พลังงานสถานะพื้นของสองแอนนิออนในศักย์ฮาร์มอนิกโดยเทคนิคอินทิเกรตตามวิถี

นาย วีระชาติ กิเลนทอง

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GROUND STATE ENERGY OF TWO ANYONS IN HARMONIC POTENTIAL BY PATH INTEGRATION

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วิทยานิพนธ์ฉบับนี้นำเสนอ การศึกษาสองแอนนิออนในศักย์ฮาร์มอนิกสองมิติโดยประยุกต์ใช้วิธีการอินทิเกรตตามวิถี จากแนวคิดของ Wilczek จะพบว่าแอนนิออนมีค่าสปินและสถิติเป็นเศษส่วน แนวคิดนี้สามารถประยุกต์ได้เป็นอย่างดีกับ ปรากฏการณ์ควอนตัมฮอลล์เศษส่วน และ สถานะตัวนำยิ่งยวดแบบแอนนิออน ยิ่งไปกว่านั้น อวกาศโครงแบบของระบบแอนนิออนมีลักษณะไม่ขาดตอนหลายส่วน และจะพบว่าวิธีการอินทิกรัลตามวิถีมีประโยชน์อย่างมากในการแก้ปัญหาดังกล่าว ถึงกระนั้นก็ตาม ยังมีอีกวิธีหนึ่งซึ่งสามารถแก้ปัญหาอันเดียวกันนี้ได้เป็นอย่างดี ซึ่งก็คือ วิธีการแบบ เชิร์น-ไซมอนซ์ โดยการเพิ่มพจน์ เชิร์น-ไซมอนซ์ จะมีผลทำให้อนุภาคปกติซึ่งมีค่าสถิติปกติกลายเป็นแอนนิออนได้

ในวิทยานิพนธ์นี้ เราแสดงการคำนวณค่าพลังงานสถานะพื้นของสองแอนนิออนในศักย์ฮาร์มอนิกโดยอาศัยวิธีการอินทิกรัลตามวิถีและประโยชน์ของทฤษฎี เชิร์น-ไซมอนซ์ และยังแสดงด้วยว่าสามารถแก้ปัญหาเกี่ยวกับความซับซ้อนของโทโพโลยีของระบบแอนนิออนได้โดยง่าย เราสามารถพบด้วยว่า ค่าพลังงานสถานะพื้นของสองแอนนิออนในศักย์ฮาร์มอนิกขึ้นอยู่กับค่าพารามิเตอร์เชิงสถิติ α ซึ่งแสดงถึงค่าสถิติของแอนนิออน

ภาควิชา ฟิสิกส์
สาขาวิชา ฟิสิกส์
ปีการศึกษา 2542

ลายมือชื่อนิสิต ....
ลายมือชื่ออาจารย์ที่ปรึกษา ....
ลายมือชื่ออาจารย์ที่ปรึกษาร่วม ....

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KEY WORD: ANYONS/ FRACTIONAL STATISTICS/ PATH INTEGRALS/ CHERN – SIMONS/ CYONS

WEERACHART KILENTHONG: GROUND STATE ENERGY OF TWO ANYONS IN HARMONIC POTENTIAL BY PATH INTEGRATION. (พลังงานสถานะพื้นของสองแอนนิออนในศักย์ฮาร์มอนิกโดยเทคนิคอินทิเกรตตามวิถี) THESIS ADVISOR: PROFESSOR VIRULH SA-YAKANIT, F. D., 101 pp. ISBN 974-333-995-7.

In this thesis, the case of two anyons in a harmonic potential in two dimensions is studied by using Feynman path integration. Following Wilczek's ideas, in the concept of the anyon the extraordinary spin and statistics are fractional. Significant applications of anyons are to the fractional quantum Hall effect and anyon superconductivity. Moreover, the configuration space of anyonic systems is multiply connected, and then the path integral is very useful to solve such problems. Nevertheless, there is the alternative approach that can be applied to the same problem, namely, the Chern-Simons construction. By adding the Chern-Simons term into the ordinary Lagrangian, normal particles, particles with the usual statistics, can become anyons.

In this thesis we show that the ground-state energy of two anyons in a harmonic potential can be evaluated both by applying the path integral technique and by the use of the Chern – Simons theory. We show that the complexity of the topology of the anyonic system can be handled readily. It is shown that the ground – state energy of two anyons in a harmonic potential depends on the statistical parameter, α, representing the statistics of the anyon.

ภาควิชา ฟิสิกส์	ลายมือชื่อนิสิต ....
สาขาวิชา ฟิสิกส์	ลายมือชื่ออาจารย์ที่ปรึกษา ....
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CHAPTER 1
INTRODUCTION

In classical physics, particles are considered as distinguishable particles and their interactions come from conventional forces only (no intrinsically statistical effect). But quantum particles regarded as indistinguishable particles can feel exotic interactions coming from the inherent property, their statistics. The statistics can help us classify identical particles into two classes, bosons and fermions, for three or more spatial dimensions.

Firstly, in the early days of quantum mechanics, identical particles, influenced not only by ordinary forces but also by the particle statistics, have exclusively two families, bosons and fermions, considered in just three spatial dimensions. In other words, since one or two spatial dimensions were neglected in those days, it was considered that only bosons and fermions are logical identical particles in the physical world. Moreover, it is well know that spin and statistics of particle have a close relation. At first, in 1925, S. N. Bose[1] introduced the new quantum concept that more than one particle can be in the same quantum state before W. Pauli[2] introduced his famous exclusion principle stating that there can never be two or more equivalent electrons in an atom. While Pauli's particles are electrons (1/2-spin particles), Bose's particles are photons (spinless particles). These concepts can be extended to cover that particles with integral spin. Unfortunately, even though the spin-statistics connection[3] is crucial in quantum physics, especially quantum field theory, we have not enough time to spend on this subject here. However, many useful details and papers on this topic are collected in Ref.[3].

Twenty three years ago, in 1977, however, J. M. Leinaas and J. Myrheim[4] studied the classical configuration space of a system of identical particles. They showed also that two possibilities, corresponding to symmetric and anti-symmetric wave functions, appear in a natural way in the formalism. But this is only the case when the particles move in a three-or higher-dimensional space. For one and two dimensional spaces a continuum of possible intermediate cases connects the boson and fermion cases. Furthermore, they proposed that particle spin can take on exotic values in a two dimensional world. Unfortunately, there was no wide interest in this subject at that time since no important physical situation in one or two dimension was known until the quantum Hall effect was discovered[5]

The exotic phenomenon of two-dimensional particle was regenerated by Frank Wilczek[6] in 1982. Although he realized that practical applications of these phenomena seemed remote, he thought that they have considerable methodological interest and shed light on the fundamental spin-statistics connection. In that work, he proposed that the flux-tube-charged-particle composites have unusual statistics. Since quantum statistics of this composite, defined by the phase of amplitude associated with slow motion of distance particle around one another, can take on any value, he called them generically "anyons". Especially, for two anyons, their energy levels are not simply related to the single anyon energy levels, as will be stated in detail later.

It is well known that the physical realization of a two-dimensional electron system are inversion layers formed at the interface between a semiconductor and an insulator or between two semiconductors, with one of them acting as an insulator. This system, shown in Fig.1, in which the quantum Hall effect (QHE) was discovered by Ando and et. al.[7] has Si for the semiconductor, SiO_2 for the insulator.

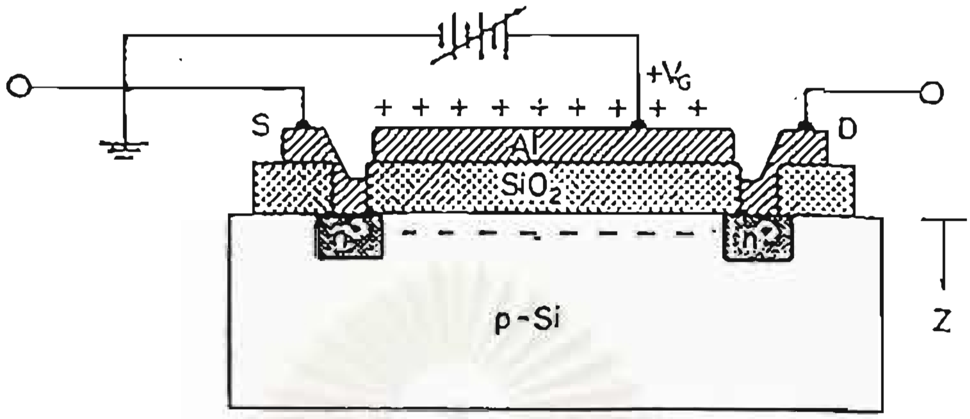

Fig. 1 Schematic side view of a silicon MOSFET[7].

In the above picture the insulating oxide layer is made relatively thick (e.g., 5000 A°) and a metallic electrode (Al) plated over it. This electrode is positively charged by application of an external gate voltage. The resulting device is called a MOSFET. Metal oxide semiconductor field effect transistor.

In 1982, Tsui, Stormer, and Gossard[8] discovered the new quantum phenomenon, the fractional quantum Hall effect (FQHE), in high quality , GaAs-AlxGa1-x As heterostructures. The primary phenomenon is considerably, if not deceptively, similar to the integer quantum Hall effect (IQHE) discovered by Klaus von Klitzing[5]. It is found that in a high quality sample (examplified by large carrier mobility at zero B fields) as very high magnetic fields are applied at very low temperatures, Hall plateaus and deep minimal structures in ρ_{xx}, the resistivity tensor, develop at fractional fillings ν=1/3[8] and 2/3[9], of the lowest Landau level, in a manner analogous to the development of the IQHE at integral fillings. The experimental results are shown in Fig.2 for ν=1/3 and in Fig.3 for ν=2/3

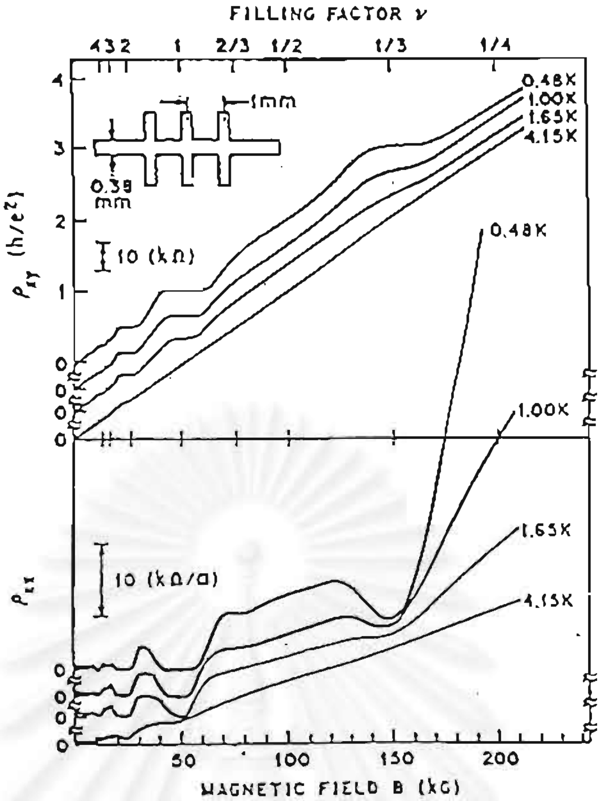

Fig.2 The FQHE at 1/3 filling of the lowest Landau level[8].

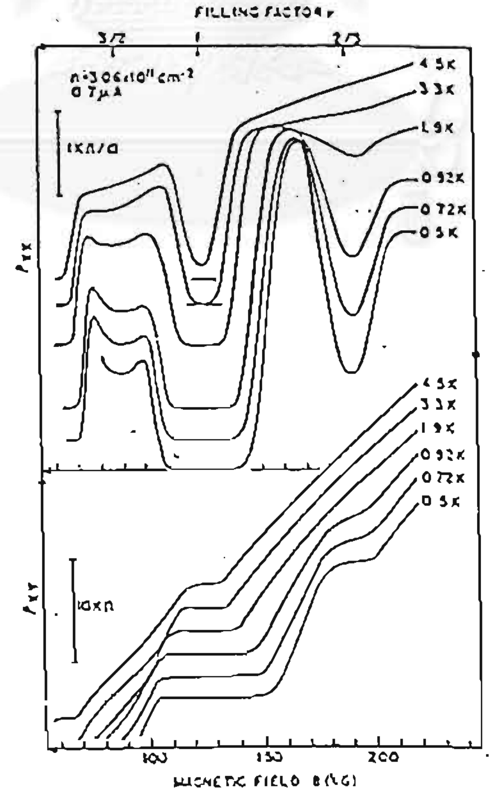

Fig.3 The FQHE at 2/3 filling of the lowest Landau Level[9].

The existence of the QHE at fractional fillings came as a great surprise since it could not be understood within the framework of the then developing theory of the QHE. It immediately suggested the possibility of fractionally charged excitations, perhaps reminiscent of the predicted fractional charges in quasi-one-dimensional systems, However, in 1982, the theory that can describe this phenomenon was developed by Laughlin[10] with important contributions from Haldane[11] and from Halperin[12].At the foundation of the theory is the idea that the new states are best described as incompressible quantum liquids, around which the low-energy excitations are located quasi-particles with non - customary quantum numbers, including strikingly fractional statistics. Using this idea, Halperin was able to augur the values of the allowed fractions in the FQHE hierarchy in a simple and convincing , as well as observationally successful way. Arovas, Schrieffer, and Wilczek[13] using the Berry phase technique[14], showed directly that the quasi-particles had the properties assumed by Halperin. They also suggested that because the statistical interaction, together of course with ordinary electromagnetism, is the dominant interaction of the quasi-particles at long distances it should be possible to write an effective Lagrangian for the long-wavelength behavior of the quasi-particle gas, using just these interactions. The formal implementation of this idea was carried through in the above mentioned paper by Arovas, Schrieffer, Wilczek, and Zee[15]. An important element of that paper, which has played a key role in the further development of the subject, is the introduction of a local implementation of fractional quantum statistics, through the Chern-Simons interaction.

Moreover, it is a very attractive problem, to figure out the behavior of such as new quantum ideal gas. The high temperature, low density behavior was addressed several years ago in a paper by Arovas, Schrieffer, Wilczek, and Zee[15]. They calculate, in particular, the value of the second virial coefficient. A simple result was discovered , that interpolates continuously between bosons and fermions. While this result was significant as a check of the consistency of the whole circle of ideas, and as an exercise for sharpening techniques, it hardly addressed the key questions regarding the new quantum ideal gases. The most important effects of quantum statistics, of course, occur only at low temperatures or high density. The existence of a cusp in the virial coefficient at Bose statistics was one of several indications that the behavior of anyon gases at low temperatures would be interesting and probably far from smooth. However, it has proved quite difficult to extend the calculations starting from the high-temperature end, and since the problem seemed both esoteric and inaccessible it was largely abandoned.

At this point it would be disingenuous not to remark that much of the stimulus for the recent upsurge in interest in the anyon gas are some theoretical speculations that quasi-particles in CuO planes, which presumably are the main actors in high temperature superconductivity, are in fact anyons. These speculations were motivated by the analysis of excitations around certain types of ordered states (chiral spin liquids)[16] that have been proposed for the electronic ground state in the planes. Needless to say, it is a fact that in models of relevance to high-temperature superconductivity there will always be an even number of anyon species.

Immediately after the experimental discovery of the new superconductors, Anderson[17] stressed their essentially two-dimensional character, the importance of strong magnetic ordering, and the possible existence of excitations with exotic quantum numbers. A relatively concrete proposal embodying one form of Anderson's vision was put forward by Kivelson, Rokshar, and Sethna[18]. They showed that the division of valence bonds on a square lattice occupied by approximately one valence electron per site into localized dimers, as suggested by the phase " resonating valence bond , could plausibly support excitations – specifically , defects in the pair – bonding of electrons, trapping a single unpaired site – which are charged, spinless bosons. The initial thought was that Bose condensation of such charged excitations was the mechanism of superconductivity. However, the microscopic basis of this picture was never clear until now.

Unfortunately, the most immediate natural consequence of all these suggestions is that, since one has direct Bose condensation instead of pairing, the flux quantum should be h/e. Experimentally, it appears to be h/2e, at least in the regimes where it has been studied so far. Various modifications of the ideas have been proposed[18], but it is difficult to know what conclusions to trust when such a seemingly straightforward one must be abandoned. Also , with the loss of the compellingly simple concept of Bose condensation as a mechanism of superconductivity, the motivation for the suggestion of exotic quantum numbers becomes much less clear.

An essentially new set of ideas was added by Laughlin and collaborators[19-22]. Kalmeyer and Langhlin[19] made an approximate mapping of certain frustrated spin models onto Bose gases with short range repulsive interactions situation and subject to strong external magnetic field. The latter situation is completely analogous to that in the quantized Hall effect, and one can therefore take battle-tested knowledge of the ground state and low-lying excitations in the Hall system over into the spin models. Given the previous discussion of the FQHE, it should not seem surprising that the quasi-particles are then found to obey fractional statistics. Wen, Wilczek, and Zee[16] have given a more abstract solution of the problem, not relying on the details of a specific wave function, indicating what sort of spin ordering is essential to obtain fractional statistics quasi-particles.

Once one has a chiral spin liquid, it is plausible that charged particles doped into the system induce or bind to the fractional statistics quasi-particles, thus themselves acquiring fractional statistics. Laughlin and his collaborators have argued, in several papers, that fractional statistics in and of itself leads to superconductivity. Later, Chen, Wilczek, Witten, and Halperin[23] extended and verified these arguments. The new concept of superconductivity described here is called "anyon superconductivity"

An important feature of most models concerning anyons is the discrete symmetries P and T violations. This is quite basic for the FQHE, taking place in an external magnetic field. It would also have to occur spontaneously in high-temperature superconductors, if they can be described by anyon models. It is, of course, characteristic of chiral spin liquids. That such symmetry breaking could occur, and can have important experimental consequences, was first emphasized by March-Russell and Wilczek[24], and considerably elaborated by these two together with Halperin[25]. Some of the issues have also been discussed after that by Wen and Zee[26]. The considerations of this paper suggest some additional possibilities, and allow us to begin to discuss them quantitatively.

Calculations of the energy of the undoped spin systems using variational wave functions of the Kalmeyer-Laughlin type have not yielded particularly good energies for simple model Hamiltonians, such as Heisenberg antiferromagnets with any combination of couplings to a few near neighbors. Moreover, for the undoped parent compounds of the actual copper-oxide superconductors (e.g., La_2CuO_4) there is compelling evidence that the planes of copper spins are well described by a nearest-neighbor Heisenberg model on a square lattice, with a ground state that has conventional antiferromagnetic order[27]. It is certainly possible that the holes also induce an effective multispin-interaction which favors a chiral spin state for the remaining copper spins. If this is the case, then it is reasonable to approach the superconducting state by starting with a model Hamiltonian where the spins form a chiral spin liquid even in the absence of free charges. Laughlin has shown that there exists in fact a model Hamiltonian (with long-range-four-spin interactions, and with explicitly broken time-reversal and chiral symmetries) for which the quantum–Hall–effect wave function is the exact ground state[28]. There is little reason to doubt that there exists also a class of Hamiltonians which only have finite range interactions, and are invariant under P and T, for which the ground state is a chiral spin liquid.

Unfortunately, up to the present there is no experimental evidence of the violation of discrete symmetries P and T. In the future, new technology may make it possible to observe this violation and after that the realization of anyons will be proved. Although the directly experimental proof is not yet realized, the concept may still be useful in low dimensional physics, especially FQHE and high-temperature superconductivity

Since the configuration space of quantum (identical) particles is multiply connected space, the quantization rules require unusual phase factor [29-31]. From mathematical point of view, the configuration space of identical particles in three dimensions consists of two classes; boson and fermion, but in the two - dimensional world there are infinite families of quantum particles; anyons . More technically , in three dimensional space the first homotopy group or the fundamental group [32] comprises two homotopy classes only, yet there are infinitely many homotopy classes in the fundamental group for two dimensional space. Therefore , the Feynman path integration is very useful for this case[29-31]. In path integrals, the propagator comes from summing over all possible paths, then in a multiply connected configuration space system we have to sum over all a historical paths and possible homotopy classes. That is why the path integral formalism helps us understand the anyon system very well [31].

