CHAPTER II

MINIMUM PROPER CONGRUENCES

L. O' Carroll has shown in [8] that every inverse semigroup has the minimum proper congruence and it is the congruence generated by $\mathcal{R} \cap \sigma$, which is denoted by $\tau(S)$ or τ. In this chapter, we give an explicit form of τ on some inverse semigroups, and also show an explicit relationship between the minimum proper congruences on an inverse semigroup and on its ideals. Moreover, a relation among the minimum group congruence σ, the maximum idempotent - separating congruence μ and the minimum proper congruence τ on an inverse semigroup is given.

Let S be an inverse semigroup. If $a \in S$, then $a = aa^{-1}a$ so that

$$Sa = Sa^{-1}a, \qquad aS = aa^{-1}S,$$

and hence $(a, a^{-1}a) \in \mathcal{L}$ and $(a, aa^{-1}) \in \mathcal{R}$. Let $a, b \in S$ such that $a\mathcal{R}b$. Then $aa^{-1}\mathcal{R}\, bb^{-1}$. Since aa^{-1} and bb^{-1} are idempotents of the inverse semigroup S and $aa^{-1}S = bb^{-1}S$, we have $aa^{-1} = bb^{-1}$ [[1], Theorem 1.17].

If ρ is a group congruence on a semigroup S, then E(S) is clearly contained in the ρ-class which represents the identity of the group S/ρ, so for any $e \in E(S)$, $e\rho$ is the identity of S/ρ, and $E(S) \subseteq e\rho$ for all $e \in E(S)$.

Recall that a congruence ρ on an inverse semigroup S is called a proper congruence on S if S/ρ is proper. An inverse semigroup S is proper if and only if for all $a \in S$, $e \in E(S)$, $ae = e$ implies $a \in E(S)$. However, the definition of proper inverse semigroups can be given in many forms as follow :

2.1 Proposition. Let S be an inverse semigroup. Then the following are equivalent :

(1) For $a \in S$, $e \in E(S)$, $ae = e$ implies $a \in E(S)$.

(2) For $a \in S$, $e \in E(S)$, $ea = e$ implies $a \in E(S)$.

(3) $e\sigma = E(S)$ for all $e \in E(S)$.

(4) $\mathcal{R} \cap \sigma = \iota$, where ι denotes the identity congruence on S.

(5) The mapping $\psi : S \longrightarrow E(S) \times S/\sigma$ defined by

$$a\psi = (aa^{-1}, a\sigma) \qquad (a \in S)$$

is one-to-one.

(6) For any $a, b \in S$, if $a\sigma = b\sigma$ and $aa^{-1} = bb^{-1}$, then $a = b$.

Proof : That (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) is obvious. The equivalence of (3) and (4) was shown by Reilly [9], and the equivalence of (3) and (6) was shown in [6]. That (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) was shown by Saitô [10]. The equivalence of (5) and (6) is trivial. #

Let S be an inverse semigroup and ρ be a congruence on S. We know that S/ρ is also an inverse semigroup and for any $a\rho \in E(S/\rho)$, there exists $e \in E(S)$ such that $a\rho = e\rho$. Hence

$$E(S/\rho) = \{e\rho / e \in E(S)\}.$$

The next proposition shows a specific property of the minimum group congruence on an inverse semigroup.

2.2 Proposition. Let S be an inverse semigroup with the minimum group congruence σ. Let η be a congruence on S such that $\eta \subseteq \sigma$. Then for any a, $b \in S$, $(a, b) \in \sigma$ if and only if $(a\eta, b\eta) \in \sigma(S/\eta)$.

Proof : Let $(a, b) \in \sigma$. Then ae = be for some $e \in E(S)$ so that

$$(a\eta)(e\eta) = (ae)\eta = (be)\eta = (b\eta)(e\eta).$$

Since $e\eta \in E(S/\eta)$, we have $(a\eta, b\eta) \in \sigma(S/\eta)$.

Conversely, let $(a\eta, b\eta) \in \sigma(S/\eta)$. Then there exists $e \in E(S)$ such that $(a\eta)(e\eta) = (b\eta)(e\eta)$ and hence $(ae)\eta = (be)\eta$. But $\eta \subseteq \sigma$, then $(ae)\sigma = (be)\sigma$ and hence

$$a\sigma = (ae)\sigma = (be)\sigma = b\sigma,$$

so that $(a, b) \in \sigma$. #

The following theorem shows the existence of the minimum proper congruence on any inverse semigroup [[8], L. O' Carroll].

