INTRODUCTION

Let S be a semigroup. An element a of S is an *idempotent* of S if $a^2 = a$. For a semigroup S, we denote by E(S) the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a *semilattice* if for all $a, b \in S$, $a^2 = a$ and $ab = ba$. An element z of a semigroup S is a *zero* of S if $xz = zx = z$ for all $x \in S$. An element e of a semigroup S is an *identity* of S if for all $x \in S$, $ex = xe = x$.

Let S be a semigroup, and let 1 be a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to one is $S \cup 1$ by defining $1 \cdot 1 = 1$ and $1a = a1 = a$ for every $a \in S$. Throughout this thesis we will adhere to the following notation :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{otherwise.} \end{cases}$$

Let S be a semigroup. An element a of S is *regular* if $a = axa$ for some $x \in S$, and S is called a *regular semigroup* if every element of S is regular.

In any semigroup, S if $a, x \in S$ such that $a = axa$, then ax and xa are idempotents of S. Hence if S is a regular semigroup, then $E(S) \neq \phi$.

Let a and x be elements of a semigroup S such that a = axa. Then

(i) $aS = aS^1$ and $S^1a = Sa$, and

(ii) $aS = axS$ and $Sxa = Sa$.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if a = axa, x = xax. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [[1], Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. For any elements a, b of an inverse semigroup S and $e \in E(S)$, we have

$$(a^{-1})^{-1} = a, \quad (ab)^{-1} = b^{-1}a^{-1} \text{ and } e^{-1} = e$$

[[1],Lemma 1.18].

Let X be a set. By a <u>one-to-one partial transformation</u> of the set X we mean a one-to-one mapping α of a subset of X onto a subset of X. Let I_X be the set of all one-to-one partial transformations of X. For $\alpha \in I_X$, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain of α and the range of α; respectively. Note that the mapping, whose domain and range are the empty subset of X, is a member of I_X, which is called the <u>empty transformation</u> and will be denoted by 0. The product $\alpha\beta$ of two elements α and β of I_X is defined as follows: If $\nabla\alpha \cap \Delta\beta = \phi$, we define $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, we define $\alpha\beta$ to be the iterate of $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{(\nabla\alpha \cap \Delta\beta)}$ in the usual sense. Under this operation, I_X becomes an inverse semigroup [[1]] and we call it the symmetric inverse semigroup on the set X. It is clearly seen that the empty transformation is the zero of I_X and the identity mapping on X is the identity of I_X.

Let T be a subset of a semigroup S. The centralizer of T in S is

$$C(T) = \{a \in S \mid at = ta \text{ for all } t \in T\}.$$

The centralizer of S in S is the center of S. It then follows that if S is an inverse semigroup, $E(S) \subseteq C(E(S))$.

Let P be a nonempty set and $\leq$ be a relation on P. If the relation $\leq$ is reflexive, antisymmetric and transitive, then $\leq$ is called a partial order on P, and $(P, \leq)$ or P is called a partially ordered set.

If a, b belong to an inverse semigroup S, then the following are equivalent [[2], Lemma 7.1] :

(i) $aa^{-1} = ab^{-1}$.

(ii) $aa^{-1} = ba^{-1}$.

(iii) $a^{-1}a = a^{-1}b$.

(iv) $a^{-1}a = b^{-1}a$.

(v) $ab^{-1}a = a$.

(vi) $a^{-1}ba^{-1} = a^{-1}$.

The relation $\leq$ defined on an inverse semigroup S by

$$a \leq b \text{ if and only if } aa^{-1} = ab^{-1}$$

is a partial order on S [[2], Lemma 7.2], and this partial order is called the natural partial order on the inverse semigroup S. We note that the restriction of the natural partial order $\leq$ on an inverse semigroup S to E(S) is as follows :

$$e \leq f \text{ if and only if } e = ef \ (= fe).$$

It then follows that if S is a semilattice, $a \leq b$ in S if and only if $a = ab$ ($= ba$).

If S is an inverse semigroup S and $a, b \in S$, then the following hold :

(i) $a \leq b$ if and only if $a = be$ for some $e \in E(S)$.

(ii) $a \leq b$ if and only if $a = fb$ for some $f \in E(S)$.

A reflexive, symmetric and transitive relation on a nonempty set X is an equivalence relation on X.