Although the case of two anyons in a harmonic potential is quite simple, it is a very vital step to first understand the single anyon. Thus, the Feynman path integration is used to study this problem in this work. Even though the configuration space of anyon systems is multiply connected space and in this case has to be treated in the way stated above, there is an equivalent approach, the Chern-Simons technique, to solve this problem too. In conclusion, in this work the Feynman path integration technique will be combined with the Chern-Simons theory to solve the problem of two anyons in a harmonic potential.

CHAPTER 2
FEYNMAN PATH INTEGRAL

Since we will formulate our problem in the form of Feynman path integrals [30,33,34] we will devote this chapter to introduce and clarify this approach. Furthermore we emphasize specifically the Feynman path integrals for multiply connected space [29,30] since the configuration space of anyon systems [31] described in chapter 3 is multiply connected too. However, to understand the last topic we need a topological background, especially the homotopy theory [30,32]. Therefore it is briefly introduced in Appendix B.

Feynman Path Integration in Elementary Quantum Mechanics

In 1926 the new great mechanics, quantum mechanics, was developted. At the time, there were two quite different mathematical formulations, the wave mechanics of Schroedinger and the matrix mechanics of Heisenberg, proved to be mathematically equivalent by Schroedinger. Later, Dirac developed the transformation theory that can transform one approach to another. Nearly two decades later, a young graduate student at Princeton University, R.P. Feynman, noted Dirac's remarks concerning the relation of classical action to quantum mechanics [33] proposing that "$\exp\left[i\int_{t_1}^{t_2}\frac{dt}{\hbar}L_{cl}(t,x,\dot{x})\right]$ corresponds to the kernel $\langle x_2,t_2|x_1,t_1\rangle$ [35]". Consequenly, he proposed an intuitive formulation of quantum mechanics, the third approach, which is mathematically equivalent to two former approaches. This Feynman path integration is based on the following two postulates [33].

" 1) If an ideal measurement is performed to determine whether a particle has a path lying in a region of space-time, then the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the region.

2) The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$); i.e., the time integral of the Lagrangian taken along the path. "

From the second postulate, Feynman could show that the contribution of any path $x(t)$ [33] is

$$\Phi[x(t)] = (\text{const})\exp\left[\frac{i}{\hbar}\int L(\dot{x}(t), x(t), t)dt\right] \tag{2.1}$$

where $L(\dot{x}(t), x(t), t)$ is the classical Lagrangian taken along the path in question. In addition, from the first postulate we will have the probability amplitude, $K(b, a)$, as the sum over paths of contribution $\Phi[x(t)]$ from each path. That is

$$\begin{aligned} K(b, a) &= \sum_{\substack{\text{over all paths} \\ \text{from a to b}}} \Phi[x(t)] \\ &= \sum_{\substack{\text{over all paths} \\ \text{from a to b}}} (\text{const})\exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(\dot{x}(t), x(t), t)dt\right] \end{aligned} \tag{2.2}$$

where $a \equiv (x_a, t_a)$ and $b \equiv (x_a, t_a)$ are points in space-time.

The physical interpretation of $|K(b, a)|^2$ is the probability that a particle will go from point x_a at the time t_a to the point x_b at the time t_b, $P(b, a)$ [34]. That is

$$P(b, a) = |K(b, a)|^2 \tag{2.3}$$

Although the above definition is expressed in the form of a sum over paths seems to be discrete, indeed it is a sum over infinite continuous paths. Then, next, we will replace the summation by the (functional) integral. Hence, the Eq.(2.2) becomes

$$K(b,a) = \lim_{N\to\infty} \iint \ldots \int (\text{const}) \exp\left[\frac{i}{\hbar} S[x(t)]\right] dx_1 dx_2 \ldots dx_{N-1} \tag{2.4}$$

where $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}(t), x(t), t) dt$ is the classical action,

and x_i is the variable defining the path.

To evaluate this (functional) integral we have to divide the time interval from t_a to t_b, into N infinitesimal intervals, ε, as shown in Fig.4. From this process we will find that there is a set of successive times t_1, t_2, ... lying between t_a and t_b, where $t_{i+1} = t_i + \varepsilon$, and there are points x_i corresponding to each t_i, i.e.

$$x_i = x(t_i) \quad , \quad x_a = x(t_a = t_0) \quad , \quad x_b = x(t_b = t_n) \tag{2.5}$$

, and defining the path $x(t)$.

Moreover, the functional action can be expressed in the discrete form as[36]

$$S_N[x_j] = \varepsilon \sum_{j=1}^{N} L\left[\frac{x_j - x_{j-1}}{\varepsilon}, \frac{x_j + x_{j-1}}{2}, j\varepsilon\right] \tag{2.6}$$

Because of the normalization condition, $\int K(b,a) dx_b = 1$, we can define the constant in Eq.(2.4) as the normalization factor A_N^{-N}. Thus, the probability amplitude, $K(b,a)$, in Eq.(2.4) can be rewritten as[34]

$$K(b,a) = \frac{1}{A_N} \lim_{N\to\infty} \iint \ldots \int \exp\left[\frac{i}{\hbar} S_N[x_j]\right] \frac{dx_1}{A_N} \frac{dx_2}{A_N} \ldots \frac{dx_{N-1}}{A_N} \tag{2.7}$$

In the general form of path integration, Eq.(2.7) will be expressed as

$$K(b,a) = \int D[x(t)] \exp\left[\frac{i}{\hbar} S[x(t)]\right] \tag{2.8}$$

known as the Feynman propagator.

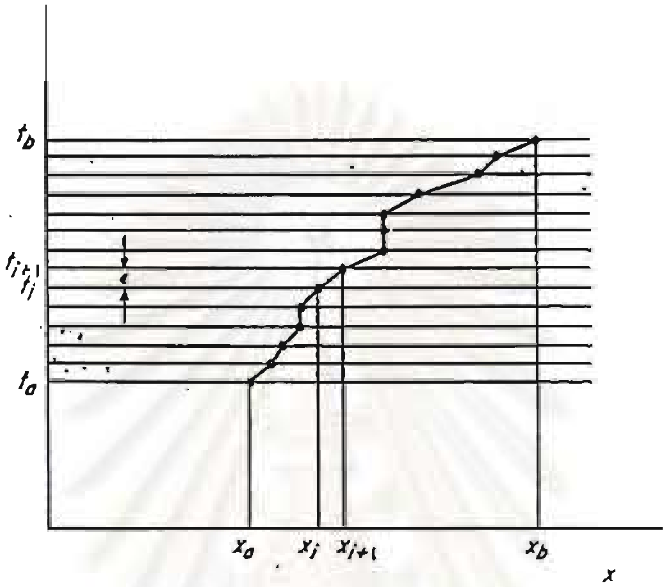

Fig.4 Diagram representing how the path integrals can be generated[34].

Propagator from Schrödinger's equation

In the preceding section, the Feynman propagator is derived from Feynman's postulates leading to the Feynman path integral. Next, the propagator will be directly developed from the Schrödinger's equation. This part will provide the proof of equivalence of the Feynman and Schrödinger formulation while in Feynman's original paper he derived the Schrödinger equation from his propagator. We believe that this point of view will be easier to understand. At first, we will start with the time-dependent Schrodinger's equation

$$\left[i\hbar\frac{\partial}{\partial t} - H\right]\psi(x,t) = 0 \tag{2.9}$$

For convenience we will show the simple case, the one-electron case, which has a Green function, $G(\mathbf{x},\mathbf{x}';t,t')$ satisfying the following differential equation

$$\left[i\hbar\frac{\partial}{\partial t}-H\right]G(\mathbf{x},\mathbf{x}';t,t')=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{2.10}$$

It is well known that this Green function can be expressed in matrix form as

$$G(\mathbf{x},\mathbf{x}';t,t')=\langle\mathbf{x}\left|\exp\left\{-\frac{i}{\hbar}H(t-t')\right\}\right|\mathbf{x}'\rangle \tag{2.11}$$

Similarly to the preceding section, we divide the time interval $t-t'$ into n equal infinitesimal subintervals, i.e. $n\varepsilon=t-t'$. We will have Eq.(2.11) as

$$\begin{aligned}G(\mathbf{x},\mathbf{x}';t,t')&=\langle\mathbf{x}\left|\exp\left\{-\frac{i}{\hbar}Hn\varepsilon\right\}\right|\mathbf{x}'\rangle\\&=\lim_{\varepsilon\to 0}\langle\mathbf{x}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\left(1-\frac{i\varepsilon H}{\hbar}\right)\ldots\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}'\rangle\end{aligned} \tag{2.12}$$

where we have used the identity

$$\exp\left(\frac{-i}{\hbar}Hn\varepsilon\right)=\lim_{\varepsilon\to 0}\left(1-\frac{i\varepsilon H}{\hbar}\right)^{n} \tag{2.13}$$

From the completeness relation of elementary quantum mechanics Eq.(2.12) will be expressed as

$$\begin{aligned}G(\mathbf{x},\mathbf{x}';t,t')=\lim_{\varepsilon\to 0}\iint\ldots\int\langle\mathbf{x}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}_{n-1}\rangle\langle\mathbf{x}_{n-1}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}_{n-2}\rangle\cdot\ldots\\\cdot\langle\mathbf{x}_{1}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}'\rangle d\mathbf{x}_{n-1}d\mathbf{x}_{n-2}\ldots d\mathbf{x}_{1}\end{aligned} \tag{2.14}$$

In general, the Hamiltonian is in the form

$$H=\frac{p^{2}}{2m}+V(\mathbf{x}) \tag{2.15}$$

where p denotes the momentum operator,

and $V(\mathbf{x})$ denotes the potential operator at $\mathbf{x}$.

Hence, we will get

$$\left\langle \mathbf{x}_{i-1}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}_i\right\rangle = \int\left\langle \mathbf{x}_{i+1}\,|\,\mathbf{p}\right\rangle\left\langle \mathbf{p}\left|1-\frac{i\varepsilon}{\hbar}\left(\frac{p^2}{2m}+V(\mathbf{x})\right)\right|\mathbf{x}_i\right\rangle d\mathbf{p}$$
$$= \int\left\langle \mathbf{x}_{i+1}\,|\,\mathbf{p}\right\rangle\left\langle \mathbf{p}\,|\,\mathbf{x}_i\right\rangle\left[1-\frac{i\varepsilon}{\hbar}\left\{\frac{p^2}{2m}+V(\mathbf{x})\right\}\right]d\mathbf{p} \qquad (2.16)$$

From ordinary quantum mechanics, it is well known that the in 3-dimensional case

$$\left\langle \mathbf{x}\,|\,\mathbf{p}\right\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\exp\left[\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}\right] \qquad (2.17)$$

Hence, from Eq.(2.17) we have

$$\left\langle \mathbf{x}_{i+1}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}_i\right\rangle = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\ \exp\left\{\frac{i}{\hbar}(\mathbf{x}_{i+1}-\mathbf{x}_i)\cdot\mathbf{p}\right\}$$
$$\cdot\left\{1-\frac{i\varepsilon}{\hbar}\left(\frac{p^2}{2m}+V(\mathbf{x})\right)\right\}$$

(2.18)

Since ε is an infinitesimal quantity, the error that comes from replacing term $\left\{1-\frac{i\varepsilon}{\hbar}\left(\frac{p^2}{2m}+V(\mathbf{x})\right)\right\}$ by the corresponding exponential can be neglected. Therefore we can rewrite Eq.(2.18) as

$$\left\langle \mathbf{x}_{i+1}\left|\left(1-\frac{i\varepsilon H}{\hbar}\right)\right|\mathbf{x}_i\right\rangle = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\exp\left\{\frac{-i\varepsilon}{\hbar}\left[\frac{p^2}{2m}-\frac{(\mathbf{x}_{i+1}-\mathbf{x}_i)}{\varepsilon}\cdot\mathbf{p}\right]\right\}$$
$$\cdot\exp\left\{-\frac{i\varepsilon}{\hbar}V(\mathbf{x})\right\}$$
$$= \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\exp\left\{\frac{-i\varepsilon}{2m\hbar}\left[\mathbf{p}-\frac{m}{\varepsilon}(\mathbf{x}_{i+1}-\mathbf{x}_i)\right]^2\right\}$$
$$\cdot\exp\left\{\frac{i\varepsilon}{2m\hbar}\left(\frac{m}{\varepsilon}(\mathbf{x}_{i+1}-\mathbf{x}_i)\right)^2\right\}\exp\left\{\frac{-i\varepsilon}{\hbar}V(\mathbf{x})\right\}$$

Using Gaussian integrals, we get

$$\left\langle \mathbf{x}_{i+1} \left| \left(1-\frac{i\varepsilon H}{\hbar}\right) \right| \mathbf{x}_i \right\rangle = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{\frac{3}{2}} \exp\left[\frac{i\varepsilon}{\hbar}\left\{\frac{m}{2}\left(\frac{\mathbf{x}_{i+1}-\mathbf{x}_i}{\varepsilon}\right)^2 - V(\mathbf{x})\right\}\right] \tag{2.19}$$

Let us rewrite Eq.(2.14) by substituting Eq.(2.19) into it, thereby

$$G(\mathbf{x},\mathbf{x}';t,t') = \lim_{\varepsilon\to 0}\frac{1}{A}\iint\ldots\int \exp\left[\frac{i\varepsilon}{\hbar}\sum_{i=1}^{n-1}\left\{\frac{m}{2}\left(\frac{\mathbf{x}_{i+1}-\mathbf{x}_i}{\varepsilon}\right)^2 - V(\mathbf{x})\right\}\right] \cdot \frac{d\mathbf{x}_1}{A}\frac{d\mathbf{x}_2}{A}\ldots\frac{d\mathbf{x}_{n-1}}{A}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{A}\iint\ldots\int \exp\left[\frac{i}{\hbar}\int_{t'}^{t}\left\{\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right\}dt\right] \cdot \frac{d\mathbf{x}_1}{A}\frac{d\mathbf{x}_2}{A}\ldots\frac{d\mathbf{x}_{n-1}}{A} \tag{2.20}$$

where $A = \left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{\frac{3}{2}}$, $\mathbf{x}_n = \mathbf{x}$, and $\mathbf{x}_0 = \mathbf{x}'$.

Next, let us take into account the exponent in Eq.(2.20) that can be rewritten as

$$\frac{i}{\hbar}\int_{t'}^{t}\left\{\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right\}dt = \frac{i}{\hbar}S[\mathbf{x}(t)] \tag{2.21}$$

thus ,Eq.(2.20) becomes

$$G(\mathbf{x},\mathbf{x}';t,t') = \lim_{\varepsilon\to 0}\frac{1}{A}\iint\ldots\int \exp\left[\frac{i}{\hbar}S[\mathbf{x}(t)]\right]\frac{d\mathbf{x}_1}{A}\frac{d\mathbf{x}_2}{A}\ldots\frac{d\mathbf{x}_{n-1}}{A} \tag{2.22}$$

Notice that Eq.(2.22) is the same as Eq.(2.7) even though they start from very different points of view : one begins from an intuitive approach and the other one begins from original quantum mechanics. This is the evidence to confirm that the Feynman path integral approach is exactly equivalent to the ordinary methods, such as Schrödinger and Heisenberg's. To boot, it is obvious that the Feynman propagator is precisely equivalent to the Green's function, $G(x, x'; t, t')$.

To understand better, we have to consider a simple example, the propagator of a free particle. Let us start with the Lagrangian of a free particle with mass, m,

$$L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 \tag{2.23}$$

Hence, the propagator or Green's function can be expressed as

$$G(x, x'; t, t') = \lim_{\varepsilon \to 0} \frac{1}{A} \iint \ldots \int \exp\left[\frac{i\varepsilon}{\hbar}\sum_{i=0}^{n-1}\frac{m}{2}\left(\frac{x_{i+1} - x_i}{\varepsilon}\right)^2\right]\frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_{n-1}}{A}$$

where $A = \left(\frac{2\pi\hbar i\varepsilon}{m}\right)^{\frac{3}{2}}$, $x_n = x$, and $x_0 = x'$.

To solve this problem, we have to utilize the Gaussian integrals [30]

$$\int_{-\infty}^{\infty} \exp\left\{a(x_0 - x_1)^2 + b(x_2 - x_1)^2\right\}dx_1 = \left(\frac{-\pi}{a+b}\right)^{\frac{3}{2}} \exp\left\{\frac{ab}{a+b}(x_2 - x_0)^2\right\}$$

Let us consider

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^3 \int_{-\infty}^{\infty} \exp\left\{\frac{im}{2\hbar\varepsilon}\left[(x_1 - x_0)^2 + b(x_2 - x_1)^2\right]\right\}dx_1$$

$$= \left[\frac{m}{2\pi\hbar(2\varepsilon)}\right]^{\frac{3}{2}} \exp\left\{\frac{im}{2\hbar(2\varepsilon)}(x_2 - x_0)^2\right\}$$

Next, multiplying the next term, $\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{3}{2}} \exp\left\{\frac{im}{2\hbar\varepsilon}(x_3 - x_2)^2\right\}$ and integrating over x_2, we have

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{9/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{\frac{im}{2\hbar\varepsilon}\left[(x_1 - x_0)^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2\right]\right\}dx_1 dx_2$$

$$= \left(\frac{m}{2\pi i\hbar 3\varepsilon}\right)^{3/2} \exp\left\{\frac{im}{2\hbar 3\varepsilon}(x_3 - x_0)^2\right\}$$

After (n-1) steps, we obtain

$$G_0(\mathbf{x},\mathbf{x}';t,t') = \left(\frac{m}{2\pi i\hbar n\varepsilon}\right)^{3/2} \exp\left\{\frac{im}{2\hbar n\varepsilon}(\mathbf{x}_n - \mathbf{x}_0)^2\right\}$$
$$= \left(\frac{m}{2\pi i\hbar T}\right)^{3/2} \exp\left\{\frac{im}{2\hbar T}(\mathbf{x} - \mathbf{x}')^2\right\} \tag{2.24}$$

where $T = n\varepsilon$.