2.3 Theorem [8]. Let S be an inverse semigroup, $\nu = \mathcal{R} \cap \sigma$ and τ be the congruence generated by ν. Then τ is the minimum proper congruence on S.

Proof : By Proposition 2.1 (3), to show S/τ is proper, it suffices to show that $(e\tau)\sigma(S/\tau) = E(S/\tau)$ for all $e \in E(S)$. Let $e \in E(S)$. Since $\sigma(S/\tau)$ is a group congruence on S/τ, $E(S/\tau) \subseteq (e\tau)\sigma(S/\tau)$. Conversely, let $t\tau \in (e\tau)\sigma(S/\tau)$. Then $tt^{-1} \in E(S)$ so that $(e\tau)\sigma(S/\tau) = ((tt^{-1})\tau)\sigma(S/\tau)$. Then by

Proposition 2.2, $(t, tt^{-1}) \in \sigma$ because $\tau \subseteq \sigma$. Since $tS = tt^{-1}S$, $(t, tt^{-1}) \in \mathcal{R}$. Therefore $(t, tt^{-1}) \in \nu \subseteq \tau$, so that $t\tau = (tt^{-1})\tau$ which is an idempotent of S/τ. This proves $(e\tau)\sigma(S/\tau) = E(S/\tau)$. Hence S/τ is proper.

Next, let η be any proper congruence on S. To show that $\nu \subseteq \eta$, let $(x, y) \in \nu$. Then $(x, y) \in \mathcal{R}$ and $(x, y) \in \sigma$, so $xx^{-1} = yy^{-1}$ and $ex = ey$ for some $e \in E(S)$. Thus

$$(x\eta)(x\eta)^{-1} = (xx^{-1})\eta = (yy^{-1})\eta = (y\eta)(y\eta)^{-1}$$

and

$$(e\eta)(x\eta) = (ex)\eta = (ey)\eta = (e\eta)(y\eta).$$

Thus $(x\eta, y\eta) \in \sigma(S/\eta)$, that is, $(x\eta)\sigma(S/\eta) = (y\eta)\sigma(S/\eta)$. Now we have $(x\eta)\sigma(S/\eta) = (y\eta)\sigma(S/\eta)$ and $(x\eta)(x\eta)^{-1} = (y\eta)(y\eta)^{-1}$. Since S/η is proper, by Porposition 2.1 (6), $x\eta = y\eta$ so that $(x, y) \in \eta$. Hence $\nu \subseteq \eta$, so $\tau \subseteq \eta$. #

2.4 Proposition [8]. Following from Theorem 2.3, let η be any congruence on S such that $\eta \subseteq \sigma$. Then η is a proper congruence on S if and only if $\nu \subseteq \eta$.

Proof : Since the minimum proper congruence on S is the smallest congruence containing ν, we have $\nu \subseteq \eta$ if η is a proper congruence on S.

Conversely, assume $\nu \subseteq \eta$. To show S/η is proper by Proposition 2.1 (3), let $e \in E(S)$ and $a\eta \in (e\eta)\sigma(S/\eta)$. Since $aa^{-1} \in E(S)$, $(aa^{-1})\eta \in E(S/\eta)$ so that $(aa^{-1})\eta \in (e\eta)\sigma(S/\eta)$. Hence $(a\eta, (aa^{-1})\eta) \in \sigma(S/\eta)$. By Proposition 2.2, we have $(a, aa^{-1}) \in \sigma$.

But $a\mathcal{R}aa^{-1}$, so $(a, aa^{-1}) \in \mathcal{R} \cap \sigma = \nu \subseteq \eta$. Hence $a\eta = (aa^{-1})\eta \in E(S/\eta)$. This proves $(e\eta)\sigma(S/\eta) \subseteq E(S/\eta)$. But $\sigma(S/\eta)$ is a group congruence on S/η, so we have $E(S/\eta) \subseteq (e\eta)\sigma(S/\eta)$. Hence $E(S/\eta) = (e\eta)\sigma(S/\eta)$ for all $e \in E(S)$. Therefore S/η is proper, and so η is a proper congruence on S. #

A reformulation of the preceeding proposition is given as follows :

2.5 <u>Proposition</u> [8]. Following Theorem 2.3, a congruence η on S such that $\eta \subseteq \sigma$ is a proper congruence if and only if $E\eta = E\sigma$ where E denotes $E(S)$.

<u>Proof</u> : Recall that $E(S/\eta) = \{e\eta / e \in E(S)\}$.