Let S be a semigroup. A relation ρ on S is left compatible if for all $a, b, c \in S$, $a\rho b$ implies $ca\rho cb$. Right compatibility is defined dually. By a congruence on S we mean an equivalence relation on S which is both right and left compatible.

Arbitary intersection of congruences on a semigroup S is a congruence on S.

Let ρ be any relation on a semigroup S. Then the intersection of all congruences containing ρ is the <u>congruence on S generated by</u> ρ.

Let ρ be an equivalence relation on a semigroup S, and the relation ρ' on S be defined as follows :

$$\rho' = \{(xay, xby) \mid (a, b) \in \rho \text{ and } x, y \in S^1\}.$$

Let the relation $\bar{\rho}$ on S be defined from ρ' by the rule :

For a, $b \in S$,

$$a\bar{\rho}b \text{ if and only if } a\rho' c_1 \rho' c_2 \ldots\ldots\ldots \rho' c_n \rho' b$$

for some $c_1, c_2, \ldots, c_n \in S$. Then $\bar{\rho}$ is the congruence on S generated by ρ [[1], Theorem 1.8].

If ρ is a congruence on a semigroup S, then the set

$$S/\rho = \{a\rho / a \in S\}$$

with the operation defined by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a, b \in S)$$

is a semigroup, and is called the <u>quotient semigroup relative to the congruence</u> ρ.

Let ρ be a congruence on a semigroup S. Then the mapping $\psi : S \longrightarrow S/\rho$ defined by

$$a\psi = a\rho \qquad (a \in S)$$

is an onto homomorphism and ψ will be denoted by $\rho^\natural$, and call it the <u>natural homomorphism</u> of S onto S/ρ.

Conversely, if $\psi : S \longrightarrow T$ is a homomorphism from a semigroup S into a semigroup T, then the relation ρ on S defined by

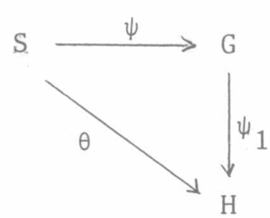
$$a\rho b \text{ if and only if } a\psi = b\psi \qquad (a, b \in S)$$

is a congruence on S and $S/\rho \cong S\psi$, and ρ is called the <u>congruence on S induced by ψ</u>.

Let ρ be a congruence on an inverse semigroup S. Then S/ρ is an inverse semigroup, and for every $a\rho \in S/\rho$, $(a\rho)^{-1} = a^{-1}\rho$. Hence for all $a, b \in S$

$$a\rho b \text{ if and only if } a^{-1}\rho b^{-1}.$$

A group G is called the <u>maximum group homomorphic image</u> of a semigroup S if there exists a homomorphism ψ from S onto G such that the following hold : For any group H and for any homomorphism θ from S onto H, there exists a unique group homomorphism ψ_1 from G onto H such that the diagram



commutes, that is, $\psi\psi_1 = \theta$.

A congruence ρ on a semigroup S is called a <u>group congruence</u> if S/ρ is a group. If ρ is a group congruence on a semigroup S, then E(S) is contained in the ρ-class which represents the identity of the group S/ρ and hence $E(S) \subseteq e\rho$ for all $e \in E(S)$.

Let σ be a group congruence on a semigroup S such that for any group congruence ρ on S, $\sigma \subseteq \rho$. Then σ is called the minimum group congruence on S.

If σ is the minimum group congruence on a semigroup S, then S/σ is the maximum group homomorphic image of S.

Munn [7] has shown that any inverse semigroup S has a minimum group congruence σ and

$$\sigma = \{(a, b) \in S \times S \mid ae = be \text{ for some } e \in E(S)\};$$

equivalently,

$$\sigma = \{(a, b) \in S \times S \mid ea = eb \text{ for some } e \in E(S)\}.$$

Hence any inverse semigroup S has a maximum group homomorphic image. Throughout this thesis, $\sigma(S)$, or σ if there is no danger of ambiguously, will be denoted for the minimum group congruence on the inverse semigroup S.