Nevertheless, even though the direct integration method can be applied to the free particle problem very well, it is useless for more complicated cases. Therefore, a different way to solve this difficult problem is required. Let us start now with the general form of the Feynman path integral

$$K(b,a) = N\int D[x(t)]\exp\left[\frac{i}{\hbar}S[\mathbf{x}(t)]\right] \tag{2.25}$$

When considered in classical physics, the action S is extremized and then it furnishes us the classical path completely fixed. Therefore, any path $\mathbf{x}(t)$ can be expressed as the sum of the classical path, $\mathbf{x}_c(t)$, and a new variable $\mathbf{y}(t)$. That is

$$\mathbf{x}(t) = \mathbf{x}_C(t) + \mathbf{y}(t) \tag{2.26}$$

and it is clear that the path differential $D[\mathbf{x}(t)]$ can be replaced by $D[\mathbf{y}(t)]$. This means that besides defining a point on the path by its distance $\mathbf{r}(t)$ from an arbitrary coordinate axis, we now give the meaning of it by its deviation $\mathbf{y}(t)$ from the classical path, as shown in Fig.2. The crucial conditions the deviations, $\mathbf{y}(t)$, have to satisfy are

$$\mathbf{y}(0) = \mathbf{y}(T) = 0 \tag{2.27}$$

In here, we start with the time $t = 0$ and end at the time $t = T$. Generally, the Lagrangian will be the quadratic form

$$L = a(t)\dot{x}^2(t) + b(t)\dot{x}(t)x(t) + c(t)x^2(t) + d(t)\dot{x}(t) + e(t)x(t) + f(t) \tag{2.28}$$

Hence, the action S can be expressed as

$$\begin{aligned} S[x(t)] &= S[x_c(t) + y(t)] \\ &= \int_0^T \left[a(t)\left\{\dot{x}_c^2(t) + 2\dot{x}_c(t)\dot{y}(t) + \dot{y}^2(t)\right\} + \ldots + f(t)\right]dt \end{aligned} \tag{2.29}$$

It is obvious that the integral of all terms involving exclusively $x_c(t)$ is exactly the classical action and the integral of all terms that are linear in $y(t)$ precisely vanishes. So, all the remaining terms in the integral are second-order terms in $y(t)$ only. That is

$$S[x(t)] = S_{cl}[x_c(t)] + \int_0^T \left[a(t)\dot{y}^2(t) + b(t)\dot{y}(t)y(t) + c(t)y^2(t)\right]dt \tag{2.30}$$

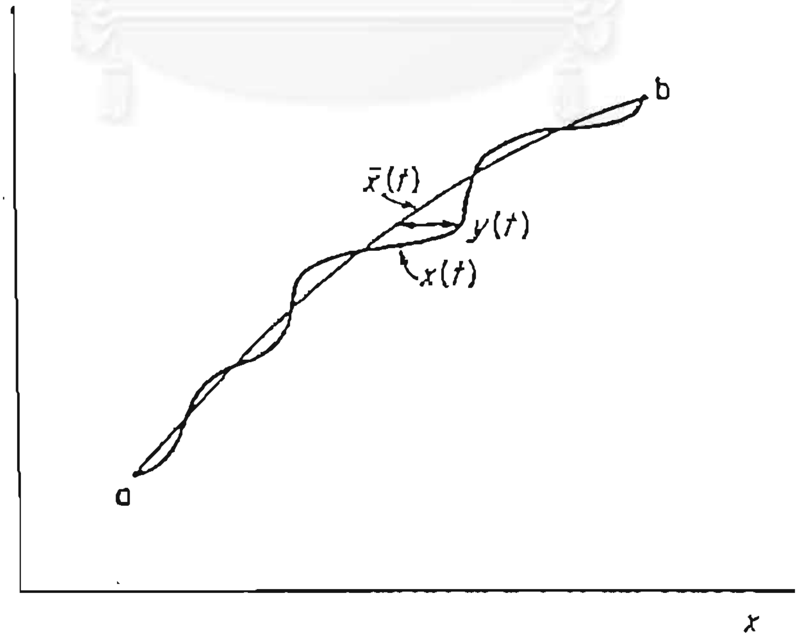

Fig.5 Diagram showing a path deviating from the classical path [34].

From Eq.(2.30), the propagator or the Green's function can be rewritten as

$$K(b,a) = N\int D[y(t)]\exp\left[\frac{i}{\hbar}\left(S_{Cl}[x_C(t)] + \int_0^T dt\,\{a(t)\dot{y}^2(t) + b(t)\dot{y}(t)y(t) + c(t)y^2(t)\}\right)\right]$$

$$= \exp\left\{\frac{i}{\hbar}S_{Cl}[x_C(t)]\right\}N\int D[y(t)]\exp\left[\frac{i}{\hbar}\int_0^T dt\,\{a(t)\dot{y}^2(t) + b(t)\dot{y}(t)y(t) + c(t)y^2(t)\}\right] \tag{2.31}$$

For the quadratic Lagrangian, customarily, the propagator can be expressed as

$$K(b,a) = F(T)\exp\left\{\frac{i}{\hbar}S_{Cl}[x_C(t)]\right\} \tag{2.32}$$

where

$$F(T) = N\int D[y(t)]\exp\left[\frac{i}{\hbar}\int_0^T dt\,\{a(t)\dot{y}^2(t) + b(t)\dot{y}(t)y(t) + c(t)y^2(t)\}\right]$$

is a prefactor.

Next, let us show the power of this approach in the problem of a one-dimensional harmonic oscillator whose Lagrangian is

$$L[x(t),\dot{x}(t)] = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \tag{2.33}$$

We will obtain the equation of motion by applying the Euler-Langrange equation to the Lagrangian

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{2.33}$$

thus we have

$$\ddot{x}_c + \omega^2 x_c = 0 \tag{2.34}$$

and the solution of Eq (2.34), with boundary condition $x(0) = x'$ and $x(T) = x$,is

$$x_c(t) = \frac{x - x'\cos\omega T}{\sin\omega T}\sin\omega t + x'\cos\omega t \tag{2.35}$$

Therefore, the corresponding action can be expressed as

$$S_{Cl}[x_c(t)] = \int_0^T \frac{m}{2}\left[\dot{x}_c^2(t) - \omega^2 x_c^2(t)\right]dt$$

$$= \frac{m}{2}\left[\dot{x}_c(T)x_c(T) - \dot{x}_c(0)x_c(0) - \int_0^T x_c\left[\ddot{x}_c - \omega^2 x_c\right]dt\right]$$

Using Eq.(2.34)

$$S_{Cl}[x_c(t)] = \frac{m}{2}\left[\dot{x}_c(T)x_c(T) - \dot{x}_c(0)x_c(0)\right] \tag{2.36}$$

Substituting Eq.(2.35) into Eq.(2.36), we have

$$S_{Cl}[x_c(t)] = \frac{m\omega}{2\sin\omega T}\left[\cos\omega T\left(x^2 + x'^2\right) - 2xx'\right] \tag{2.37}$$

that is the classical action of the harmonic oscillator.

From Eq.(2.32) and Eq.(2.37), it is only the prefactor, F(T), that is required to complete the propagator. Start now with

$$F(T) = N\int D[y(t)]\exp\left[\frac{i}{\hbar}\int_0^T dt\,\frac{m}{2}\left\{\dot{y}^2 - \omega^2 y^2\right\}\right] \tag{2.38}$$

at the boundary condition $y(0) = y(T) = 0$.

By expressing $y(t)$ in the form of a Fourier series,

$$y(t) = \sum_n a_n \sin\frac{n\pi t}{T} \tag{2.39}$$

it is obviously seen that we can change the integration variables from y's to the new variable a_n's, and then with the use of the identity

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left(1 - \left(\frac{\omega T}{n\pi}\right)^2\right)^{-1/2} = \sqrt{\frac{\omega T}{\sin\omega T}} \tag{2.40}$$

we will obtain the result as

$$F(T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} \tag{2.41}$$

Finally, the time-independent propagator or Green's function of the harmonic oscillator is

$$K(b,a) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} \exp\left\{\frac{im\omega}{2\hbar \sin \omega T}\left[\cos \omega T\left(x^2 + x'^2\right) - 2xx'\right]\right\} \tag{2.42}$$

Path Integrals for Multiply Connected Space.

After the ordinary Feynman path integration in elementary quantum mechanics has been presented in the preceding section of this chapter, the more advanced subject of the path integrals for multiply connected space is given in this portion. Unfortunately, since there is no enough time, it is solely the introductory level of this topic that is discussed here. The quantisation rules for a system whose classical configuration space is multiply connected are shown here as exclusively fundamental concepts and for a simple example while the more profound concepts, involving topology, especially homotopy theory, are neglected here. Nevertheless, details of this topic can be found in Ref. [37] and Ref.[38].

In this part, many details are quoted from Schulman's book [30] which is the classic and famous book for path integrals both in its foundations and applications. This subject is one of many applications contained in Schulman's book and the example shown in there is the problem of a rotator. However, this important application was proposed in 1971 by Laidlaw and de Witt [39]. In that paper a system of indistinguishable particles in a three-dimensional world, multiply connected, is quantized and the result shows that solely two kinds of particles in 3-D, bosons and fermions, are possible. This means that the topology of configuration space of physical systems determines the possibility of particles in this system. In 1984 two year after Wilczek introduced the anyon, Wu [31] showed that the Feynman path-integration formulation allows unusual statistics. This result comes from the complexity of the topology of the configuration space for identical particles in two dimensions. While in three dimensions the fundamental group of the configuration space has only two classes, in two dimensions there are infinite classes in the fundamental group. Nevertheless, the relation between anyonic systems and systems whose configuration space is multiply connected is deferred to the next chapter.

Let us start now with the basic definition of path integrals stating that the propagator can be expressed as

$$K(b,a) = \sum_{\substack{\text{all possible paths} \\ \text{from a to b}}} (\text{const}) \exp iS[q(t)] \tag{2.43}$$

where q(t) is the coordinate function defining any path from a to b, by which $q(t_a) = a$ and $q(t_b) = b$. For convenience $\hbar$ is set to be 1.

In multiply connected space, M, the fundamental group or the first homotopy class, $\pi_1(M)$, has an infinite number of elements. This means that there are an infinite number of equivalent classes whose elements can be continuously transformed from one to the other if and only if they belong to the same class. Therefore, we must divide all possible classes into all equivalent classes, the elements of $\pi_1(M)$.

The crucial point is that in multiply connected space (or manifold) the paths belonging to different classes have different weights in the sum over histories a problem which does not exist in standard quantum mechanics.

Hence, it is essential to reformulate the quantum theory taking in account this fact. Actually, this topic was discussed for the first time by Schulman [29] in 1968, rigorously formulated by Laidlaw and de Witt[39] in 1971, and eventually reviewed by Gamboa [40] in 1996. Let $\alpha \in \pi_1(M)$ be the equivalence class and $\chi(\alpha)$ the weight factor corresponding to class α, thus

$$K(b,a) = \sum_{\alpha \in \pi_1(M)} \chi(\alpha) \sum_{q(t) \in \alpha} \exp iS[q(t)] \tag{2.44}$$

For convenience, we have to define the partial propagator, corresponding to the class α, as

$$K_\alpha(b,a) = \sum_{q(t) \in \alpha} \exp iS[q(t)] \tag{2.45}$$

Therefore, the total propagator

$$K(b,a) = \sum_{\alpha \in \pi_1(M)} \chi(\alpha) K_\alpha(b,a) \tag{2.46}$$

To understand more clearly let us consider the simple example of a particle constrained to a circle. Starting with its Lagrangian

$$L = \frac{1}{2} I \dot{\varphi}^2 \tag{2.47}$$

where φ is angle coordinate ; $0 \le \varphi \le 2\pi$,

and I is the moment of inertia.

A continuous path of this system is a continuous function φ with the identification of 0 and 2π. The set of such paths can be classified into classes labeled by their "winding number", the number of times the path goes past some specific point. Generally the sign of moving clockwise is positive and the other is negative or vice versa. Two different paths having different winding numbers cannot be continuously transformed into each other. Now let us define α_n as the class whose members are paths having the winding numer as n, and thus the propagator can be rewritten as

$$K(b,a) = \sum_{n=-\infty}^{\infty} \chi_n K_n(b,a) \tag{2.48}$$

It is clear that if $\varphi(t_b)$ goes through a complete circle, the K_n's become the K_{n+1}'s while $K(b,a)$ itself must show no physical change, which is to say it can exclusively be multiplied by a phase factor, $\exp(i\theta)$. From this fact and the linear independence of the K_n's we will have

$$\chi_{n+1} = \exp(i\theta)\chi_n \tag{2.49}$$

The magnitude of χ_0 is fixed by unitarily to be 1, or taking its (arbitrary) phase to be zero we get

$$\chi_n = \exp(in\theta) \tag{2.50}$$

Next, to perform the path integral for (2.47) let us consider the mapping from the real line R to the circle

$$f : R \to S^1 \quad ; \quad f(x) = x - \left[\frac{x}{2\pi}\right]2\pi \tag{2.51}$$

where [x] means the integral part of x.

It is obvious that $0 \le p(x) \le 2\pi$. The mapping p is illustrated in Fig.6. It is clear that p is locally invertible, but, however, its invertible property is ill defined globally. Then the problem can be seen easily in the case that two paths having $p(x_1') = p(x_2') = \varphi'$ or $p(x_1'') = p(x_2'') = \varphi''$ but $x_1' \neq x_2'$ or $x_1'' \neq x_2''$. To solve the problem, the preimage $p^{-1}(\varphi')$ must be fixed at first, yet it can not completely solve the problem. It is clear that paths $\varphi(t)$ having the same end points and same choice of $p^{-1}(\varphi')$ can end at different $p^{-1}(\varphi'')$ To solve the last problem, the winding number part of $p^{-1}(\varphi'')$ should be excluded and then the preimage can be rewritten as $p^{-1}(\varphi'') + 2\pi n$ for paths with winding number n. Therefore it follows that to calculate K_n can one do the path integral on R for the path that begins at the definite $p^{-1}(\varphi')$ which has been selected and ends at $p^{-1}(\varphi'') + 2\pi n$ where $p^{-1}(\varphi'')$ is fixed by the choice of $p^{-1}(\varphi')$. The map p can be used to carry the classical Lagrangian from S^1 to R since it is smooth.

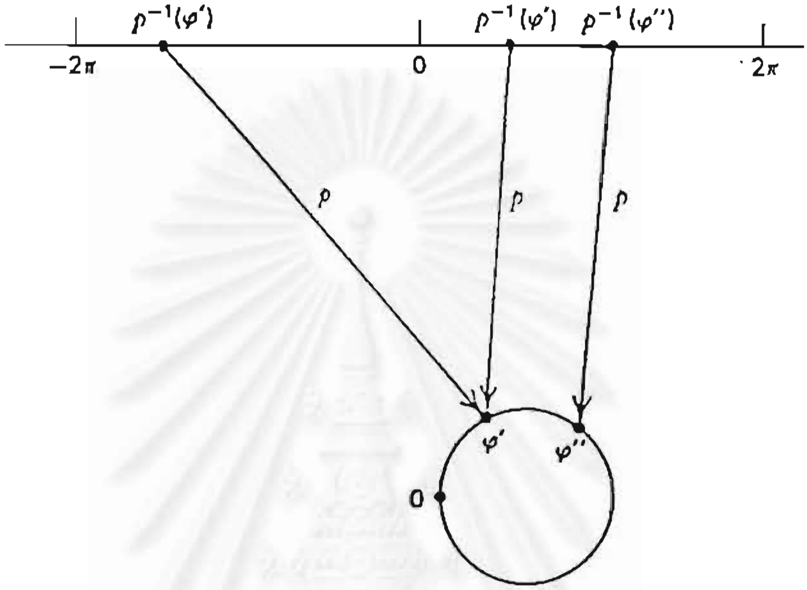

Fig.6 The map p for points on the line to some φ and to some φ″ [30].

Let us now go back to the Lagrangian (2.47), and consider it on R. It is clear that it is that of a free particle, and we immediately have

$$K_n(b,a) = \sqrt{\frac{I}{2\pi i T}} \exp\left[\frac{iI}{2T}(\varphi - 2n\pi)^2\right] \tag{2.52}$$

with

$$\varphi = \varphi(t_b) - \varphi(t_a),\ T = t_a - t_b \tag{2.53}$$

and the point for reckoning number has been taken to be zero. From Eq. (2.52), (2.50), and (2.48), we will get

$$K(b,a) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i T}} \exp\left[in\theta + \frac{iI}{2T}(\varphi - 2n\pi)^2\right] \tag{2.54}$$

Using the identity involving the Jacobi theta function [30]

$$\theta_3(z,t) = \sum_{n=-\infty}^{\infty} \exp\left[i\pi t n^2 + i2nz\right]$$
$$\theta_3(z+\pi,t) = \theta_3(z,t) \tag{2.55}$$
$$\theta_3(z+\pi t,t) = \exp\left[-i\pi t - i2z\right]\theta_3(z,t)$$

we have

$$K(b,a) = \sqrt{\frac{I}{2\pi i T}} \exp\left[\frac{iI}{2T}\varphi^2\right]\theta_3\left(\frac{\pi\varphi I}{T} - \frac{\theta}{2}, \frac{2\pi I}{T}\right) \tag{2.56}$$

Using the additional identity [30]

$$\theta_3(z,t) = \frac{1}{\sqrt{it}} \exp\left(\frac{z^2}{i\pi t}\right)\theta_3\left(\frac{z}{t}, -\frac{1}{t}\right) \tag{2.57}$$

we have

$$K(b,a) = \frac{1}{2\pi} \exp\left[\frac{i\theta}{2\pi}\varphi - \frac{i\theta^2 T}{8I\pi^2}\right]\theta_3\left(\frac{\varphi}{2} - \frac{T\theta}{4\pi I}, \frac{-T}{2\pi I}\right) \tag{2.58}$$

From this example, it is seen clearly that a system with non-trivial topology should be carefully treated in an unusual way. Firstly, all paths must be divided into equivalent classes in the fundamental group. Secondly, in each class the partial propagator, K_n should be multiplied by the weight factor $\exp(i\theta)$. This method is used again in the next chapter in connection with the problem of the anyon system.

CHAPTER 3

ANYONS: CONCEPTS AND APPLICATIONS

In chapter 1, both the introductory concepts and applications have been discussed, but, however, it is necessary to expand both of them here. Especially, the mathematical details of the simple model of anyon will be shown in precise detail in this episode. The second part of this chapter is devoted to the Feynman path integration for the configuration space of anyons that is multiply connected. Furthermore, in the third portion, the relation between the Chern–Simons theory and anyons will be analyzed. A similar form of the exotic phase factor occurs in both the second and third sections. The last section is reserved for the important application of anyon theory, the concept of anyons for the Fractional Quantum Hall Effect (FQHE). Unfortunately, because it is so advanced, the anyon superconductivity has to be neglected here. Nevertheless, many details and topics concerning anyon superconductivity can be found in Lerda's book [41] and Wilczek's book [42] even though they are disregarded in this work.

Simple Model of the Anyon

The simplest model of the anyon is a charged point-particle interacting with an infinitely long magnetic solenoid of flux-tube-charge composites, called cyon[6, 43]. It is obvious that the cyon will obey the law of a two-dimensional world and its dynamics is essentially planar if the motion along the solenoid is disregarded.

Now let us consider a non-relativistic particle with mass m and electric charge e that moves in the magnetic field B generated by an infinitely long and thin solenoid passing through the origin and directed along the z-axis. Since the (free) motion along the solenoid is neglected, the pertinent dynamics takes place in the (x, y) - plane and is governed by the non-relativistic Lagrangian

$$L = \frac{1}{2} m\mathbf{v}^2 + \frac{e}{c}\mathbf{v}\cdot\mathbf{A}(\mathbf{r}) \tag{3.1}$$

where $\mathbf{r} = (x, y) \in R^2$ denotes the particle position, $\mathbf{v} = \dot{\mathbf{r}}$ its velocity and $\mathbf{A}$ the vector potential for the solenoid configuration. In a convenient symmetric gauge, $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2\pi}\left(\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}\right) = \frac{\Phi}{2\pi}\frac{\mathbf{z}\times\mathbf{r}}{|\mathbf{r}|^2} \tag{3.2}$$

where **i** and **j** are the unit vectors along the x- and y- axis respectively, and Φ is the flux of the solenoid. From elementary electromagnetism, the magnetic field associated with $\mathbf{A}(\mathbf{r})$ in Eq.(3.2) can be expressed as

$$B(\mathbf{r}) = \vec{\nabla}\wedge\mathbf{A} = \epsilon_{ij}\,\partial_i A_j \tag{3.3}$$

where $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, $\epsilon_{11} = \epsilon_{22} = 0$.