Let $\eta \subseteq \sigma$ be a congruence on S. Assume that η is a proper congruence. Since $\eta \subseteq \sigma$, $E\eta \subseteq E\sigma$. Let $x \in E\sigma$. Let $e \in E(S)$. Then $e\sigma = E\sigma$, so that $x \in e\sigma$. Hence by Proposition 2.2, $(x\eta, e\eta) \in \sigma(S/\eta)$. Since S/η is proper, by Proposition 2.1 (3), $(e\eta)\sigma(S/\eta) = E(S/\eta)$ and hence $x\eta \in E(S/\eta) = \{f\eta / f \in E(S)\}$. Thus, $x\eta = f\eta$ for some $f \in E(S)$, that is, $x \in f\eta$. This proves $E\sigma \subseteq E\eta$. Therefore $E\eta = E\sigma$.

Conversely, assume that $E\eta = E\sigma$. Let $e \in E(S)$ and $x\eta \in (e\eta)\sigma(S/\eta)$. Then by Proposition 2.2, $x \in e\sigma \subseteq E\eta$, so $x \in f\eta$ for some $f \in E(S)$ so that $x\eta = f\eta \in E(S/\eta)$. Hence $E(S/\eta) = (e\eta)\sigma(S/\eta)$ for all $e \in E(S)$. Therefore S/η is proper, so η is a proper congruence on S. #

Let A be an ideal of an inverse semigroup S. Then for $a \in A$, 2.7 Lemma. Let A be an ideal of an inverse semigroup S. Let s, $t \in S$. If $s\tau'(S)t$, then $csd\tau'(A)ctd$ for all c, $d \in A$.

Proof : Let s, $t \in S$, c, $d \in A$ and $s\tau'(S)t$. Then $s = u_1v$, $t = u_2v$ for some $(u_1, u_2) \in \mathcal{R}(S) \cap \sigma(S)$ and $v \in S^1$. Since $\mathcal{R}(S) \cap \sigma(S)$ is a left congruence and A is an ideal of S, $(cu_1, cu_2) \in \mathcal{R}(S) \cap \sigma(S) \cap (A\times A) = \mathcal{R}(A) \cap \sigma(A)$ [Lemma 2.6]. Since $vd \in A$, $(cst, ctd) = (cu_1vd, cu_2vd) \in \tau'(A)$. #

2.8 Theorem. Let A be an ideal of an inverse semigroup S. Then $\tau(A) = \tau(S) \cap (A\times A)$.

Proof : For this proof, let σ, $\mathcal{R}$, τ and τ' denote $\sigma(S)$, $\mathcal{R}(S)$, $\tau(S)$ and $\tau'(S)$; respectively. From Lemma 2.6,

$$\sigma(A) \cap \mathcal{R}(A) \quad = \quad \sigma \cap \mathcal{R} \cap (A\times A) \subseteq \tau \cap (A\times A).$$

Because τ is a congruence on S and A is a subsemigroup of S, $\tau \cap (A\times A)$ is a congruence on A. Hence $\tau(A) \subseteq \tau \cap (A\times A)$.

Since τ is the congruence generated by $\mathcal{R} \cap \sigma$ and $\mathcal{R} \cap \sigma$ is a left congruence, τ can be obtained from τ' as follows :
For a, $b \in S$,

$$a\tau b \text{ if and only if } a\tau' c_1 \tau' c_2 \ldots\ldots c_n \tau' b$$

for some positive integer n and for some $c_1, c_2, \ldots\ldots, c_n$ in S. Let $(a, b) \in \tau \cap (A\times A)$. Then $(a^{-1}, b^{-1}) \in \tau \cap (A\times A)$. Hence there exist $s_1, s_2, \ldots\ldots, s_n, t_1, t_2, \ldots\ldots, t_m \in S$ such that

$$a\tau' s_1 \ldots\ldots\ldots\ldots\ldots\ldots s_{n-1} \tau' s_n \tau' b \qquad (1)$$

and

$$a^{-1} \tau' t_1 \ldots\ldots\ldots\ldots\ldots\ldots t_{m-1} \tau' t_m \tau' b^{-1} \qquad (2)$$

By compatibility of τ', we get the following :

we have $a^{-1} = a^{-1}aa^{-1} \in A$. Hence A is an inverse subsemigroup of S, so that the minimum proper congruence on A exists.