Let S be a semigroup. A nonempty subset A of S is a left ideal of S if $sa \in A$ for all $s \in S$, $a \in A$. A right ideal of S is defined dually. A nonempty subset of S is an ideal (or two-sided ideal) of S if it is both a left ideal and a right ideal of S. An arbitary intersection of left ideals, of right ideals and of ideals of a semigroup S is a left ideal, a right ideal and an ideal of S; respectively.

An ideal of an inverse semigroup S is an inverse subsemigroup of S.

Let A be a nonempty subset of a semigroup S. The left ideal of S generated by A is the intersection of all left ideals of S containing A. The right ideal of S generated by A is defined dually. The ideal of S generated by A is the intersection of all ideals of S containing A. If A contains only one element, say a, the left ideal of S generated by A is called the principal left ideal of S generated by a, the principal right ideal of S generated by a and the principal ideal of S generated by a are defined similarly.

Let a be an element of a semigroup S. Then we have S^1a, aS^1 and S^1aS^1 are the principal left ideal of S generated by a, the principal right ideal of S generated by a and the principal ideal of S generated by a; respectively.

If S is a regular semigroup, then

$$S^1a = Sa, \quad aS^1 = aS \quad \text{and } S^1aS^1 = SaS$$

for all $a \in S$. If S is a semilattice, then an ideal I of S is principal if and only if $I = aS = Sa = SaS$ for some $a \in S$.

Let S be a semigroup. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ on S are defined as follow :

$$a\mathcal{L}b \text{ if and only if } S^1a = S^1b.$$

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1.$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Note that $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are equivalence relations on S and $\mathcal{H} \subseteq \mathcal{L}$, $\mathcal{H} \subseteq \mathcal{R}$. Moreover, $\mathcal{L}$ is a right congruence on S and $\mathcal{R}$ is a left congruence on S. These relations are called Green's relations on S. Equivalent definitions of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S are given as follow :

$a \mathcal{L} b$ if and only if $a = xb$, $b = ya$ for some $x, y \in S^1$.

$a \mathcal{R} b$ if and only if $a = bx$, $b = ay$ for some $x, y \in S^1$.

Any $\mathcal{H}$-class of S containing an idempotent is a subgroup of S [[1], Theorem 2.16].

Let S be a regular semigroup and $a, b \in S$. Then

$$a \mathcal{L} b \iff Sa = Sb$$
$$\iff a = xb,\ b = ya \text{ for some } x, y \in S,$$

and

$$a \mathcal{R} b \iff aS = bS$$
$$\iff a = bx,\ b = ay \text{ for some } x, y \in S.$$

Let S be an inverse semigroup. For $a \in S$, $a = aa^{-1}a$ so that $Sa = Sa^{-1}a$ and $aS = aa^{-1}S$ and hence $a \mathcal{L} a^{-1}a$ and $a \mathcal{R} aa^{-1}$. Every $\mathcal{L}$-class and every $\mathcal{R}$-class of the inverse semigroup S contains exactly one idempotent [[1], Theorem 1.17]. Then for any $a, b \in S$, we have the following :

$$a \mathcal{L} b \text{ if and only if } a^{-1}a = b^{-1}b.$$
$$a \mathcal{R} b \text{ if and only if } aa^{-1} = bb^{-1}.$$

A congruence ρ on a semigroup S is called an <u>idempotent-separating congruence</u> if each ρ-class contains at most one idempotent of S. An idempotent-separating congruence μ on a semigroup S is the <u>maximum idempotent-separating congruence</u> on S if it contains every idempotent-separating congruence of S.

Howie [4] has proved that the maximum idempotent-separating congruence μ on an inverse semigroup S always exists and

$$\mu = \{(a, b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\};$$

equivalently,

$$\mu = \{(a, b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\}.$$

Moreover, $\mu \subseteq \mathcal{H}$. The maximum idempotent-separating congruence on an inverse semigroup will be denoted by $\mu(S)$ or μ.

A relation between μ and σ on an inverse semigroup S has been given by Howie in [4] as follows :

On an inverse semigroup S, $\mu \cap \sigma = \iota$ if and only if $C(E(S)) \cap E\omega = E(S)$ where ι denotes the identity congruence on S, $E\omega = \{x \in S \mid x \geq e \text{ for some } e \in E(S)\}$ and $C(E(S))$ is the centralizer of $E(S)$ in S.