<u>Case I</u> $\mathbf{r} \neq 0$.

$$\begin{aligned} B(\mathbf{r}) &= \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x \\ &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \\ &= 0 \end{aligned} \tag{3.4}$$

<u>Case II</u>

Using a regularization scheme with

$$A_i = -\frac{\Phi}{2\pi}\lim_{\epsilon\to 0}\frac{\epsilon_{ij}\, r_j}{r^2+\epsilon^2} = -\frac{\Phi}{2\pi}\lim_{\epsilon\to 0}\frac{\epsilon_{ij}\, r_j}{(x^2+y^2)}$$

$$B(\mathbf{r}) = \lim_{\epsilon \to 0} \epsilon_{ij}\, \partial_i A_j$$

$$= \frac{-\Phi}{2\pi} \lim_{\epsilon \to 0} \left\{ \frac{\left(x^2 - y^2 - \epsilon^2\right) - \left(x^2 - y^2 + \epsilon^2\right)}{\left(x^2 + y^2 + \epsilon^2\right)^2} \right\}$$

and considering the case of $r \ll \epsilon$,

$$B = \frac{\Phi}{2\pi} \lim_{\epsilon \to 0} \frac{2\epsilon^2}{\epsilon^4} = \frac{\Phi}{\pi} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2}$$

$$\int d^2 r B = \left(\int d^2 r B\right)_{r \neq 0} + \left(\int d^2 r B\right)_{r \ll \epsilon}$$

$$= \frac{\Phi}{\pi} \lim_{\epsilon \to 0} \int d^2 r \frac{1}{\epsilon^2}$$

$$= \frac{\Phi}{\pi} \left(\frac{\pi \epsilon^2}{\epsilon^2} \right) = \Phi \tag{3.5}$$

From Eq. (3.4) and Eq. (3.5), we can deduce that

$$B(\mathbf{r}) = \Phi \delta^{(2)}(\mathbf{r}) \tag{3.6}$$

where $\delta^{(2)}(\mathbf{r})$ is the Dirac-delta function in two-dimensional space.

It is Eq. (3.6) that shows that the cyon, a charged point-particle interacting with an infinitely long magnetic solenoid of flux-tube-particle composites, has the magnetic flux of the solenoid, Φ, locating at the position of the charge particle, $\mathbf{r}$.

Before discussing the spin and statistics of the cyon, a new form of the vector potential in Eq. (3.2) will be derived. This form will be very useful in next section where we discuss the path integration of two anyons by using the Chern–Simons term. For convenience we shall represent the vector potential, $\mathbf{A}$, in each component as

$$A_i = -\frac{\Phi}{2\pi} \frac{\epsilon_{ij}\, r_j}{r^2} \tag{3.7}$$

Let us consider that $\theta = \arctan\left(\frac{r_2}{r_1}\right)$

$$\partial_k\left\{\arctan\left(\frac{r_2}{r_1}\right)\right\} = \frac{r_1^2}{r_1^2 + r_2^2}\partial_k\left(\frac{r_2}{r_1}\right)$$

$$= \frac{r_1^2}{r^2}\left\{\frac{\delta_{k2}}{r_1} - \delta_{k1}\frac{r_2}{r_1^2}\right\}$$

$$\partial_k\left\{\arctan\left(\frac{r_2}{r_1}\right)\right\} = \frac{r_1}{r^2} \qquad \text{if } \ k = 2$$

$$= -\frac{r_2}{r^2} \qquad \text{if } \ k = 1$$

$$\partial_i\theta = -\epsilon_{ij}\frac{r_j}{r^2} \tag{3.9}$$

where $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j}$. Therefore, we can rewrite Eq.(3.7) as

$$A_i = \frac{\Phi}{2\pi}\partial_i\theta \tag{3.10}$$

where $\theta = \arctan\left(\frac{r_2}{r_1}\right)$ is the angle made by the vector r with an arbitrary axis.

Now, we go back to the crucial point, the spin and statistics of the cyon and then we will show that the cyon is generally an anyon. In this part the basic concepts of elementary quantum mechanics, the quantization rules of angular momentum and spin, are used to verify these exotic quantum properties of the cyon or generically anyon. Next, let us consider the canonical momentum **p** which can be readily obtained from the Lagrangian (3.1). The momentum **p** differs from the kinetic momentum m**v** by a term involving the vector potential **A** ,namely

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

$$= m\mathbf{v} + \frac{e}{c}\mathbf{A} \tag{3.11}$$

The Hamiltonian of this system is

$$H = \mathbf{p}\cdot\mathbf{v} - L$$
$$= \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 = \frac{1}{2}m\mathbf{v}^2 \tag{3.12}$$

Notice that the magnetic field and the vector potential are invisible if the Hamiltonian is written in terms of the kinetic momentum. Actually, the classical equations of motion are the same as those of a free particle, yet non-trivial features are present in the quantum theory.

If the Lagrangian L does not change when the dynamical variables are rotated in the plane, the Lagrangian is rotationally invariant. Thus we have to pay more attention to the constant of motion associated with this rotational symmetry, the canonical orbital angular momentum, J_c,

$$J_c = \mathbf{r}\wedge\mathbf{p}$$
$$= \mathbf{r}\wedge m\mathbf{v} + \frac{e}{c}\mathbf{r}\wedge\mathbf{A} \tag{3.13}$$
$$= J + \frac{e\Phi}{2\pi c}$$

where $J = m\mathbf{r}\wedge\mathbf{v}$ is the gauge invariant kinetic angular momentum. It is well known that J_c has a conventional spectrum. In other words, the eigenvalues of J_c are always integers in units of $\hbar$. This is true despite the fact that the algebra of the two-dimensional rotation is abelian and in principle an arbitrary constant could be added to the angular momentum operator, thus obtaining arbitrary eigenvalues.

Since, both in the absence and the presence of Φ, the canonical angular momentum is always expressed by the same quantum operator

$$J_c = -i\hbar\frac{\partial}{\partial\varphi} \tag{3.14}$$

, its eigenvalues are always integers in units of $\hbar$ as below

$$J_c = \hbar m \qquad , m \in Z \tag{3.15}$$

where Z is the set of integers.

The interesting quantity is the kinetic angular momentum, J, not the canonical angular momentum. However, they are indentical when the magnetic flux, Φ, is equal to zero. Now, let us consider J in the case of $\Phi \neq 0$. From Eq.(3.13) and (3.14) we can show that

$$\begin{aligned} J &= J_c - \frac{e\Phi}{2\pi c} \\ &= -i\hbar\frac{\partial}{\partial\varphi} - \frac{e\Phi}{2\pi c} \end{aligned} \tag{3.16}$$

If the kinetic angular momentum J is applied to single-valued wavefunctions with angular dependence exp(imφ), we will get

$$J = \hbar\left(m - \frac{e\Phi}{hc}\right) \quad , m \in Z \tag{3.17}$$

Therefore, the spectrum of J is composed of integers shifted by -eΦ/hc.

Commonly in quantum mechanics, only the conserved quantity, the canonical one, is considered, but, however, for the present case, A. S. Goldhaber and R. Mackenzie [44] lucidly pointed out that the integer canonical angular momentum is divided into two parts: One localized near the cyon and in general fractional, and one located at the spatial infinity and also fractional. Moreover they argued that this diffused angular momentum is irrelevant in describing local phenomena, and identified the piece localized on the cyon with the kinetic angular momentum. Unfortunately, we have to skip the details of their argument, but these details can be found in Ref. [44] and Ref. [45].

The spin of the cyon is defined as the (kinetic) angular momentum of the cyon in the case of m = 0, that is the spin

$$S = \frac{J(m=0)}{\hbar} = -\frac{e\Phi}{hc} \tag{3.17}$$

This equation means that in general the spin of the cyon can take on any values. This is one relevant reason that the cyon is generally an anyon.

Next, we will discuss the statistics of the cyon closely relating them to its spin. It is well known that if some connection between spin and statistics exists, we should expect that the cyon be generally an anyon. To establish its statistical properties, let us consider two identical cyons with a wavefunction $\psi(1,2)$ and assume that the magnetic flux and electric charge are tightly bound on each particle. To evaluate the statistics we have to move one cyon around the other by a full loop and neglect both charge-charge and vortex-vortex interactions. Using the Aharonov-Bohm effect [45], the charged particle moving around the other attached to the magnetic flux Φ feels the effect of the vortex on its quantum motion. Let us consider the case that particle 1 is moved around the vortex 2 on a closed loop Γ, therefore the wave function acquires a phase

$$\exp\left(-i\frac{e}{\hbar c}\oint_{T} d\mathbf{r}\cdot\mathbf{A}\right) \tag{3.19}$$

Using Stokes' theorem, it can be rewritten as

$$\begin{aligned}\exp\left(-i\frac{e}{\hbar c}\int d^2r\nabla\wedge\mathbf{A}\right) &= \exp\left(-i\frac{e}{\hbar c}\int d^2rB\right)\\ &= \exp\left(-i\frac{e\Phi}{\hbar c}\right)\end{aligned} \tag{3.20}$$

In two-cyon system, the effect of particle 1 on particle 2 is equal to the effect of particle 2 on particle 1. Thereby, the phase picked up should be twice that in Eq. (3.20). This means that the phase factor in Eq. (3.20) can be rewritten as

$$\exp(2\pi i\nu) = \exp\left(-2\pi i\frac{2e\Phi}{hc}\right) \tag{3.21}$$

Thus the statistics of the cyon is

$$\nu = -\frac{2e\Phi}{hc} \tag{3.22}$$

Furthermore the spin s and the statistics ν appear to be related in the conventional way

$$\nu = 2s \tag{3.23}$$

Therefore generally the cyon is an anyon and the standard spin-statistics connection is satisfied.

In an alternative approach, we can define the quantum statistics of the cyon by the phase of the amplitude associated with slow motion of distant particle around one another. Hence the quantum statistics in this case can be obtained by rewriting Eq.(3.21) as

$$\exp(i\pi\nu) = \exp\left(-i2\pi\frac{e\Phi}{hc}\right) \tag{3.24}$$

We can see that if $\nu = \pm 1$, the quantum statistics is ± 1 that is the fermionic statistics. This means that the bosonic particle with magnetic flux $\Phi = \frac{1}{2}\Phi_0$ ($\Phi_0 = \frac{hc}{e}$ is the fundamental flux unit) is a fermion. In this thesis, after this portion, the above definition of the quantum of the quantum statistics will be used unless denoted otherwise. Furthermore, we have to define the statistical parameter α as

$$\alpha = \pi\nu = 2\pi\frac{e\Phi}{hc} \tag{3.25}$$

Hence, the quantum statistics can be expressed as $\exp(i\alpha)$.

Finally, from the previous discussion of the cyon, we can deduce that the cyon is generally the simple model of an anyon. Since both its spin and statistics can take on any values, as shown in Eq.(3.18) and Eq.(3.24), it is called “anyon”.

Path Integral for Multiply Connected Space and Anyons

This part is devoted to the relation of the path integral for multiply connected space with the anyon. Especially, its statistics is discussed from the viewpoint of the braid group, a group classified by the winding number. Furthermore, the relation between the additional phase factor in path integrals and the representation of the generator in the braid group is presented here also.

Let us review the definition of quantum statistics. From the preceding section, it is well known that the quantum statistics in two dimensions can be defined as the phase factor of the amplitude associated with slow motion of distant particle around one another. In the previous part, the quantum statistics is expressed as $\exp(i\alpha)$.

The configuration space for distinguishable particles is exclusively $R^{dn} \equiv R^d \times \ldots \times R^d$ (n factors), where d is the dimension of space [31]. There is something different for the system of identical particles. For indistinguishable particles the different ordering of particle indices make no physical distinction. This means that $(\mathbf{r}_1, \ldots, \mathbf{r}_i, \mathbf{r}_j, \ldots, \mathbf{r}_n)$ is equivalent to $(\mathbf{r}_1, \ldots, \mathbf{r}_j, \mathbf{r}_i, \ldots, \mathbf{r}_n)$. In our problem, we have to exclude the diagonal points, $D = \{(\mathbf{r}_1, \ldots, \mathbf{r}_n) \text{ with } \mathbf{r}_i = \mathbf{r}_j \text{ for some } i \neq j\}$ because any two particles are prohibited to coincide with each other. Therefore, the configuration space of n identical particles is $M_n^d = \left(R^{dn} - D\right)/S_n$ where S_n is the symmetric group of n objects that shows the symmetric property of ordering in the system.

It is reasonable to assume that M_n^d is multiply connected. Hence, the path integral for multiply connected space, shown is Chapter 2, can be applied here. To do path integrals, we have to divide all possible paths into the homotopic classes $\alpha \in \pi_1(M_n^d)$. Let us rewrite Eq.(2.46) as

$$K(q,t';q,t) = \sum_{\alpha\in\pi_1(M_n^d)} \chi(\alpha)\int Dq_\alpha \exp\left[\frac{i}{\hbar}\int_t^{t'} d\tau L\{q_\alpha(\tau),\dot{q}_\alpha(\tau)\}\right] \tag{3.26}$$

Now, we consider the paths that start and end at the same point, called a loop and put back the factor $\hbar$ here. For Eq.(3.26) to make sense as a probability amplitude, the weights $\chi(\alpha)$ cannot be arbitrary. Actually, since the usual rule for combining probabilities

$$K(q,t';q,t) = \int_{M_n^d} dq'' K(q,t';q'',t'')K(q'',t'';q,t) \tag{3.27}$$

must be correct, thus the weights $\chi(\alpha)$ have to satisfy

$$\chi(\alpha_1)\chi(\alpha_2) = \chi(\alpha_1 \cdot \alpha_2) \tag{3.28}$$

for and loops α_1 and α_2. From Eq.(3.28) it can be concluded that $\chi(\alpha)$ must be a one-dimensional representation of the fundamental group $\pi_1(M_n^d)$.

To see the group structure of the system of particles the fundamental group of the configuration space, $\pi_1(M_n^d)$ must be identified, Unfortunately, since this topic requires so much background in algebraic topology, we quote only the result here. It turns out that the fundamental group of M_n^d is given by

$$\begin{aligned} \pi_1\left(M_n^d\right) &= S_n \quad , \quad d \geq 3, \\ &= B_n \quad , \quad d = 2 \end{aligned} \tag{3.29}$$

where B_n is Artin's braid group of n objects containing the permutation group S_n as a finite subgroup [46].

Next, to be clearer let us consider the system of two particles as an example. In this example, the important difference between two and higher dimensions is presented as well. Rather than assigning the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for the two particles, they can be replaced by the relative and center-of-mass coordinates, **r** and **R** respectively. Their definitions are

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \in R^d, \tag{3.30}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad \in R^d - \{0\} \tag{3.31}$$

where d is the dimension of space.

Now, let us start with the case of two particles in three dimensions. The configuration space M_2^3 can be decomposed as

$$M_2^3 = R^3 \times r_2^3 \tag{3.32}$$

where the space r_2^3 describes the three degrees of freedom of the relative motion consisting of the length and the two angles of relative coordinate **r**. Since they are identical, **r** and **-r** are identified. The topology of r_2^3 is represented by the product of the semi-infinite line describing $|\mathbf{r}|$ times the projective space P_2 describing the orientation of $\pm\frac{\mathbf{r}}{|\mathbf{r}|}$ [41].

It is well known that the projection space P_2 is doubly connected and admits merely two classes of loops. One class comprises loops, which can be shrunk to a point by a continuous transformation, and the other consists of loops, which cannot be shrunk to a point. The loop, which can be shrunk to a point, is a contractible loop and the other is non-contractible loop. Both of them are shown in Fig 7. In addition it is easy to find that there are no other classes because the square of a non-contractible loop is contractible as shown in Fig 8.

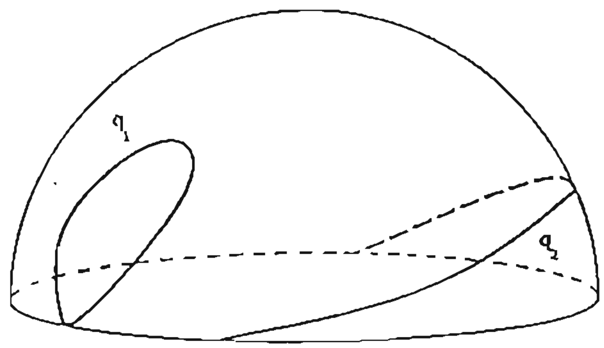

Fig.7 Examples of contractible (q_1) and non-contractible (q_2) loops [41].

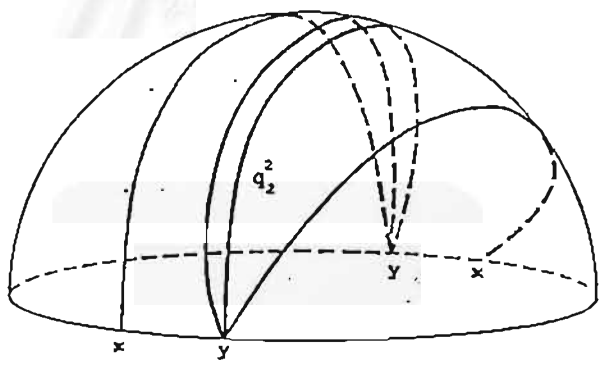

Fig.8 The square of non-contractible loop, q_2^2 is contractible M_2^3 [41].

From the composition (3.32) and the topology of r_2^3, we obtain the fundamental group of M_2^3

$$\pi_1\left(M_2^3\right) = \pi_1\left(R^3 \times r_2^3\right) = S_2 \tag{3.33}$$

This means that there are sole two classes of particles, bosons and fermions, in three dimensions. The former class corresponds to contractible loops and the latter corresponds to non-contractible loops.

Let us now turn to our problem, the anyon in two dimensions. Thus we have to discuss two dimensions here. Like the system in three dimensions, the configuration space in two dimensions can be expressed as

$$M_2^2 = R^2 \times r_2^2 \qquad (3.34)$$

where r_2^2 is some space describing the two degrees of freedom of the relative motion. Since **r** and -**r** must be identified, the space r_2^2 is then the upper-half plane without the origin and with the positive x-axis identified with the negative one. Thus the topology of r_2^2 is a cone without the tip [41] shown in Fig.9. If the vertex was included, it is easy to see that any loops can be contractible. This means that there is only one class, bosonic class. However, in physical situations since the hard-core assumption is reasonable, the tip of the cone should be removed. According to Eq.(3.34) any loop in M_2^2 can be classified by the number of times it winds around the cone r_2^2. Since the vertex of the cone has been removed, two loops, q and q', with different winding numbers are homotopically inequivalent. This means that these two loop cannot be continuously transformed to each other. Therefore, it can be deduced that

$$\pi_1(M_2^2) = \pi_1(R_2^2 \times r_2^2) = Z = B_2 \qquad (3.35)$$

where Z is the set of integer number. From Eq. (3.35), we can conclude that the configuration space M_2^2 is infinitely connected.

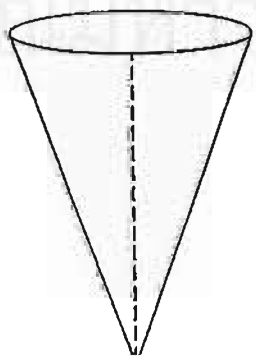

Fig. 9 The topology of r_2^2 is a cone without the tip [41].

It has been seen that in two dimensions $\pi_1(M_n^2)$ is much more complicated; it is an infinite non-Abelian group, the braid group [31]. Moreover since the anyon exists in two dimensions, the braid group should be briefly reviewed here. Unfortunately, the difficulty of this subject makes it impossible to clarify this subject in detail. Thus it is shown only essential relations and properties here.

To illustrate a closed path in M_n^2, it can be represented by n curves in the three-space (x, y, t) with no intersection and with the final positions in R^2 in t' being exclusively permutations of the initial ones at t. The equivalence classes of these curves are displayed by projecting them on a fixed x - t plane and the projections on the plane shall be called strings. Without loss of generality, we can assume that (1) the initial positions of the strings are all different (i.e. $x_1 < ... < x_n$), (2) at each time slice there is at most one intersection of two neighboring string, and (3) the strings are always parallel to the t axis, with x values being permuted initial ones, except in the neighborhood of an intersection [31]. Since how the curves in three-space wind have to be seen in this process, we will consider that one of the strings at the intersection be in front if the corresponding curve in three-space has a smaller ordinate at that point. Such a configuration of string discussed above is called a braid [46], and examples are shown in Fig.10 and Fig. 11.