The next theorem shows a natural relation between the minimum proper congruences on an inverse semigroup and on its ideals. To give the theorem, we need the following two lemmas:

2.6 Lemma. Let A be any ideal of an inverse semigroup S. Then $\mathcal{R}(A) = \mathcal{R}(S) \cap (A \times A)$ and $\sigma(A) = \sigma(S) \cap (A \times A)$ where $\mathcal{R}(S)$ and $\mathcal{R}(A)$ denote the Green's relations $\mathcal{R}$ on S and on A; respectively.

Proof : Clearly, $\mathcal{R}(A) \subseteq \mathcal{R}(S) \cap (A \times A)$. Let $(a, b) \in \mathcal{R}(S) \cap (A \times A)$. Then $a = bx$, $b = ay$ for some $x, y \in S$. Since A is an ideal and $a, b \in A$, we get $b^{-1}bx, a^{-1}ay \in A$. Thus $a = b(b^{-1}bx)$ and $b = a(a^{-1}ay)$, so that $(a, b) \in \mathcal{R}(A)$. Therefore $\mathcal{R}(A) = \mathcal{R}(S) \cap (A \times A)$.

Clearly, $\sigma(A) \subseteq \sigma(S) \cap (A \times A)$. Let $(a, b) \in \sigma(S) \cap (A \times A)$. Then $ae = be$ for some $e \in E(S)$. Let $f \in E(A)$. Then $aef = bef$ and $ef \in E(A)$, so $(a, b) \in \sigma(A)$. Therefore $\sigma(A) = \sigma(S) \cap (A \times A)$. #

For convenience, on an inverse semigroup S, let $\tau'(S)$ denote the set

$$\{(ax, bx) \mid (a, b) \in \mathcal{R}(S) \cap \sigma(S) \text{ for all } x \in S^1\}.$$

We note that in any inverse semigroup S, $\mathcal{R} \cap \sigma$ is a left congruence but not in general a congruence.

From (2), we have

$$a = aa^{-1}a\tau'at_1a\tau'at_2a \ldots\ldots\ldots \tau'at_ma\tau'ab^{-1}a, \quad (3)$$

$$ba^{-1}b\tau'bt_1b\tau'bt_2b \ldots\ldots\ldots \tau'bt_mb\tau'bb^{-1}b = b, \quad (4)$$

and $$bb^{-1}a\tau'bt_ma\tau'bt_{m-1}a \ldots\ldots\ldots \tau'bt_1a\tau'ba^{-1}a. \quad (5)$$

From (1), we have

$$ab^{-1}a = aa^{-1}ab^{-1}a\tau'aa^{-1}s_1b^{-1}a \ldots \tau'aa^{-1}s_nb^{-1}a\tau'aa^{-1}bb^{-1}a = bb^{-1}a, \quad (6)$$

and

$$ba^{-1}a = ba^{-1}ab^{-1}b\tau'ba^{-1}s_1b^{-1}b \ldots \tau'ba^{-1}s_nb^{-1}b\tau'ba^{-1}bb^{-1}b = ba^{-1}b. \quad (7)$$

Combining (3), (6), (5), (7), (4) and Lemma 2.7, we obtain $a\tau(A)b$.

Hence we get $\tau(A) = \tau \cap (A\times A)$ as required. #

If S is a commutative inverse semigroup, then the Green's relation $\mathcal{R}$ on S is a congruence so that $\tau = \mathcal{R} \cap \sigma$. A commutative inverse semigroup is a semilattice of groups by Lemma 1.3, but the converse is not true in general. However, in a semilattice of groups, its Green's relation $\mathcal{R}$ is also a congruence and so its minimum proper congruence is $\mathcal{R} \cap \sigma$.

The next proposition gives an explicit form of τ on a semilattice of groups.

2.9 <u>Proposition</u>. Let S be a semilattice Y of groups G_α. Then

$$\tau = \{(a, b) \in G_\alpha \times G_\alpha \mid \alpha \in Y \text{ and } ae_\beta = be_\beta \text{ for some } \beta \in Y\}.$$

In particular, if Y has the zero 0, then

$$\tau = \{(a, b) \in G_\alpha \times G_\alpha \mid \alpha \in Y \text{ and } ae_0 = be_0\}.$$

Proof : From Introduction page 11, S is an inverse semigroup, $E(S) = \{e_\alpha \mid \alpha \in Y\}$ and $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in Y$. Let

$$\delta = \{(a, b) \in G_\alpha \times G_\alpha \mid \alpha \in Y \text{ and } ae_\beta = be_\beta \text{ for some } \beta \in Y\}.$$

Since $\mathcal{L} = \mathcal{R} = \mathcal{H}$ on S [Introduction page 11], $\mathcal{R}$ is a congruence on S, and for each $\alpha \in Y$, G_α is an $\mathcal{R}$-class. Hence $\tau = \mathcal{R} \cap \sigma = \delta$.

