ca(ee') = (cb)(ee'),$$

and

$$(e'f)ac = (e'f)bc.$$

Hence $ca\, \sigma_{\alpha\beta}\, cb$ and $ac\, \sigma_{\alpha\beta}\, bc$. Therefore $ac\tau_{\alpha\beta}\, bc$ and $ca\, \tau_{\alpha\beta}\, cb$. #

4.4 __Proposition.__ Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α such that the Green's relation $\mathcal{R}$ on S is a congruence. Then the operation * is defined on $\overline{S} = \bigcup_{\alpha \in Y} S_\alpha / \tau_\alpha$ as before is well-defined, and $(\bar{S}, *)$ is a semilattice Y of proper inverse semigroups S_α/τ_α.

<u>Proof</u> : Recall that the operation * on $\bar{S} = \bigcup_{\alpha\in Y} S_\alpha/\tau_\alpha$ is defined by

$$(a\tau_\alpha)*(b\tau_\beta) = (ab)\tau_{\alpha\beta} \qquad (\alpha, \beta \in Y, a\in S_\alpha, b\in S_\beta).$$

To show * is well-defined, let $\alpha, \beta \in Y$, $a, c\in S_\alpha$ and $b, d\in S_\beta$ such that $a\tau_\alpha = c\tau_\alpha$ and $b\tau_\beta = d\tau_\beta$. By Lemma 4.3, $ab\tau_{\alpha\beta} = cb\tau_{\alpha\beta}$ and $cb\tau_{\alpha\beta} = cd\tau_{\alpha\beta}$, so $ab\tau_{\alpha\beta} = cd\tau_{\alpha\beta}$. Hence * is well-defined. Since $(S_\alpha/\tau_\alpha)*(S_\beta/\tau_\beta) \subseteq S_{\alpha\beta}/\tau_{\alpha\beta}$ for all $\alpha, \beta \in Y$, and for each $\alpha \in Y$, S_α/τ_α is a proper inverse semigroup, we have $(\bar{S}, *)$ is a semilattice Y of proper inverse semigroups S_α/τ_α. #

By Proposition 4.4, we then have

4.5 <u>Corollary</u>. Following Proposition 4.4, $\bar{S}$ is a homomorphic image of S by the homomorphism $\psi : S \to \bar{S}$ defined by

$$a\psi = a\tau_\alpha$$

for all $\alpha \in Y$, $a \in S_\alpha$.

Let δ be the congruence on S induced by the homomorphism $\psi : S \to \bar{S}$ defined in Corollary 4.5. Then for all $a, b \in S$,

$a\delta b$ if and only if $a, b\in S_\alpha$ for some $\alpha \in Y$ and $a\tau_\alpha b$.

Therefore $S/\delta \cong (\bar{S}, *)$. Let σ be the minimum group congruence on S. To show $\delta \subseteq \sigma$, let $a, b \in S$ such that $a\delta b$. Then $a, b \in S_\alpha$ for some $\alpha \in Y$ and $a\tau_\alpha b$. Since $\tau_\alpha \subseteq \sigma_\alpha$, we have $a\sigma_\alpha b$, so $ae = be$ for some

$e \in E(S_\alpha) \subseteq E(S)$. Hence $a\sigma b$. Therefore, by Proposition 2.2, we then have

4.6 Lemma. Following Proposition 4.4, and let δ be as above. Then for all $a, b \in S$, $(a, b) \in \sigma$ if and only if $(a\delta, b\delta) \in \sigma(S/\delta)$. Hence

$$S/\sigma(S) \cong (S/\delta)/\sigma(S/\delta),$$

and so S and $(\bar{S}, *)$ have the same maximum group homomorphic image.

The next theorem follows directly from Proposition 4.4, Corollary 4.5 and Lemma 4.6.

4.7 Theorem. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α such that the Green's relation $\mathcal{R}$ on S is a congruence. Then $(\bar{S}, *)$, defined from S as before, is a semilattice Y of proper inverse semigroups S_α/τ_α and it is a homomorphic image of S. Moreover, the two semigroups have isomorphic maximum group homomorphic images.

A semigroup S is completely regular if for every element a in S, there exists x in S such that a = axa and ax = xa. It follows that if an inverse semigroup S is completely regular, then $aa^{-1} = a^{-1}a$ for all $a \in S$. To see this, let S be an inverse semigroup which is completely regular. Let $a \in S$. Then there exists $x \in S$ such that a = axa and ax = xa. Hence

$$a = a(xax)a$$

and $\qquad xax = (xax)a(xax).$

Since S is an inverse semigroup, $a^{-1} = xax$ so that

$$aa^{-1} = axax = xaxa = a^{-1}a.$$

Let an inverse semigroup S be completely regular. Let $a \mathcal{R} b$ in S. Since $a \mathcal{R} a^{-1}a$ and $b \mathcal{R} b^{-1}b$, $a^{-1}a \mathcal{R} b^{-1}b$. But $a^{-1}a = aa^{-1}$ and $b^{-1}b = bb^{-1}$. Then $aa^{-1} \mathcal{R} bb^{-1}$. Since $\mathcal{R}$ is left compatible on S, $a^{-1}aa^{-1} \mathcal{R} b^{-1}bb^{-1}$ so that $a^{-1} \mathcal{R} b^{-1}$. This proves for any $a, b \in S$, $a \mathcal{R} b$ if and only if $a^{-1} \mathcal{R} b^{-1}$. To show $\mathcal{R}$ is right compatible on S, let $a \mathcal{R} b$ in S and $c \in S$. Then $a^{-1} \mathcal{R} b^{-1}$ and $c^{-1} \in S$. Since $\mathcal{R}$ is left compatible, $c^{-1}a^{-1} \mathcal{R} c^{-1}b^{-1}$, so $(ac)^{-1} \mathcal{R} (bc)^{-1}$. From above proof, $ac \mathcal{R} bc$. Hence $\mathcal{R}$ is a congruence on S.

Thus, from the above proof and Proposition 4.4, we have

4.8 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If S is completely regular, then $(\bar{S}, *)$ defined as before, is a semilattice Y of proper inverse semigroups S_α/τ_α and $(\bar{S}, *)$ is a homomorphic image of S. Moreover,

$$S/\sigma(S) \cong \bar{S}/\sigma(\bar{S}).$$

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. It is clearly seen that if for each $\alpha \in Y$, S_α is completely regular, then S is completely regular.

Hence the following corollary follows :

4.9 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If for each $\alpha \in Y$, S_α is completely regular, then $(\bar{S}, *)$, defined from S as before, is a semilattice Y of proper inverse semigroups S_α/τ_α and it is a homomorphic image of S, and also

$$S/\sigma(S) \quad \cong \quad \bar{S}/\sigma(\bar{S}).$$