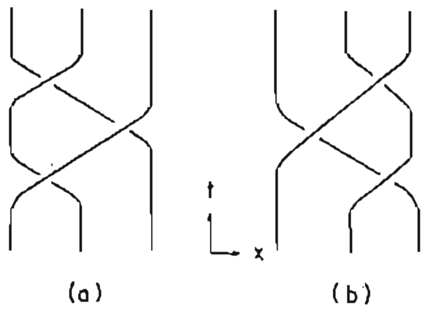

Fig.10 Two braids for n = 3 [31].

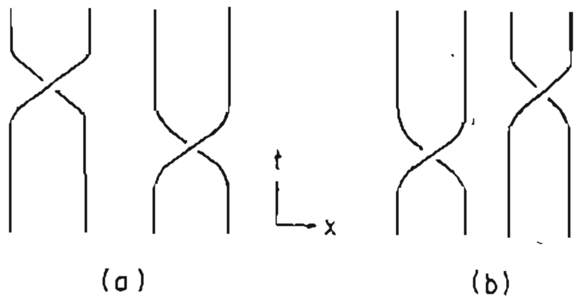

Fig.11 Two braids for n = 4 [31].

The equivalence classes of braids under continuous deformation also form a group, called the braid group $B_n(R^2)$. This group can be analogous to the fundamental group $\pi_1(M_n^2)$ generated by the multiplication of closed paths in M_n^2, so that is roughly that $B_n(R^2)$ is isomorphic to $\pi_1(M_n^2)$.

The crucial quantities in a group are the generators. The generators of the braid group are σ_i's. The operation σ_i denotes the interchange of two neighboring strings at x_i and x_{i+1} with the left one in front. This means that a braid can be expressed algebraically as a product of a sequence of $\sigma_i^{\pm}(1 \le i \le n-1)$ as shown in Fig. 11. From Fig. 9 and Fig. 10 it is clear that the essential relations among σ_i's are

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_k &= \sigma_k\sigma_i \qquad (k \ne i \pm 1) \end{aligned} \tag{3.36}$$

Moreover, it has been proved that there are no further relations among σ_i's [47].

It can be deduced from Eq. (3.36) that all one-dimensional unitary representations of $\pi_1(M_n^2)$ satisfy [31]

$$\chi_\theta(\sigma_1) = \ldots = \chi_\theta(\sigma_{n-1}) = \exp(-i\theta) \tag{3.37}$$

and are labeled by $\theta(0 \leq \theta \leq 2\pi)$. This means that the term $\chi_\theta(\alpha)$ in Eq.(3.26) represents the exotic statistics, called θ-statistics. The θ-statistics can take on any values between the Bose - Einstein $(\theta = 0)$ and the Fermi - Dirac $(\theta = \pi)$ statistics. Therefore the θ-statistics is the statistics of anyons and is called fractional statistics.

Now, let us go back to phase factor $\chi(\alpha)$ in Eq.(3.26). The homotopic class α, in general, is a product of a sequence of $\sigma_k^{\pm 1}$. To generalize to $\chi(\alpha)$ the case of $\chi(\sigma_k^{\pm})$ should be rewritten here in novel form. Since, physically, σ_k denotes the interchange of only the two particles at $\mathbf{r}_k$ and $\mathbf{r}_{k+1}$ along a counter-clockwise loop with, the other particles kept outside, hence it is equivalent to the homotopic class whose winding number is exactly minus one. Thus, we can rewrite Eq. (3.37) as

$$\chi_\theta\left(\sigma_k^{\pm 1}\right) = \exp(\mp i\theta) = \exp\left\{-i\frac{\theta}{\pi}\sum_{i<j}\Delta\phi_{ij}\right\} \tag{3.38}$$

where $\Delta\phi_{ij}$ is the change of the azimuthal angle of particle i relative to particle j. It is obviously seen that, for each σ_k, only a single term in the sum is nonvashing and its value is exactly π. This term can be readily generalized to arbitrary $\alpha \in \pi_1(M_n^2)$:

$$\chi_\theta(\alpha) = \exp\left\{-\frac{i\theta}{\pi}\int dt\frac{d}{dt}\sum_{i<j}\phi_{ij}(t)\right\} \tag{3.39}$$

Let us put Eq.(3.39) into Eq.(3.26) , we will obtain

$$\begin{aligned} K(q,t';q,t) &= \sum_{\alpha\in\pi_1(M_n^2)} Dq_\alpha(\tau)\exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\left[L - \frac{\hbar\theta}{\pi}\frac{d}{d\tau}\sum_{i<j}\phi_{ij}(\tau)\right]\right\} \\ &= \sum_{\alpha\in\pi_1(M_n^2)} Dq_\alpha(\tau)\exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau L'\right\} \end{aligned} \tag{3.40}$$

where $L' = L - \frac{\hbar\theta}{\pi}\frac{d}{d\tau}\sum_{i<j}\phi_{ij}(\tau)$.

Notice that instead of taking care of the multiply connectedness property of the anyon in two dimensions, we will deal with the sum of ordinary path integrals with a new Lagrangian. The additional term in this Lagrangian can be analogous to the Chern – Simons term discussed in the next chapter. Furthermore, it is clearly seen that this exotic term in the Lagrangian comes from the multiply connectedness of the configuration space of the anyon.

Chern-Simons Theory and Anyons

As mentioned in the previous section, the additional term in the Lagrangian comes from the complicated topology of the configuration space. In this part we will put the Chern-Simons term by hand into the ordinary Lagrangian before it is proved in a later part that it is related to the complicated topology of the configuration space. This part is devoted to a simple Chern-Simons term of fractional statistics.

Chern-Simons theory is a very important and interesting subject both in physics and mathematics. The connection between Chern-Simons gauge theory and the theory of knot and link invariant was established by Edward Witten eleven years ago [48]. Moreover, these topics are an importat and advanced subject in quantum field theory [49]. Unfortunately, only Abelian Chern-Simons gauge theory is presented here. Now let us start with the Chern-Simons Lagragian in two dimensions expressed as

$$L_{CS} = \frac{\kappa}{2c}\int d^2x \, \epsilon^{\alpha\beta\upsilon} \, A_\alpha \partial_\beta A_\upsilon \qquad (3.41)$$

where $\in^{\alpha\beta\gamma}$ is the completely antisymmetric tensor density, κ is the coupling constant of Chern-Simons term and A_α is a dynamical (2+1)- dimensional U(1) gauge field. Moreover, here and in the following, Greek indices, such as α, take values 0,1,2 while Latin indices, such as i, take values 1,2. For example, the three-vector $x^\alpha \equiv (ct, x^i)$ is denoted for short also as x, so that $d^3x = cdtd^2x$. Nevertheless, the crucial quantity in field theory is the action. The Chern-Simons action can be written as

$$S_{CS} = \frac{\kappa}{2c}\int d^3x \in^{\alpha\beta\upsilon} A_\alpha \partial_\beta A_\upsilon \qquad (3.42)$$

This action is gauge invariant even if the Lagrangian contains an undifferentiated gauge field A_α. This is very easy to show that the action is gauge invariant up to the surface term. Consider a gauge transform

$$A_\alpha \to A_\alpha + \partial_\alpha \Lambda \qquad (3.43)$$

where Λ is a space-time dependent parameter, thereby the Chern-Simons action changes only by a surface term

$$\delta S_{CS} = -\frac{\kappa}{2c}\int d^3x \in^{\alpha\beta\upsilon} \partial_\alpha \left[(\partial_\beta \Lambda) A_\upsilon\right] \qquad (3.44)$$

Thus the action (3.42) remains invariant

To see the effect of the Chern - Simons term in a many-body system in (2+1)–dimensional space, let us now couple the gauge field A_α to a matter system consisting of N non-relativistic point particles of mass m and charge e. The interesting point is that this charge is not the the standard electric charge because the gauge field A_α is not standard vector potential in electrodynamics. Their coordinates are denoted by dynamical variables, $r_I(t)$ where I take values 1, ..., N. Now, let us define the current as

$$j^\alpha(x) = \sum_{I=1}^{N} e v_I^\alpha(t) \delta^{(2)}(x - r_I(t)) \equiv (c\rho, j) \qquad (3.45)$$

where the velocity $v_I(t) \equiv (c, \mathbf{v}_I(t))$ and $\delta^{(2)}(\mathbf{x} - \mathbf{r}_I(t))$ is the Dirac-delta function in 2 dimensions. Moreover, it is easy to show that the current satisfies the continuity equation.

$$\partial_\alpha j^\alpha = \partial_t \rho + \nabla \cdot \mathbf{j} = 0 \tag{3.46}$$

Next, to understand more clearly let us couple the conserved current j^α to the gauge field A_α in the standard minimal way

$$\begin{aligned} S_{int} &= -\frac{1}{c^2}\int d^3x j^\alpha A_\alpha \\ &= \frac{1}{c}\int dt\left\{\sum_{I=1}^{N} e\left[\mathbf{v}_I(t)\cdot \mathbf{A}(t,\mathbf{r}_I(t)) - A_0(t,\mathbf{r}_I(t))\right]\right\} \end{aligned} \tag{3.47}$$

where the second line is obtained from substituting Eq. (3.45) into the first line. The kinetic action of this system can be expressed in the conventional way as

$$S_{matter} = \int dt\left(\sum_{I=1}^{N}\frac{1}{2}m\mathbf{v}_I^2\right) \tag{3.48}$$

Thus, the total action will be expressed as

$$\begin{aligned} S &= S_{matter} + S_{int} + S_{CS} \\ &= \int dt L \end{aligned} \tag{3.49}$$

where the total Lagrangian is

$$\begin{aligned} L = \sum_{I=1}^{N}&\left[\frac{1}{2}m\mathbf{v}_I^2 + \frac{e}{c}\mathbf{v}_I(t)\cdot \mathbf{A}(t,\mathbf{r}_I(t)) - \frac{e}{c}A_0(t,\mathbf{r}_I(t))\right] \\ &- \frac{\kappa}{2}\int d^2x\left[\mathbf{E}(t,\mathbf{x})\wedge \mathbf{A} + A_0(t,\mathbf{x})B(t,\mathbf{x})\right] \end{aligned} \tag{3.50}$$

in Eq. (3.50) the Chern-Simons magnetic field

$$B = \nabla \wedge \mathbf{A} = \partial_1 A^2 - \partial_2 A^1 \equiv -F_{12} \tag{3.51}$$

and the Chern-Simons electric field is

$$E^i = -\frac{1}{c}\partial_t A^i - \partial_i A^0 \equiv F_{0i} \tag{3.52}$$

and here the metric is $\eta_{\alpha\beta} = (1,-1,-1)$.

Next, we will explore the equations of motion of our system. By varying L with respect to $\mathbf{r}_I(t)$, the equations of motion for matter variables will be in the form of the standard Lorentz force equations of particle mass m and charge e moving in an electric field E and in magnetic field B. These equations can be expressed as

$$m\dot{v}_I^i(t) = e\left[E^i(t,\mathbf{r}_I(t)) + \frac{1}{c}\in^{ij} v_I^j B(t,\mathbf{r}(t))\right] \tag{3.53}$$

If L is varied with respect to A_α, we will get instead the equations of motion for the gauge fields written as

$$j^\alpha = \frac{\kappa c}{2}\in^{\alpha\beta\upsilon} F_{\beta\upsilon} \tag{3.54}$$

or in components

$$E^i = \frac{1}{\kappa c}\in^{ik} j^k \tag{3.55a}$$

$$B = -\frac{1}{\kappa}\rho \tag{3.55b}$$

These equations are crucial for the anyon system. Firstly, they tell us that while in the cyon system the magnetic field B is an externally given field and could be chosen to be that of a vortex, it is given by the field strength of the Chern-Simons term as shown in Eq.(3.55b). Secondly, it is not difficult to prove that in a single particle system.

$$\begin{aligned}\frac{m}{e}\dot{v}^i &= E^i(t,\mathbf{r}_I(t)) + \frac{1}{c}\in^{ij} v_I^j(t)B(t,\mathbf{r}(t)) \\ &= \frac{1}{\kappa c}\in^{ik}\left[j^k(x) - v^k(t)\rho(x)\right] \\ &= \frac{1}{\kappa c}\in^{ik}\left[v^k(t) - v^k(t)\right]\delta^{(2)}(x - \mathbf{r}(t)) = 0\end{aligned} \tag{3.56}$$

This means that a particle that moves in a Chern-Simons background does not feel any self-interaction because the Lorentz force becomes obsolete.

Nevertheless, the presence of the Chern-Simons electric and magnetic fields is not at all unimportant. It leads to the non-trivial relationship between the velocity **v** and the canonical momentum **p**, i.e.

$$\mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{r}) \tag{3.57}$$

From Eq. (3.55b) a significant result can be obtained by integrating this equation over a small two-dimensional disc C_I including solely the I-th particle. The result is

$$\Phi_I = \int_{C_I} d^2xB = -\frac{e}{\kappa}\int_{C_I} d^2x\sum_{I=1}^{N}\delta^{(2)}(\mathbf{x} - \mathbf{r}(t)) = -\frac{e}{\kappa} \tag{3.58}$$

where Φ_I is the magnetic flux attached to that particle whose charge is e.

Eq.(3.59) tells us that the Chern-Simons naturally binds charge and flux to the particle. This means that in this system if a particle has a charge e, then it has also a flux $\Phi_I = -\frac{e}{\kappa}$. In this situation, it is easy to see that each particle possesses both a charge and a flux, and moves in the background fields created by the other particles, the Chern-Simons electric and magnetic fields conspire to cancel all self - interactions and the resulting Lorentz force is zero [41]. Nevertheless, at the quantum level there still exists the Aharonov-Bohm interaction between charges and fluxes in this system.

From the total Lagrangian in Eq.(3.50) we derive the corresponding Hamiltonian

$$H = \sum_{I=1}^{N}\frac{1}{2}m\mathbf{v}_I^2 + \int d^2xA_0(x)\{\kappa B(x) + \rho(x)\} \tag{3.59}$$

Choosing the Weyl gauge, A_0 may be set equal to zero if a constraint $\rho = -\kappa B$ is imposed. Therefore,

$$H = \sum_{I=1}^{N} \frac{1}{2} m \mathbf{v}_I^2 \tag{3.60}$$

that is the Hamiltonian of N non-interaction particles. However, the condition $A_0 = 0$ is not enough to fix the gauge completely because we are still free to perform any time-independent gauge transformation without leaving the Weyl gauge. To this end, let us impose the condition

$$\partial_i A^i = 0 \tag{3.61}$$

From both these conditions and the constraint $\rho = -\kappa B$, we obtain

$$\begin{aligned} A_I^i(\mathbf{r}_1, \dots, \mathbf{r}_N) &= \frac{1}{2\pi\kappa} \int d^2x \, \epsilon^{ij} \frac{r_I^j - x_J^j}{|\mathbf{r}_I - \mathbf{x}_J|^2} \rho(t, \mathbf{x}) \\ &= \frac{e}{2\pi\kappa} \sum_{J \neq I} \epsilon^{ij} \frac{r_I^j - r_J^j}{|\mathbf{r}_I - \mathbf{r}_J|^2} \end{aligned} \tag{3.62}$$

where we have used the two-dimensional Green's function satisfying

$$\nabla^2 \left(\frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{y}| \right) = \delta^{(2)}(\mathbf{x} - \mathbf{y}) \tag{3.63}$$

Therefore, the Hamiltonian can be expressed as

$$H = \sum_{I=1}^{N} \frac{1}{2m} \left[\mathbf{p}_I - \frac{e}{c} \mathbf{A}_I(\mathbf{r}_1, \dots, \mathbf{r}_N) \right]^2 \tag{3.64}$$

Furthermore, since Eq.(3.62) is similar to Eq.(3.2), the magnetic field be can be derived similarly as

$$B_I = -\frac{e}{\kappa} \sum_{J \neq I} \delta^{(2)}(\mathbf{r}_I - \mathbf{r}_J) \tag{3.65}$$

This means that each particle feels the effect of the (N-1) others as if they are vortices carrying a flux $\Phi_I = -\frac{e}{\kappa}$.

Let us now go back to Eq.(3.62) and derive the alternative form of $A_I^i(\mathbf{r}_1, \dots, \mathbf{r}_n)$ as

$$A_I^i(r_1,\dots,r_n) = \frac{e}{2\pi\kappa} \in^{ij} \frac{\partial}{\partial r_I^j} \sum_{J\neq I} \phi_{IJ} \tag{3.66}$$

where we have used the identity $\partial_i\left(\arctan \frac{r_2}{r_1}\right) = -\in_{ij} \frac{r^j}{|r|^2}$ and defined

$\phi_{IJ} = \arctan\left(\frac{r_I^2 - r_J^2}{r_I^1 - r_J^1}\right)$. Next, let us consider the Lagragian corresponding to Eq. (3.64) that is

$$\begin{aligned} L &= \sum_{I=1}^{N}\left[\frac{1}{2}m v_I^2 + \frac{e}{c} v_I \cdot A_I(r_{1,\dots,} r_N)\right] \\ &= \sum_{I=1}^{N}\frac{1}{2}m v_I^2 - \frac{e^2}{2\pi c\kappa}\sum_{I=1}^{N}\sum_{J\neq I} v_I^i \frac{\partial}{\partial r_I^i}\phi_{IJ} \end{aligned} \tag{3.67}$$

Using the property

$$\frac{\partial}{\partial r_I^i}\phi_{IJ} = -\frac{\partial}{\partial r_J^i}\phi_{IJ} \tag{3.68}$$

we get

$$L = \sum_{I=1}^{N}\frac{1}{2}m v_I^2 - \frac{e^2}{2\pi c\kappa}\sum_{I<J}\left(v_I^i - v_J^i\right)\frac{\partial}{\partial r_I^i}\phi_{IJ} \tag{3.69}$$

From the fact that

$$\begin{aligned} \frac{d}{dt}\phi_{IJ} &= \left(\frac{\partial r_I^i}{\partial t}\frac{\partial}{\partial r_I^i} + \frac{\partial r_J^i}{\partial t}\frac{\partial}{\partial r_J^i}\right)\phi_{IJ} \\ &= v_I^i \frac{\partial}{\partial r_I^i}\phi_{IJ} + v_J^i \frac{\partial}{\partial r_J^i}\phi_{IJ} \\ &= \left(v_I^i - v_J^i\right)\frac{\partial}{\partial r_J^i}\phi_{IJ} \end{aligned} \tag{3.70}$$

The Lagrangian in Eq.(3.69) can be rewritten as

$$L = \sum_{I=1}^{N}\frac{1}{2}m v_I^2 - \frac{e^2}{2\pi c\kappa}\frac{d}{dt}\sum_{I<J}\phi_{IJ}(t) \tag{3.71}$$

It is clearly seen that the additional term in Eq.(3.71) coming from the Chern-Simons term takes a similar form to that in Eq.(3.40) coming from the complicated topology of the configuration space. This means that adding the topological term, the Chern-Simons term, into the Lagrangian is equivalent to taking care of the multiply connectedness of the configuration space. In other words, instead of considering anyons in multiply connected space, we can study them by adding the Chern-Simons term into their Lagrangian.