Assume more that Y has the zero 0. Let $(a, b) \in \sigma$. Then $ae_\beta = be_\beta$ for some $\beta \in Y$, so

$$ae_0 = ae_\beta e_0 = be_\beta e_0 = be_0.$$

Therefore $(a, b) \in \sigma$ implies $ae_0 = be_0$, so $(a, b) \in \sigma$ if and only if $ae_0 = be_0$. Hence from the first part of the proof, we have

$$\tau = \{(a, b) \in G_\alpha \times G_\alpha \mid \alpha \in Y \text{ and } ae_0 = be_0\}. \quad \#$$

2.10 Corollary. Following Proposition 2.9, assume that Y has the zero 0. Then $\tau = \sigma$ if and only if $S = G_0$.

Proof : Assume that $\tau = \sigma$. Let $s \in S$. Since $(s, se_0) \in \sigma = \tau$, by Proposition 2.9, we have $s, se_0 \in G_\alpha$ for some $\alpha \in Y$. But $se_0 \in G_0$. Then $s \in G_0$. Therefore $S \subseteq G_0$, so that $S = G_0$.

Conversely, assume that $S = G_0$. Then $\mathcal{R}$ is the universal congruence, and thus $\tau = \mathcal{R} \cap \sigma = \sigma$. #

J.M. Howie [4] has proved the existence of the maximum idempotent-separating congruence on any inverse semigroup S, and denote it by $\mu(S)$ or μ and

$$\mu = \{(a, b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\};$$

equivalently,

$$\mu = \{(a, b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\}.$$

Moreover, $\mu \subseteq \mathcal{H}$.

On any inverse semigroup S, a relation among σ, τ and μ is as follows :

2.11 Proposition. Let S be an inverse semigroup. Then $\mu \cap \tau = \mu \cap \sigma$.

Proof : Since $\tau \subseteq \sigma$, $\mu \cap \tau \subseteq \mu \cap \sigma$. Because $\mu \subseteq \mathcal{H} \subseteq \mathcal{R}$, we have $\mu \cap \sigma \subseteq \mathcal{R} \cap \sigma \subseteq \tau$. Then $\mu \cap \sigma \subseteq \mu \cap \tau$. Therefore $\mu \cap \tau = \mu \cap \sigma$. #

From this fact, the next theorem follows immediately [[4], Theorem 3.2].

2.12 Theorem. Let τ be the minimum proper congruence and μ be the maximum idempotent-separating congruence on S. Let

$$E\omega = \{x \in S \mid x \geq e \text{ for some } e \in E(S)\},$$

and $C(E(S))$ be the centralizer of $E(S)$ in S. Then $\tau \cap \mu = \iota$ if and only if $E\omega \cap C(E(S)) = E(S)$, where ι denote the identity congruence on S.

Let A be an ideal of an inverse semigroup S. To show a natural relation between $\mu(S) \cap \sigma(S)$ and $\mu(A) \cap \sigma(A)$, we need the following lemma :

2.13 Lemma. Let A be an ideal of an inverse semigroup S. Then $\mu(A) = \mu(S) \cap (A \times A)$.

Proof : Clearly, $\mu(S)\cap(A\times A)\subseteq\mu(A)$. Let $(x, y)\in\mu(A)$. Then $x^{-1}ex = y^{-1}ey$ for all $e\in E(A)$. Let $f\in E(S)$. Then $xx^{-1}fxx^{-1}\in E(A)$ so that

$$x^{-1}fx = x^{-1}(xx^{-1}fxx^{-1})x = y^{-1}(xx^{-1}fxx^{-1})y.$$

Since $(x, y)\in\mu(A)\subseteq\mathcal{H}(A)\subseteq\mathcal{R}(A)$, $xx^{-1} = yy^{-1}$ so that

$$x^{-1}fx = y^{-1}(yy^{-1}fyy^{-1})y = y^{-1}fy.$$

Therefore $(x, y)\in\mu(S)\cap(A\times A)$. Hence the proof is completed. #

2.14 Theorem. Let A be an ideal of an inverse semigroup S. Then $\mu(A)\cap\tau(A) = (\mu(S)\cap\tau(S))\cap(A\times A)$.

Proof : It follows directly from Theorem 2.8 and Lemma 2.13. #