An inverse semigroup S is <u>proper</u> if for all $a \in S$, $e \in E(S)$, $ae = e$ implies $a \in E(S)$. An inverse subsemigroup of a proper inverse semigroup is clearly proper. Every group is proper, also every semilattice is proper.

Let S be an inverse semigroup. S is an <u>F-inverse semigroup</u> if every σ-class of S has a maximum element.

McFadden [5] has shown that any F-inverse semigroup is proper and has an identity. But the converse is not generally true.

Let ρ be a congruence on an inverse semigroup S. ρ is called a <u>proper congruence</u> on S if S/ρ is proper. An <u>F-inverse congruence</u> on an inverse semigroup is defined similary. The definitions of the <u>minimum proper congruence</u> and the <u>minimum F-inverse congruence</u> on an inverse semigroup are given as similary as the definition of the minimum group congruence on an inverse semigroup.

Let Y be a semilattice and a semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a disjoint union of the subsemigroups S_α of S. S is a _semilattice_ Y _of semigroups_ S_α if $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of semigroups S_α. If for each $\alpha \in Y$, S_α is an inverse subsemigroup of S, then $S = \bigcup_{\alpha \in Y} S_\alpha$ is called a _semilattice_ Y _of inverse semigroups_ S_α. A _semilattice of groups_, a _semilattice of regular semigroups_, etc. are defined similarly.

A semilattice of inverse semigroups is an inverse semigroup [[2], Theorem 7.5]. Then a semilattice Y of groups is an inverse semigroup.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. To each $\alpha \in Y$, let e_α denote the identity of the group G_α. Then

$$E(S) \quad = \quad \{e_\alpha | \alpha \in Y\},$$

and E(S) is contained in the center of S [[1], Lemma 4.8]. Because S is an inverse semigroup, $e_\alpha e_\beta = e_{\alpha\beta}$ for all α, β in Y and hence $E(S) \cong Y$ by the isomorphism $e_\alpha \longmapsto \alpha (\alpha \in Y)$. Moreover, S has an identity if and only if Y has an identity.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then for each $\alpha \in Y$, G_α is an $\mathcal{H}$-class. Moreover, $\mathcal{H} = \mathcal{R} = \mathcal{L}$, and then it is a congruence on S.

An injective homomorphism $\psi : S \longrightarrow T$ from an inverse semigroup S into another inverse semigroup T is called a _full_ _σ-embedding_ of S into T if each $\sigma(T)$-class contains exactly one $\sigma(S\psi)$-class, S is then said to be <u>fully</u> <u>σ-embedded</u> into T.

L. O' Carroll [8] has shown how to construct an F-inverse semigroup M(S) from an arbitary proper inverse semigroup S such that S can be fully σ-embedded into M(S), moreover, they have isomorphic maximum group homomorphic images.

Let S be a proper inverse semigroup. In the first chapter, it is shown that if S is a semilattice of groups, then the extension M(S) is also a semilattice of groups. Moreover, we show that this is true for the case of semilattices of inverse semigroups.

The minimum proper congruence on any inverse semigroup S always exists, which will denoted by τ(S) or τ, and it is the congruence generated by $\mathcal{R} \cap \sigma$ [[8], O' Carroll]. An explicit form of the minimum proper congruence on a semilattice of groups is given in the second chapter. Including in this chapter, we show that the minimum proper congruence on an ideal A of an inverse semigroup S is the restriction of the minimum proper congruence of S to A. Moreover, a relation among σ, μ and τ on an inverse semigroup is given.

In the third chapter, minimum F-inverse congruences on inverse semigroups are studied. An example to show that the minimum F-inverse congruence on an inverse semigroup need not exist is given. Any inverse semigroup with zero and identity always has the minimum F-inverse congruence and it is the minimum proper congruence. It is proved that if an inverse semigroup S has the minimum F-inverse congruence η, then any congruence on S which lies between η and σ is an F-inverse congruence on S. Some kinds of inverse semigroups whose their minimum F-inverse congruences always exist are studied in this chapter.

In the last chapter, we construct a semilattice Y of proper inverse semigroups from a given semilattice Y of inverse semigroups, with a certain condition, such that the semilattice Y of proper inverse semigroups which we construct is a homomorphic image of the given semilattice Y of inverse semigroups. Moreover, the two semigroups have isomorphic maximum group homomorphic images.