Now, let us go back to consider the statistics of anyons in the Chern-Simons construction. The statistics can be derived from the Ahanorov-Bohm phase or the Berry's phase, shown in detail in Appendix A, picked up when two neighboring anyons are interchanged [23]. Therefore, the statistics is

$$\begin{aligned} \exp(i\alpha) &= \exp\left\{\frac{-ie}{2\hbar c}\oint_{\Gamma} d\mathbf{r}\cdot\mathbf{A}\right\} = \exp\left\{\frac{-ie}{2\hbar c}\int d^2 r B\right\} \\ &= \exp\left\{\frac{-ie}{2\hbar c}\right\} = \exp\left\{\frac{i}{\hbar}\frac{e^2}{2c\kappa}\right\} \end{aligned} \tag{3.72}$$

When we compare Eq.(3.72) and Eq.(3.25), it is clearly seen that the statistical parameter, α, in Eq.(3.72) is a half of that in Eq.(3.25). The factor $1/2$ come from the fact that the Noether current in this system is [41]

$$\begin{aligned} J^{\alpha} &= \frac{1}{c} j^{\alpha} - \frac{\kappa}{2}\in^{\alpha\beta\gamma} F_{\beta\gamma} \\ &= \frac{1}{2c} j^{\alpha} \end{aligned} \tag{3.73}$$

where in the last step we use Eq.(3.54). Therefore, The Noether charge can be derived from

$$q_I = \int_{C_I} d^2 x J^0 = \frac{1}{2}\int_{C_I} d^2 x \rho = \frac{1}{2} e \tag{3.74}$$

where C_I is a small disc containing only the I-th particle. Thus, we can rewrite Eq. (3.72) as

$$\exp(i\alpha) = \exp\left\{-i\frac{q\Phi}{\hbar c}\right\} \tag{3.75}$$

Here we assume all particles have the same charge e. Notice that a bosonic particle with a flux $\Phi = \frac{1}{2}\tilde{\Phi}_o = \frac{1}{2}\frac{hc}{q}$ is a fermion. This implies that the Chern- Simons flux unit $\tilde{\Phi}_o = \frac{hc}{q}$ is twice the ordinary electrodynamics flux unit $\Phi_o = \frac{hc}{e}$.

The last topic in this section is devoted to the significant properties of anyons, parity and time reversal violations. Prior to working out any detail, we have to introduce the Dirac gamma matrices in (2+1) dimensional space as [50]

$$\begin{aligned} \gamma^o &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \gamma^1 &= i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \gamma^2 &= i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{3.76}$$

where σ_1, σ_2 and σ_3 are Pauli's matrices and the Minkowski metric $\eta^{\mu\nu} = (1,-1,-1)$. Let us now consider the parity transformation, P. In (2+1) dimensions the normal operation of the parity transformation, $\mathbf{x} \rightarrow -\mathbf{x}$, is equivalent to a rotation. Therefore, the novel parity transformation should be defined particularly in the plane as the reflection in just one of spatial axes [50]. Without losing generality, we can choose the parity transformation as

$$\begin{aligned} x^1 &\rightarrow -x^1 \\ x^2 &\rightarrow x^2 \end{aligned} \tag{3.77}$$

From the kinetic part of the Dirac Lagrangian, it is clear that the spinor field ψ transforms as

$$\psi \to \gamma^1 \psi \qquad (3.78)$$

This leads to the fact that a fermion mass term breaks parity [50]

$$\overline{\psi}\psi \to -\overline{\psi}\psi \qquad (3.79)$$

Moreover, the gauge fields transform under P as

$$A^1 \to -A^1,\ A^2 \to A^2,\ A^0 \to A^0 \qquad (3.80)$$

thus

$$\in^{\mu\upsilon\rho} A_\mu \partial_\upsilon A_\rho \to - \in^{\mu\upsilon\rho} A_\mu \partial_\upsilon A_\rho \qquad (3.81)$$

This means that the Chern-Simons term violates the parity symmetry [51].

Next, we will consider the time reversal, T, defined as

$$i \to -i \qquad (3.82)$$

Under T the spinor and gauge fields transform as

$$\Psi \to \gamma^2 \Psi \qquad (3.83)$$

$$\mathbf{A} \to -\mathbf{A} \quad , \quad A^0 \to A^0 \qquad (3.84)$$

Moreover, we see that under T $x^0 \to -x^0$ without taking $P^0 \to -P^0$ where P is an energy-momentum three vector. Hence it is clearly seen that both the fermion mass term and the gauge field Chern-Simons term change sign under time reversal [51].

To conclude, in (2+1) dimensional space, the anyonic world, parity and time reversal will be violated. Unfortunately, there has been no experiment that confirms this phenomenon until now. However, since it can be applied to many phenomena very well, anyon is still an the interesting and important subject in physics.

Fractional Quantum Hall Effect and Anyons

This part of this chapter is devoted to the applicability of the concept of anyons to the fractional quantum Hall effect (FQHE). The important point of this part is that it is the most remarkable application of anyons. This confirms that the concept of anyons is very important in modern theoretical physics. Unfortunately, many details and topics are disregarded here. This means that we can present only an introduction of these topics. However, relevant reviews can be found readily in Lerda's book [41] and Wilczek's book [42].

Let us now start with QHE characterized by the fact that the Hall conductance, σ_H, is quantized in units of e^2/h.

$$\sigma_H = \upsilon \frac{e^2}{h} \tag{3.85}$$

where (-e) is the electron change and h is Plank constant while υ can be an integer for integer QHE and a simple fraction for fractional QHE. To understand this better, let us consider the quantization rules of two-dimensional particles with the external magnetic filed perpendicular to their plane. This field, B, organizes the energy spectrum of the particles into Landau level [51] and forces particles to fill such levels from the bottom up. For samples of finite area A, it is not difficult to show that the degeneracy in each level is finite and thus the number of available levels is given by

$$\deg = \frac{A}{2\pi l_0^2} \tag{3.86}$$

where $l_o = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length [52].

Since a quantity playing an important role in QHE is the factor, υ, defined as the number of electrons, N, per number of Landau levels available [41], the filling factor is

$$\upsilon = \frac{N}{A/2\pi l_o^2} = 2\pi l_o^2 \rho \tag{3.87}$$

where $\rho = N/A$ it's the electron density. Furthermore, Eq.(3.87) can be rewritten as

$$\upsilon = \frac{N}{AB\left(\frac{e}{hc}\right)} = \frac{N}{\Phi/\phi_o} \tag{3.88}$$

where $\Phi = AB$ is the magnetic flux through the area A, and $\phi_0 = hc/e$ is the flux unit. This means that the number states is equal to the number of external magnetic flux in unit of hc/e. From a knowledge of the classical Hall effect, it is well known that the transverse Hall conductance is defined as[41]

$$\sigma_H \equiv \sigma_{xy} = \frac{j_x}{E_y} \tag{3.89}$$

where $j_x \sim e\rho v_x$ is the electric current in the x-direction. Therefore,

$$\sigma_H = \frac{ec}{B}\rho \tag{3.90}$$

To get the quantum result, we have to use Eq.(3.88), thus the quantized Hall conductance

$$\sigma_H = \upsilon \frac{e^2}{h} \tag{3.91}$$

As in Eq.(3.85) it can be interpreted that for the case of υ is integer correspondingly integer QHE, an integer number of Landau levels are completely filled. Moreover, this system is a non-interacting-electron system in a magnetic field while the fractional QHE results from the condensation of the two-dimensional electron system into a new type of collective ground state driven by the Coulomb repulsion.

In the case of FQHE, the first corresponding ground state wavefunction, a variational wavefunction, was proposed by Laughlin [10] in 1983. This wavefunction can describe only the case of $\upsilon = 1/m$ with m and odd integer and is given by

$$\Psi_m = N_m \prod_{I<J} (z_I - z_J)^m \exp\left\{ \frac{-1}{4l_o^2} \sum_I |z_I|^2 \right\} \tag{3.92}$$

where z_I is the complex coordinate for the I-th electron and N_m is a normalization factor. It can be proved from Eq.(3.92) that the statistics of particles in Eq.(3.92) is $\exp(i\pi\nu) = \exp(i\pi m)$. Moreover, since the prefactor $\prod_{I<J} (z_I - z_J)^m$ is purely analytic, it can be deduced that all particles are in the lowest Landau level. The very important aspect of Ψ_m is that it is not simply the product of single particle wavefunctions, yet is a perplexing combination of such products. From Eq.(3.92), using the plasma analogy [10], our system can be seen as a system of a plasma of particles with charge

$$q = \sqrt{2m} \tag{3.93}$$

In a neutralizing background of density

$$\rho = \frac{1}{2\pi l_o^2} \cdot \frac{1}{m} \tag{3.94}$$

Hence, from Eq.(3.88) we conclude that the filling factor corresponding to Ψ_m is given by

$$\upsilon = 2\pi l_o^2 \rho = \frac{1}{m} \tag{3.95}$$

To see the effect of anyons, Let us now go beyond the ground state to consider excitations in our system. From Eq.(3.94) we find that any deviations from the density ρ can be accommodated in the system as localized quasi-hole or quasi-particle across a gap. To prove that these excitations turn out to have fractional statistics and fractional charge, let us now begin by considering the case of a quasi-hole. It can be predicted reasonably that the wavefunction for the state at filling factor $\upsilon = 1/m$ with a quasi-hole at the point z_α is given by

$$\Psi_m^{+z_\alpha} = N_+ \prod_I (z_I - z_\alpha) \Psi_m \tag{3.96}$$

where N_+ is a normalization factor and Ψ_m is the ground state wavefunction in Eq.(3.92). From Eq.(3.95) and (3.88) we can conclude that each particle carries m units of flux, and similarly the quasi-hole described by Eq.(3.96) will carry only one unit of flux. Hence, the quasi-hole behaves like $1/m$ of an electron. To see more clearly, Berry's phase [14], shown in detail in Appendix A, will be applied to evaluate the fractional charge of the quasi-hole [13]. Since

$$\frac{d\Psi_m^{+z_\alpha}(t)}{dt} = \sum_I \frac{d}{dt}\left[\ln(z_I - z_\alpha(t))\right]\Psi_m^{+z_\alpha}(t) \tag{3.97}$$

therefore the geometric phase

$$\begin{aligned} \gamma_\alpha &= -i\int dt \langle \Psi_\alpha(t) | \frac{d}{dt} | \Psi_\alpha(t) \rangle \\ &= -i\int_\Gamma dt \left\langle \Psi_m^{+z_\alpha}(t) \middle| \sum_I \frac{d}{dt}\left[\ln(z_I - z_\alpha(t))\right] \middle| \Psi_m^{+z_\alpha}(t) \right\rangle \end{aligned} \tag{3.98}$$

We can rewrite this expression in term of the electron density in the state $\Psi_m^{+z_\alpha}$,

$$\rho(z) = \left\langle \Psi_m^{+z_\alpha}(t) \middle| \sum_I \delta^{(2)}(z - z_I) \middle| \Psi_m^{+z_\alpha}(t) \right\rangle \tag{3.99}$$

as

$$\gamma = \int_{\Gamma} dt \int d^2z \frac{d}{dt}[\ln(z - z_\alpha(t))]\rho(z)$$
$$= -i \int d^2z \oint_{\Gamma} dz_\alpha \frac{1}{z_\alpha - z}\rho(z) \tag{3.100}$$

where Γ is a closed loop starting at time to and ending at time t_1. From Eq. (3.99) , it can be assumed reasonably that $\rho(z)$ is a regular function. Thus, we can evaluate γ as

$$\gamma = -i \int_{S(\Gamma)} d^2z \oint_{\Gamma} dz_\alpha \frac{1}{z_\alpha - z}\rho(z)$$
$$= 2\pi \int_{S(\Gamma)} d^2z\rho(z) = 2\pi N_\Gamma \tag{3.101}$$

where $S(\Gamma)$ is the surface confined by the closed path Γ and N_Γ is the average number of electrons containing in there. On the other hand, from Eq.(3.75), using the Aharonov-Bohm effect, a particle of charge q moving along a closed loop Γ encircling a flux Φ_Γ will have the additional phase

$$\exp\left(-\frac{iq\Phi_\Gamma}{\hbar c}\right) \tag{3.102}$$

Comparing Eq.(3.101) to Eq.(3.102), we find that

$$\frac{q\Phi_\Gamma}{\hbar c} = 2\pi N_\Gamma \tag{3.103}$$

Hence, the particle has charge given by

$$q = hc\frac{N_\Gamma}{\Phi_\Gamma} = \frac{1}{m}e \tag{3.104}$$

This means hat the quasi-holes have charge $q \equiv e^* = e/m$. Similarly, the case of quasi-particles can be solved as well, but, unfortunately, it is not considered here. Next, let us consider the statistics of the quasi-holes. Reconsidering Eq.(3.101), we will find that if two quasi-holes are exchanged also, the Eq.(3.101) have to be rewritten as

$$\gamma = 2\pi \int_{s(r)} d^2\rho(z) = 2\pi\left(N_r - \frac{1}{m}\right) \quad (3.105)$$

Comparing Eq.(3.105) and (3.101) we find that when z_α encircles z_β the wavefunctions picks up an additional phase

$$\exp[-i\Delta\gamma] = \exp\left[-i2\pi\frac{1}{m}\right] \quad (3.106)$$

Similarly to Eq (3.21), we deduced that their statistics is

$$\exp(i\alpha) = \exp\left(i\frac{\pi}{m}\right) \quad (3.107)$$

From Eq.(3.104) and (3.107), we can conclude that the quasi-holes in FQHE are anyons with fractional charges and fractional statistics. Next, let us consider the case of two quasi-holes. Hence, the effective wavefunction for them can be predicted as

$$\Psi^{+}_{m_1} \approx (z_\alpha - z_\beta)^{m_1} \exp\left\{\frac{-1}{4ml_o^2}\left(|z_\alpha|^2 - |z_\beta|^2\right)\right\} \quad (3.108)$$

where $m_1 = 1/m + 2p_1$ Using the approach similar to the method applied in the case of only single quasi-hole, we can find that the quasi-holes carry a total charge

$$q_1 = \frac{e}{m}\left(\frac{1}{mm_1}\right) \quad (3.109)$$

and the electron filling factor is given by

$$\begin{aligned} \upsilon_1 &= \upsilon - \frac{q_1}{e} \\ &= \frac{2p_1}{1 + 2p_1 m} = \frac{1}{m + \frac{1}{2p_1}} \end{aligned} \quad (3.110)$$

From Eq.(3.110), it is easy to find that if m = 3 and $p_1 = 1$, then

$$\upsilon_1 = \frac{2}{7} \quad (3.111)$$

, one of the experimentally observed fractions in the QHE. Eventually, considering excitations in a recursive way, we can find that the generic-filling factor is represented by [41]

$$\upsilon = \cfrac{1}{m + \cfrac{\alpha_1}{2p_1 + \cfrac{\alpha_2}{2p_2 + \ldots}}} \tag{3.112}$$

where $\alpha_i = +1$ if the i-th generation comprises quasi – hole and $\alpha_i = -1$ if the i-th generation consists of quasi – particle. This method is well known as the hierarchy construction proposed by Haldane [12]. However, Eq.(3.112) sometimes leads also to fractions that are not experimentally observed. Nevertheless, it has been seen that the concept of anyons is very useful to understand the mechanism of FQHE.

CHAPTER 4

RESULTS, DISCUSSION AND CONCLUSION

This chapter is devoted to presenting the results of applying the path integral to calculate the ground–state energy of two anyons in a harmonic potential and to discuss the results and also to conclude this work as well. In the first part, we would like to show the lengthy results of calculating the ground-state energy of two anyons in a harmonic potential through the path integration technique. Finally, in the last portion, the results are discussed and compared with Wu's work [55].

Results

This section is devoted to showing the details of calculating the propagator of a two-anyon system in a harmonic potential. However, since in the two-body problem center-of-mass and relative coordinates can describe the system, the Lagrangian of this system can be separated into two parts, center-of –mass and relative Lagrangian. Let us now start with the Lagrangian of two particles with mass M in a harmonic potential with frequency ω,

$$\begin{aligned} L_o &= \frac{1}{2}M\dot{r}_1^2 + \frac{1}{2}M\dot{r}_2^2 - \frac{1}{2}M\omega^2 r_1^2 - \frac{1}{2}M\omega^2 r_2^2 \\ &= M\dot{\mathbf{R}}^2 - M\omega^2\mathbf{R}^2 + \frac{1}{4}M\dot{\mathbf{r}}^2 - \frac{1}{4}M\omega^2\mathbf{r}^2 \end{aligned} \tag{4.1}$$

where $\mathbf{R} = \frac{r_1 + r_2}{2}$ is their center-of-mass coordinate of them.

$\mathbf{r} = r_1 - r_2$ is their relative coordinate of them.

Since in the path integral form the center-of-mass contribution can be directly integrated out and does not lead to the extraordinary result [15], in this part only the relative contribution is considered in detail. As mentioned in chapter 3, especially in two dimensions, to change an ordinary particle into an anyon the Chern-Simons term must be added to the original Lagrangian. Hence, the relative action corresponding to our system is

$$\begin{aligned} S &= \int dt \left\{ \frac{1}{4} M\dot{\mathbf{r}}^2 - \frac{1}{4} M\omega^2 \mathbf{r}^2 \right\} - \frac{\kappa}{2c} \int d^3x \varepsilon^{\mu\nu\beta} a_\mu \partial_\nu a_\beta \\ &= \int dt \left\{ \frac{1}{2} m\dot{\mathbf{r}}^2 - \frac{1}{2} m\omega^2 \mathbf{r}^2 - \alpha\dot{\theta} \right\} \end{aligned} \tag{4.2}$$

where $m = \dfrac{M}{2}, \alpha = \dfrac{e^2}{2\pi c\kappa}$, and to complete the second line we have used Eq. (3.71) for the two-body problem.

As shown in chapter 3, α is called the statistical parameter. Let us now consider the last term in Eq.(4.2) leading to the constraint

$$\int_{t'}^{t''} dt\dot{\theta} = \theta'' - \theta' + 2\pi n \equiv \phi \tag{4.3}$$

where $\theta(t') = \theta'$ and $\theta(t'') = \theta''$. The crucial step is expressing this constraint in the form of a Dirac-delta function. That is,

$$\delta\left(\phi - \int_{t'}^{t''} dt\dot{\theta} \right) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp\left[i\lambda(\phi - \int_{t'}^{t''} dt\dot{\theta}) \right] \tag{4.4}$$

The constraint propagator can be defined as

$$K_\phi(\mathbf{r}'', \mathbf{r}'; T) = \int Dr(t) \exp\left[\frac{i}{\hbar} S(\mathbf{r}'', \mathbf{r}') \right] \delta\left(\phi - \int_{t'}^{t''} dt\dot{\theta} \right) \tag{4.5}$$

Thus, it is clear that the complete propagator can be expressed as

$$K(\mathbf{r}'', \mathbf{r}'; T) = \int d\phi K_\phi(\mathbf{r}'', \mathbf{r}'; T) \tag{4.6}$$

Let us now go back to Eq.(4.5),

$$
\begin{aligned}
K_\phi(\mathbf{r}'',\mathbf{r}';T) &= \int Dr(t)\exp\left[\frac{i}{\hbar}S\right]\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp\left[i\lambda\left(\phi-\int_{t'}^{t''}dt\dot{\theta}\right)\right] \\
&= \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\int Dr(t)\exp\left[\frac{i}{\hbar}S_\lambda\right]
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
S_\lambda &= \int dtL_\lambda \\
&= \int dt\left[\frac{m}{2}\dot{\mathbf{r}}^2-\frac{m}{2}\omega^2\mathbf{r}^2-\bar{\lambda}\dot{\theta}\hbar\right] \quad ;\bar{\lambda}=\lambda+\frac{\alpha}{\hbar}
\end{aligned}
\tag{4.8}
$$

Eq.(4.7) can be rewritten in the term of infinitesimal time intervals as

$$
\begin{aligned}
K_\phi(\mathbf{r}'',\mathbf{r}';T) = \lim_{N\to\infty}&\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^N\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\left(\prod_{j=1}^{N-1}\int d^2r_j\right) \\
&\cdot\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}S_\lambda(\mathbf{r}_j,\mathbf{r}_{j-1})\right]
\end{aligned}
\tag{4.9}
$$

where $\varepsilon=\frac{T}{N}=\frac{t''-t'}{N}$, $\mathbf{r}_0=\mathbf{r}'$, $\mathbf{r}_N=\mathbf{r}''$ and, the action will be expressed as

$$
\begin{aligned}
S_\lambda[\mathbf{r}_j,\mathbf{r}_{j-1}] &\approx \varepsilon L_\lambda\left[\frac{\mathbf{r}_j-\mathbf{r}_{j-1}}{\varepsilon},\mathbf{r}_j\right] \\
&= \varepsilon\left[\frac{m}{2}\left(\frac{\mathbf{r}_j-\mathbf{r}_{j-1}}{\varepsilon}\right)^2-\frac{m}{2}\omega^2 r_j^{\,2}-\bar{\lambda}\hbar\left(\frac{\theta_j-\theta_{j-1}}{\varepsilon}\right)\right] \\
&= \frac{m}{2\varepsilon}\left[r_j^2+r_{j-1}^2-r_j^2\omega^2\varepsilon^2\right]-\frac{m}{\varepsilon}r_jr_{j-1}\cos(\theta_j-\theta_{j-1}) \\
&\quad -\bar{\lambda}(\theta_j-\theta_{j-1})\hbar
\end{aligned}
\tag{4.10}
$$

From the relation [58]

$$
\cos\Delta\theta+a\varepsilon\Delta\theta\approx\cos(\Delta\theta-a\varepsilon)+\frac{1}{2}a^2\varepsilon^2
\tag{4.11}
$$

Using Eq.(4.11), the action can be rewritten as

$$S_\lambda[r_j, r_{j-1}] \approx \frac{m}{2\varepsilon}\left[r_j^2 + r_{j-1}^2 - r_j^2\omega^2\varepsilon^2\right] - \frac{m}{\varepsilon} r_j r_{j-1} \cdot \left[\cos\left(\theta_j - \theta_{j-1} - \frac{\lambda\varepsilon\hbar}{m r_j r_{j-1}}(\theta_j - \theta_{j-1})\right) + \frac{1}{2}\left(\frac{\lambda\hbar}{m r_j r_{j-1}}\right)^2 \varepsilon^2\right] \tag{4.12}$$

Therefore,

$$\frac{i}{\hbar} S_\lambda[r_j, r_{j-1}] \approx \frac{im}{2\varepsilon\hbar}\left[\left(1 - \omega^2\varepsilon^2\right) r_j^2 + r_{j-1}^2\right] - \frac{i}{2}\frac{\lambda^2 \varepsilon\hbar}{m r_j r_{j-1}} - \frac{im}{\varepsilon\hbar} r_j r_{j-1} \cos\left(\theta_j - \theta_{j-1} - \frac{\lambda\varepsilon\hbar}{m r_j r_{j-1}}(\theta_j - \theta_{j-1})\right) \tag{4.13}$$

From the identity [53]

$$\exp\left[\frac{1}{2} z\left(s + s^{-1}\right)\right] = \sum_{m=-\infty}^{\infty} s^m I_m(z) \tag{4.14}$$

where $I_m(z)$ is the modified Bessel function of the first kind and its asymptotic behavior can be expressed as [58]

$$I_\nu(x) = \frac{\exp(x)}{\sqrt{2\pi x}}\left\{1 - \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right)\frac{1}{x} + \ldots\right\} \tag{4.15}$$

in the limit ε approaches zero or N approaches infinity we can show that

$$\lim_{\varepsilon\to 0} \exp\left(-\frac{i}{2}\lambda^2 \frac{\varepsilon}{x}\right) \exp\left[-i\frac{x}{\varepsilon}\cos\left(\theta - \lambda\frac{\varepsilon}{x}\right)\right] = \sum_{m=-\infty}^{\infty} \exp(im\theta) I_{|m+\lambda|}\left(\frac{x}{i\varepsilon}\right) \tag{4.16}$$

Using Eq.(4.16), it is clear that

$$\lim_{\varepsilon\to 0} \exp\left(-\frac{i}{2}\lambda^2 \frac{\varepsilon\hbar}{m r_j r_{j-1}}\right) \exp\left[-i\frac{m r_j r_{j-1}}{\varepsilon\hbar}\cos\left(\left(\theta_j - \theta_{j-1}\right) - \lambda\frac{\varepsilon\hbar}{m r_j r_{j-1}}\right)\right] = \sum_{n=-\infty}^{\infty} \exp\left[in\left(\theta_j - \theta_{j-1}\right)\right] I_{|n+\lambda|}\left(\frac{m r_j r_{j-1}}{i\varepsilon\hbar}\right) \tag{4.17}$$

Therefore, applying Eq.(4.17), the constraint propagator can be expressed as

$$K_\phi(\mathbf{r}'',\mathbf{r}';T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^N \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\left(\prod_{j=1}^{N-1}\int d^2r_j\right)\prod_{j=1}^{N}\sum_{n_j=-\infty}^{\infty}$$
$$\left\{\exp\left[in_j(\theta_j-\theta_{j-1})\right]\exp\left[\frac{im}{2\hbar\varepsilon}\left(r_j^2+r_{j-1}^2-r_j^2\omega^2\varepsilon^2\right)\right]\right.$$
$$\left.\cdot I_{|n_j+\lambda|}\left(\frac{mr_jr_{j-1}}{i\varepsilon\hbar}\right)\right\}$$
$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^N \exp\left(-\frac{i}{\hbar}\alpha\phi\right)\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\left(\prod_{j=1}^{N-1}\int d^2r_j\right)$$
$$\cdot\prod_{j=1}^{N}\sum_{n_j=-\infty}^{\infty}\left\{\exp\left[in_j(\theta_j-\theta_{j-1})\right]I_{|n_j+\lambda|}\left(\frac{mr_jr_{j-1}}{i\varepsilon\hbar}\right)\right.$$
$$\left.\cdot\exp\left[\frac{im}{2\hbar\varepsilon}\left(r_j^2+r_{j-1}^2-r_j^2\omega^2\varepsilon^2\right)\right]\right\}$$

(4.18)

From

$$\int d^2r_j = \int_0^{\infty}dr_j\int_{-\pi}^{\pi}d\theta_j r_j \tag{4.19}$$

and

$$\int_{-\pi}^{\pi}\frac{d\theta_j}{2\pi}\exp\left[i(n'-n)\theta_j\right] = \delta_{n,n'} \tag{4.20}$$

we find that the constraint propagator become

$$K_\phi(\mathbf{r}'',\mathbf{r}';T) = \lim_{N\to\infty}\frac{1}{2\pi}\left(\frac{m}{i\hbar\varepsilon}\right)^N \exp\left(-\frac{i}{\hbar}\alpha\phi\right)\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\left(\prod_{j=1}^{N-1}\int dr_jr_j\right)$$
$$\cdot\prod_{j=1}^{N}\left\{\sum_{n=-\infty}^{\infty}\exp\left[in(\theta''-\theta')\right]I_{|n+\lambda|}\left(\frac{mr_jr_{j-1}}{i\varepsilon\hbar}\right)\right.$$
$$\left.\cdot\exp\left[\frac{im}{2\hbar\varepsilon}\left(r_j^2+r_{j-1}^2-r_j^2\omega^2\varepsilon^2\right)\right]\right\}$$

(4.21)

Next, to complete this calculation we have to consider the identity [58]

$$\int_0^\infty dxx\exp\left(i\sigma x^2\right)I_\nu(-iax)I_\nu(-ibx)=\frac{i}{2a}\exp\left[-i\frac{\left(a^2+b^2\right)}{4\sigma}\right]I_\nu\left(-\frac{iab}{2\sigma}\right) \tag{4.22}$$

Hence, we can find that

$$\int_0^\infty dr_1 r_1\exp\left(iar_1^2\right)I_\nu(-ibr_0r_1)I_\nu(-ibr_2r_1)$$
$$=\frac{i}{2a}\exp\left[-i\frac{b^2}{4a}\left(r_0^2+r_2^2\right)\right]I_\nu\left(-\frac{ib^2}{2a}r_0r_2\right) \tag{4.23}$$

and then

$$\int_0^\infty dr_2r_2\int_0^\infty dr_1r_1\exp\left[ia\left(r_1^2+r_2^2\right)\right]I_\nu(-ibr_0r_1)I_\nu(-ibr_2r_1)I_\nu(-ibr_2r_3)$$
$$=\exp\left[-i\frac{b^2}{4a}r_0^2-\frac{i}{4}\left(a-\frac{b^2}{4a}\right)^{-1}\left\{\frac{b^2}{2a}r_0^2+b^2r_3^2\right\}\right]$$
$$\cdot\left[\frac{i}{2a}\cdot\frac{i}{2a_2}I_\nu\left(-i\frac{b}{2a}\frac{b^2}{2a_2}r_0r_3\right)\right] \tag{4.24}$$

where $a_1=a\;,\;a_{j+1}=a-\dfrac{b^2}{4a_j}$ for $j\geq 1$

$b_1=b\;,\;b_{j+1}=b\displaystyle\prod_{k=1}^{j}\frac{b}{2a_k}$ for $j\geq 1$.

Then it is obviously seen that

$$a_2=a-\frac{b^2}{4a}\quad,\quad b_2=b\frac{b}{2a}\quad,\text{and}\quad b_3=b\frac{b}{2a}\frac{b}{2a_2} \tag{4.25}$$

Hence, Eq.(4.24) can be rewritten as

$$\int_0^\infty \prod_{k=1}^{2} dr_k r_k \exp\left[ia\sum_{j=1}^{2} r_j^2\right]\prod_{j=1}^{3} I_\nu\left(-ibr_j r_{j-1}\right)$$

$$= \prod_{k=1}^{2}\left(\frac{i}{2a_k}\right)\exp\left[-i\left\{r_0^2\sum_{j=1}^{2}\frac{b_j^{\,2}}{4a_j} - r_3^2\frac{b^2}{4a_2}\right\}\right] \cdot I_\nu\left(-ib_3 r_0 r_3\right) \quad (4.26)$$

To prove the general form of Eq.(4.26), let us now assume that

$$\int_0^\infty \prod_{k=1}^{N} dr_k r_k \exp\left[ia\sum_{j=1}^{N} r_j^2\right]\prod_{j=1}^{N+1} I_\nu\left(-ibr_j r_{j-1}\right)$$

$$= \prod_{k=1}^{N}\left(\frac{i}{2a_k}\right)\exp\left[-i\left\{r_0^2\sum_{j=1}^{N}\frac{b_j^{\,2}}{4a_j} - r_{N+1}^2\frac{b^2}{4a_N}\right\}\right] \cdot I_\nu\left(-ib_{N+1} r_0 r_{N+1}\right) \quad (4.27)$$

It is seen obviously that the case of N=1 and 2 is exactly correct, form Eq. (4.23) and (4.26). Next, let us consider

$$\prod_{k=1}^{N}\left(\frac{i}{2a_k}\right)\int_0^\infty dr_{N+1} r_{N+1} \exp\left[i\left\{r_{N+1}^2\left(a-\frac{b^2}{a_N}\right)\right\}\right]\exp\left[-i\left(r'^2\sum_{j=1}^{N}\frac{b_j^{\,2}}{4a_j}\right)\right] \cdot I_\nu\left(-ib_{N+1} r' r_{N+1}\right) I_\nu\left(-ibr_{N+2} r_{N+1}\right) \quad (4.28)$$

From $a_{N+1} = a - \dfrac{b^2}{4a_N}$, Eq. (4.28) becomes

$$\prod_{k=1}^{N+1}\left(\frac{i}{2a_k}\right)\exp\left[-i\left\{r'^2\sum_{j=1}^{N+1}\frac{b_j^{\,2}}{4a_j} + r_{N+2}^2\frac{b^2}{4a_{N+1}}\right\}\right] I_\nu\left(-ib_{N+2} r' r_{N+2}\right) \quad (4.29)$$

where

$$b_{N+2} = \frac{b}{2a_{N+1}} b_{N+1} = b\prod_{k=1}^{N+1}\frac{b}{2a_k}$$

Actually it is clearly seen that Eq.(4.28) is exactly

$$\int_0^\infty \prod_{k=1}^{N+1} dr_k r_k \exp\left(ia\sum_{j=1}^{N+1} r_j^2\right)\prod_{j=1}^{N+2} I_\nu\left(-ibr_j r_{j-1}\right) \quad (4.30)$$

Therefore, by mathematical induction, we can deduce that

$$\int_0^\infty \prod_{k=1}^{N-1} dr_k r_k \exp\left(ia\sum_{j=1}^{N-1} r_j^2 \right) \prod_{j=1}^{N} I_\nu\left(-ibr_jr_{j-1}\right)$$
$$= \prod_{k=1}^{N-1} \frac{i}{2a_k} \exp\left\{ -i\left[r'^2 \sum_{j=1}^{N-1} \frac{b_j^2}{4a_j} + r''^2 \frac{b^2}{a_{N-1}} \right] \right\} I_\nu\left(-ib_N r'r''\right) \tag{4.31}$$

where

$$r' = r_0 \quad , \quad r'' = r_N ,$$
$$a_1 = a \quad , \quad a_{j+1} = a - \frac{b^2}{4a_j} \qquad \text{for } j \geq 1,$$
$$b_1 = b \quad , \quad b_{j+1} = b\prod_{k=1}^{j} \frac{b}{2a_k} \qquad \text{for } j \geq 1,$$

Substituting Eq.(4.31) into Eq.(4.21), it is obvious that the constraint propagator can be expressed as

$$K_\phi(\mathbf{r}'', \mathbf{r}'; T) = \lim_{N\to\infty} \frac{1}{2\pi}\left(\frac{m}{i\varepsilon\hbar}\right)^N \exp\left(-\frac{i}{\hbar}\alpha\phi\right) \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp(i\lambda\phi)$$
$$\cdot \sum_{n=-\infty}^{\infty} \exp[in(\theta''-\theta')] \left\{ \int_0^\infty \prod_{k=1}^{N-1} dr_k r_k \exp\left[\frac{im}{\varepsilon\hbar}\left(1 - \frac{\omega^2\varepsilon^2}{2}\right) \sum_{j=1}^{N-1} r_j^2 \right] \right.$$
$$\left. \cdot \prod_{j=1}^{N} I_{|n+\lambda|}\left(-\frac{im}{\varepsilon\hbar} r_j r_{j-1}\right) \right\} \exp\left[\frac{im}{2\varepsilon\hbar}\left(r'^2 + r''^2\right) \right] \tag{4.32}$$
$$= \lim_{N\to\infty} \left\{ \frac{b}{2\pi i} \prod_{k=1}^{N-1} \frac{b}{2a_k} \right\} \exp\left(-\frac{i}{\hbar}\alpha\phi\right) \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp(i\lambda\phi)$$
$$\cdot \sum_{n=-\infty}^{\infty} \left\{ \exp\left[ir'^2 \left(\frac{b}{2} - \sum_{j=1}^{N-1} \frac{b_j^2}{4a_j} \right) + ir''^2 \left(\frac{b}{2} - \frac{b^2}{a_{N-1}} \right) \right] \right.$$
$$\left. \cdot \exp[in(\theta''-\theta')] I_{|n+\lambda|}\left(-ib_N r'r''\right) \right\}$$

where

$$a = \frac{m}{\varepsilon\hbar}\left(1 - \frac{\omega^2\varepsilon^2}{2}\right) \qquad \text{and} \qquad b = \frac{m}{\varepsilon\hbar}$$

From the relations [53]

$$
\begin{aligned}
\lim_{N\to\infty}\left(b\prod_{j=1}^{N-1}\frac{b}{2a_j}\right) &= \frac{m\omega}{\hbar}\csc\omega T \\
\lim_{N\to\infty} b_N &= \frac{m\omega}{\hbar}\csc\omega T \\
\lim_{N\to\infty}\left(\frac{b}{2}-\sum_{j=1}^{N-1}\frac{b_j^2}{4a_j}\right) &= \frac{m\omega}{2\hbar}\cot\omega T \\
\lim_{N\to\infty}\left(\frac{b}{2}-\frac{b^2}{4a_{N-1}}\right) &= \frac{m\omega}{2\hbar}\cot\omega T
\end{aligned}
\tag{4.33}
$$

the constraint propagator is

$$
\begin{aligned}
K_\phi(\mathbf{r}'',\mathbf{r}';T) &= \frac{m\omega}{2\pi i\hbar\sin\omega T}\exp\left(-\frac{i}{\hbar}\alpha\phi\right)\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi) \\
&\quad\cdot\sum_{n=-\infty}^{\infty}\left\{\exp\left[i\frac{im\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega T\right]\right. \\
&\quad\left.\cdot\exp[in(\theta''-\theta')]I_{|n+\lambda|}\left(\frac{mr'r''}{i\hbar\sin\omega T}\right)\right\} \\
&= \exp\left(-\frac{i}{\hbar}\alpha\phi\right)\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\exp(i\lambda\phi)\sum_{n=-\infty}^{\infty}\{\exp[in(\theta''-\theta')] \\
&\quad\cdot Q_{|n+\lambda|}(\mathbf{r}'',\mathbf{r}';T)\}
\end{aligned}
\tag{4.34}
$$

where

$$
\begin{aligned}
Q_{|n+\lambda|}(\mathbf{r}'',\mathbf{r}';T) &= \frac{m\omega}{2\pi i\hbar\sin\omega T}\exp\left[i\frac{im\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega T\right] \\
&\quad\cdot I_{|n+\lambda|}\left(\frac{mr'r''}{i\hbar\sin\omega T}\right)
\end{aligned}
\tag{4.35}
$$

Hence, using Eq.(4.6) and Eq.(4.34)

$$K(\mathbf{r}'',\mathbf{r}';T)=\int_{-\infty}^{\infty}d\phi K_{\phi}(\mathbf{r}'',\mathbf{r}';T)$$

$$=\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}\frac{d\phi}{2\pi}\exp\left[i\phi\left(\lambda-\frac{\alpha}{\hbar}\right)\right]\sum_{n=-\infty}^{\infty}\{\exp[in(\theta''-\theta')]$$

$$\cdot Q_{|n+\lambda|}(\mathbf{r}'',\mathbf{r}';T)\}$$

$$=\int_{-\infty}^{\infty}d\lambda\delta\left(\lambda-\frac{\alpha}{\hbar}\right)\sum_{n=-\infty}^{\infty}\exp[in(\theta''-\theta')]Q_{|n+\lambda|}(\mathbf{r}'',\mathbf{r}';T)$$

$$=\sum_{n=-\infty}^{\infty}\exp[in(\theta''-\theta')]Q_{\left|n+\frac{\alpha}{\hbar}\right|}(\mathbf{r}'',\mathbf{r}';T)$$

$$=\frac{m\omega}{2\pi i\hbar\sin\omega T}\exp\left[\frac{im\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega T\right]$$

$$\cdot\sum_{n=-\infty}^{\infty}\exp[in(\theta''-\theta')]I_{\left|n+\frac{\alpha}{\hbar}\right|}\left(\frac{mr'r''}{i\hbar\sin\omega T}\right)$$

$$=\sum_{n=-\infty}^{\infty}\exp[in(\theta''-\theta')]K_n(\mathbf{r}'',\mathbf{r}';T) \qquad (4.36)$$

where the partial propagator, corresponding to the n-winding number equivalence class, is

$$K_n(\mathbf{r}'',\mathbf{r}';T)=\frac{m\omega}{2\pi i\hbar\sin\omega T}\exp\left[\frac{im\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega T\right]$$

$$\cdot I_{\left|n+\frac{\alpha}{\hbar}\right|}\left(\frac{mr'r''}{i\hbar\sin\omega T}\right)$$

(4.37)

Using the Hille-Hardy formula[54]

$$\frac{t^{-\alpha'/2}}{1-t}\exp\left[-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right]I_{\alpha'}\left(\frac{2\sqrt{xyt}}{1-t}\right)$$

$$=\sum_{m=0}^{\infty}\frac{t^m m!\exp\left[-\frac{1}{2}(x+y)\right]}{\Gamma(m+\alpha'+1)}(xy)^{-\alpha'/2}L_m^{\alpha'}(x)L_m^{\alpha'}(y)$$

(4.38)

if we set

$$t = \exp\left(-2i\omega T\right), x = \frac{m\omega}{\hbar}r'^2 \quad , y = \frac{m\omega}{\hbar}r''^2 \quad , \alpha' = \frac{\alpha}{\hbar} + n$$

the partial propagator will become

$$K_n(\mathbf{r}'', \mathbf{r}'; T) = \sum_{N=0}^{\infty} \frac{m\omega}{\pi\hbar} \frac{\left[\exp\left(-2i\omega T\right)\right]^N N! \exp\left[-\frac{im\omega}{2\hbar}\left(r'^2 + r''^2\right)\right]}{\Gamma\left(N + \left|n + \frac{\alpha}{\hbar}\right| + 1\right)}$$

$$\cdot\left(\frac{m\omega}{\hbar}r'r''\right)^{\left|\frac{\alpha}{\hbar}+n\right|} \exp\left[-i\omega\left(\left|n + \frac{\alpha}{\hbar}\right| + 1\right)\right]$$

$$\cdot L_N^{\left|n+\frac{\alpha}{\hbar}\right|}\left(\frac{m\omega}{\hbar}r'^2\right) L_N^{\left|n+\frac{\alpha}{\hbar}\right|}\left(\frac{m\omega}{\hbar}r''^2\right)$$

(4.39)

Since the partial propagator can be expanded in the form of [30]

$$K_n(\mathbf{r}'', \mathbf{r}'; T) = \sum_{N=0}^{\infty} \exp\left(-\frac{i}{\hbar}E_N^n T\right) R_N^n(r') R_N^n(r'') \tag{4.40}$$

thus we can find that the radial eigenenergies are

$$E_N^n = \hbar\omega\left(2N + 1 + \left|n + \frac{\alpha}{\hbar}\right|\right) \qquad \text{N and n are integers} \tag{4.41}$$

and the radial eigenfunctions are

$$R_n(\mathbf{r}) = \sqrt{\frac{m\omega N!}{\pi\hbar\Gamma\left(N + \left|n + \frac{\alpha}{\hbar}\right| + 1\right)}} \sqrt{\frac{m\omega}{\hbar}r^2}^{\left|n+\frac{\alpha}{\hbar}\right|}$$

$$\cdot \exp\left[-\frac{im\omega}{2\hbar}r^2\right] L_N^{\left|n+\frac{\alpha}{\hbar}\right|}\left(\frac{m\omega}{\hbar}r^2\right)$$

(4.42)

where $\Gamma(x)$ is the gamma function,

$L_N^{(\alpha)}(x)$ is the Laguerre polynomials.

Moreover, it is easy to show that the eigenenergies of the center-of-mass part are [54]

$$E_L = \hbar\omega\left(2L + |l| + 1\right) \qquad \text{L and l are integers} \qquad (4.43)$$

Therefore, the total energies of two anyons in a harmonic potential are

$$E = \hbar\omega\left(2L + |l| + 2N + \left|n + \frac{\alpha}{\hbar}\right| + 2\right) \qquad (4.44)$$

Let us now go back to chapter 3, specifically Eq.(3.72). We can find that the statistical parameter α in here is slightly different from α in Eq. (3.72). Nevertheless, it can be seen readily that if the latter α is zero, we will have a bosonic particle but if it is π, we will have a fermionic particle. Hence in this section we can show easily that $\alpha/\hbar$ varies from zero to 1. This means that if α equals zero, the anyon will become a boson, but if α equal 1, it will become a fermion. In other words, the anyon interpolates between boson and fermion. Next, let us consider the possibilities for n, the winding number, Since in our problem, all anyons are indistinguishable, when two anyons are interchanged the system remains unchanged. In a mathematical sense, we can deduce the boundary condition [54]

$$\psi_{rel}\left(r, \pi + \theta\right) = \psi_{rel}\left(r, \theta\right) \qquad (4.45)$$

This boundary condition is very important since it also leads to the condition of eigenenergies. From Eq. (4.36) we can find that when θ is transformed to $\pi+\theta$, only the phase factor term $\exp(in\theta)$ is changed. That is

$$\exp(in\theta) \rightarrow \exp(in\theta + in\pi) \qquad (4.46)$$

Hence, it is very easy to see that to satisfy the boundary condition in Eq. (4.45), n has to be an even integer [54,55]. Let us now go back to Eq.(4.41), and then from the above discussion we can show that

$$E_{2j} = \hbar\omega\left(2j + 1 + \frac{\alpha}{\hbar}\right) \qquad ; j \geq 0$$

$$E_{2j-1} = \hbar\omega\left(2j + 1 - \frac{\alpha}{\hbar}\right) \qquad ; j > 0 \qquad (4.47)$$

where j can be both zero and a positive integer. Furthermore, from Eq.(4.47), it can be readily seen that the relative ground-state energy of two anyons in harmonic potential is

$$E_0 = \hbar\omega\left(1 + \frac{\alpha}{\hbar}\right) \qquad (4.48)$$

as shown in Fig.12

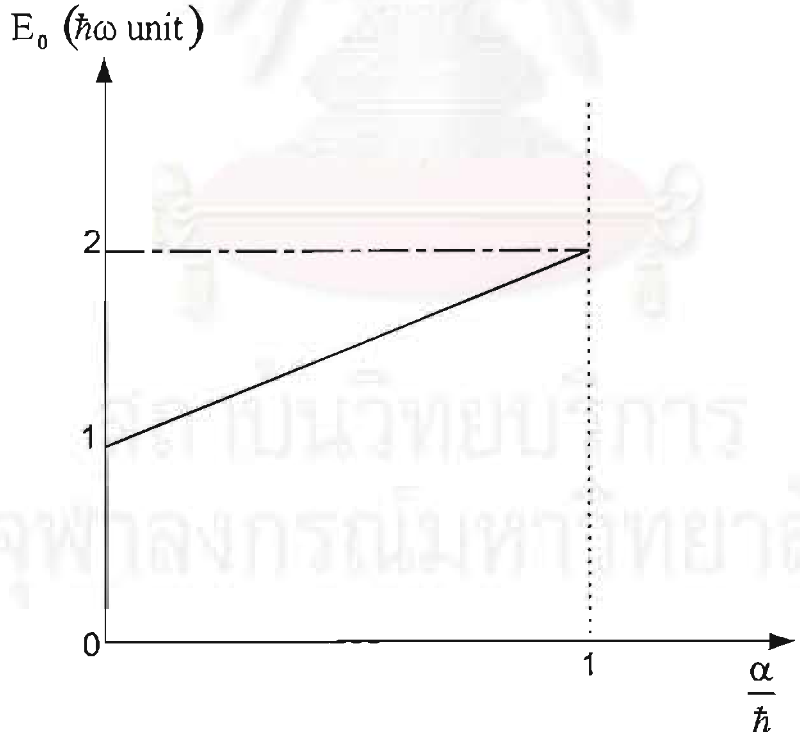

Fig.12 The relative ground-state energy of two anyons in a harmonic potential.

Moreover, it is easy to show that the relative energy can be transformed as shown in Fig. 13.

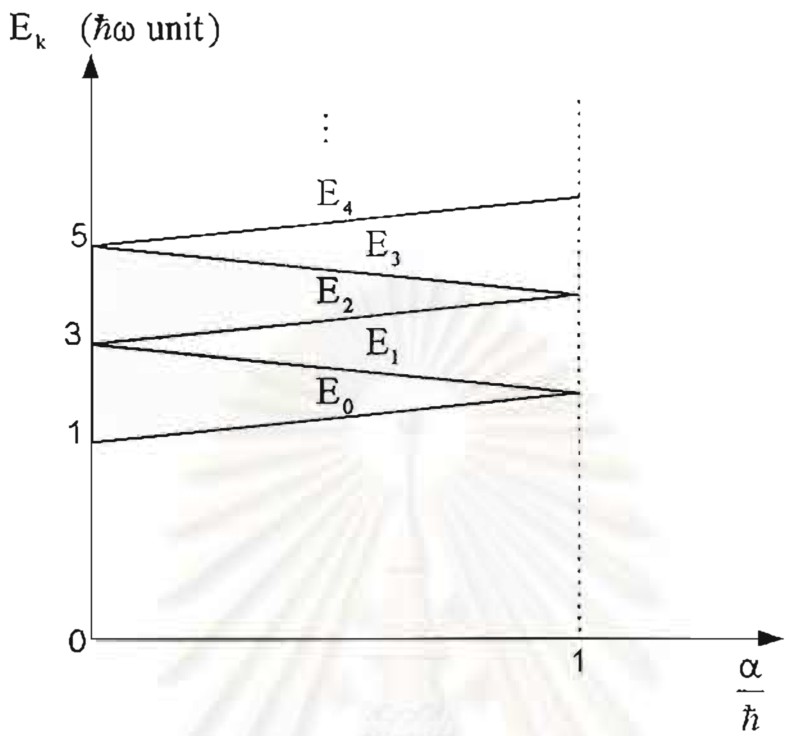

Fig.13 Relative energy of two anyons in a harmonic potential.

From the above figures, it is clear that the energy of an anyonic system can be interpolated continuously between that of bosonic and fermion system when $\alpha/\hbar$ varies from 0 to 1 [55]. This means that an anyon is a particle whose properties are intermediate between those of a boson and a fermion.

Discussion and Conclusion

Since several phenomena in modern condensed matter physics exist in two-dimensional space, properties of particles in the plane should be understood very well if physicists would like to comprehend those phenomena. As mentioned in preceding chapters, particles in two-dimensions have exotic properties, such as their spins and statistics, and since their spins and statistics can take on any values, they are called "anyons" [6]

The fractional quantum Hall effect (FQHE) [9,10] is the most important evidence of applicability of the anyonic concept. As stated in chapter 3, the quasiparticles in FQHE are anyons having fractional charge and fractional statistics. This result comes from the fact that FQHE is a phenomenon occurring in the plane. Furthermore, we can derive the generic filling factors represented in Eq. (3.121) by applying the anyonic idea. These filling factors were constructed first by Haldane [12] and he called his approach the hierarchy construction. However, this method can be understood very well when the anyonic concept is applied to discuss the mechanism as shown in chapter 3. Apart from FQHE, anyon superconductivity [23] is very important also, but, unfortunately, it is not presented in this thesis. It is crucial since it is expected that it can describe the high-temperature superconductivity. Therefore since it is very important in both FQHE and high-Tc superconductivity, the anyon should be studied in detail. In this thesis, we decided to study the ground - state energy of two – anyons in a harmonic potential using the path integration technique.

In our problem, the path integration technique is chosen because the configuration space of indistinguishable anyons is a multiply connected space. In ordinary quantum mechanics, it is clear that the multiply connectedness of the configuration space of our system can be neglected. However, in the path integral formulation it can be treated as shown in chapter 2. Introduced by Schulman [29] in 1968, the path integral for multiply connected space is very useful to solve problems connected with anyons. The difficulty of applying this approach is that we have to know the fundamental group of the configuration space of our system. The following problem is that the contribution of each equivalence class must be identified. In general it is very difficult to determine this contribution. However in our system it is seen clearly that the contribution is proportional to $\exp(in\theta)$ corresponding to one – dimensional representations of the braid group [46]. Therefore, the corresponding Lagrangian can be expressed as (modified from Eq. (3.40)),

$$L = \frac{1}{2}\sum_{I=1}^{N}\frac{1}{2}m\bar{v}_I^2 - \frac{\hbar\theta}{\pi}\frac{d}{dt}\left(\sum_{I<J}\phi_{IJ}(t)\right) \tag{4.49}$$

The structure of the braid group helps us understand why in the plane there are infinite kind of particles, On the other hand, the permutation group is required to understand particles in three or more dimensions, bosons and fermions. As mentioned in chapter 3, the permutation group is a subgroup of the braid group. It can be seen readily from the fact that the one–dimensional representations of the permutation group are 1 and –1, while those of the braid group are exp(iθ), which equals 1 if $\theta = 0$ and –1 if $\theta = \pi$.

Let us now go back to discuss the configuration space of N identical particles. It is well known that in d dimensions the configuration space M_N^d can be given by

$$M_N^d = \frac{\left(R^{dN} - \Delta\right)}{S_N} \tag{4.50}$$

As shown in chapter 3, it is found that in three or more dimensions the fundamental group of the configuration space, $\pi_1\left(M_N^d\right)$ is isomorphic to the permutation group S_N, while in two dimensions the fundamental group $\pi_1\left(M_N^2\right)$ is isomorphic to the braid group. Therefore, due to the topological structure of the configuration space as mentioned here, we can see clearly that in three dimensions there are only bosons and fermions while in two dimensions there are anyons that have infinitely many possible classes. The most important conclusion is that the infinite connectedness of the configuration space of identical anyons leads to infinite possibilities of quantum statistics in two dimensions.

In field theory, many physicists have shown how to solve the problem of the anyon by adding the Chern-Simons term into the ordinary Lagrangian. As shown is chapter 3, the corresponding Lagrangian is given by (cited from Eq, (3.72))

$$L=\sum_{I=1}^{N}\frac{1}{2}m\bar{v}_I^2-\frac{e^2}{2\pi c\kappa}\frac{d}{dt}\left(\sum_{I<J}\phi_{IJ}(t)\right) \tag{4.51}$$

Consequently, it can be easily seen that the Lagrangian in Eq. (4.51) is similar to that in Eq. (4.50). This means that adding the Chern–Simons term into the ordinary Lagrangian is equivalent to considering the connected property of the configuration space of the system. Moreover, adding the Chern–Simons term leads to the physical effect that makes the charged particle attach to the magnetic flux. As mentioned in the first section of chapter 3, the charged particle attached to the magnetic flux, called the flux–tube–charge composites, is the anyon. This means that adding the Chern–Simons term can transform an ordinary particle into an anyon. Therefore, in our problem, we use the Chern - Simons term to make two ordinary particles in a harmonic potential become two anyons in a harmonic potential.

It is well known that for the two–body problem its ground–state energy has to be the sum of single–particle–ground–state energy. For our problem, it should be expected that the ground–state energy should be [54]

$$E_0=2\hbar\omega \tag{4.52}$$

, but, actually, as shown in Eq. (4.44) the ground state of our problem is exactly

$$E_0 = \left(\frac{\alpha}{\hbar} + 2 \right) \hbar\omega \tag{4.53}$$

This means that the ground – state energy of two anyons in a harmonic potential does not equal the sum of two ground – state energies of an anyon in a harmonic potential. This result comes from the statistical interaction through the Chern – Simons term. Moreover, the two – body wavefunction of this problem is also different from the product of two wavefunctions of an anyon in a harmonic potential [54]. Moreover, considering Fig.12 and Fig.13 we can conclude that the energy level of anyons will interpolate between that of bosons and fermions, and depend on the statistical parameter, α.

Actually, this problem was solved by Wu [56] in 1984 and our result is similar to his result. However, in that paper he used the path integral only to formulate the general theory, but used the algebraic (operator) method to solve this problem. It is easy to evaluate this problem by using his method, but it is difficult to solve the problem by applying the path integration technique. Nevertheless, in our approach, we both formulate and evaluate the problem with the path integration technique. In conclusion, although we have used a different approach, the path integral method, we still obtain the same result as Wu [56].

Finally, from the preceding discussions, it can be deduced that when an anyonic system is considered, special treatment should be applied. The wavefunction of the many – body problem is not simply a product of single – particle wavefunctions, and the energy levels are also not the sum of single – particle energies. Until now, the system of three or more anyons does not have an exact solution, although Wu [55] would argue that he can solve the problem of three anyons in a harmonic potential. His result has a problem when the statistical parameter $\alpha = \pi\hbar$, equivalent to a fermion system, the ground – state energy of them does not equal the ground – state energy of a fermion system.

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APPENDICES

APPENDIX A

BERRY'S PHASE

The original idea of Berry's phase or Geometric phase started in the classic paper of Pancharatnam [57] in 1956, but it was well known in the beautiful paper of Berry [14] in 1984. In Berry's work the inherent universality and beauty of the geometric phase are presented. Moreover, Simon [58] pointed out the deep geometrical significance of Berry's phase as well.

The Berry's phase is what mathematician would call a U(1) holonomy, and the mathematical context for holonomy is the theory of fiber bundles [59]. The theory of fiber bundles is very important in gauge theory, but, unfortunately, it is not presented here. Nevertheless, it is contained in Nakahara's book [32] and, moreover, several topics of the geometric phase are presented in the book of Shapere and Wilczek [58]. In this part, many details are quoted from Sakurai's book [35].

Since in our approach we have to assume that the quantum mechanical system changes under adiabatic change, hence the meaning of the adiabatic change should be presented here. The adiabatic change is the change describing the response of the system in time to changes of the external parameter performed infinitesimally slowly. It can be deduced that certain features of the eigenfunctions remain unaltered. Such staple behaviors are called adiabatic invariants [60].

Let us now consider the Hamiltonian $H(\mathbf{R}(t))$ of the system with an external time-dependent parameter $\mathbf{R}(t)$. Consider the Schrodinger equation,

$$H(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle \tag{A.1}$$

where $|n(\mathbf{R}(t))\rangle$ is the ket of the nth normalized energy eigenstate corresponding to $\mathbf{R}(t)$. Next, we will try to change $H(\mathbf{R}(t))$ under adiabatic condition from $H(\mathbf{R}(t_0))$ to $H(\mathbf{R}(t))$ such that $\mathbf{R}(t_0) = \mathbf{R}_0$.

Consider the time-dependent Schrodinger equation

$$H(\mathbf{R}(t))|n(\mathbf{R}_0), t_0; t\rangle = i\hbar\frac{\partial}{\partial t}|n(\mathbf{R}_0), t_0; t\rangle \tag{A.2}$$

where $|n(\mathbf{R}_0), t_0; t\rangle$ would be proportional to the nth energy state $|n(\mathbf{R}(t))\rangle$ of $H(\mathbf{R}(t))$ at time t. Therefore, it can be represented as

$$|n(\mathbf{R}_0), t_0; t\rangle = \exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} E_n(\mathbf{R}(t))dt\right\}\exp(i\gamma_n(t))|n(\mathbf{R}(t))\rangle \tag{A.3}$$

If we replace Eq.(A.3) into Eq.(A.2), we will find the equation of γ_n being in the second phase term in Eq.(3). That is

$$\begin{aligned}\dot{\gamma}_n(t) &= i\langle n(\mathbf{R}(t))|\frac{\partial}{\partial t}|n(\mathbf{R}(t))\rangle \\ &= i\langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}} n(\mathbf{R}(t))\rangle\dot{\mathbf{R}}(t)\end{aligned} \tag{A.4}$$

The the geometric phase $\gamma_n(t)$ can be represented in the form of a path integral as

$$\begin{aligned}\gamma_n(t) &= i\int_C \langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}} n(\mathbf{R}(t))\rangle\dot{\mathbf{R}}(t)dt \\ &= i\int_C \langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}} n(\mathbf{R}(t))\rangle d\mathbf{R}\end{aligned} \tag{A.5}$$

where the path of integration C is that of the adiabatic change as the external parameter $\mathbf{R}$ change from $\mathbf{R}_0$ to $\mathbf{R}(t)$. From the normalization, $\langle n(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle = 1$, it is easy to show that

$$\mathrm{Re}\{\langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}} n(\mathbf{R}(t))\rangle\} = 0 \tag{A.6}$$

This means that the term $\langle n|\nabla n\rangle$ is pure imaginary. Although the practical evaluation here might be expected to be complicated because of the presence of $|\nabla_R n\rangle$ term, the geometric phase can be solved easily if we consider the closed loop; i.e. $\mathbf{R}(t_0)=\mathbf{R}(T)=\mathbf{R}_0$. Therefore, the Berry's phase

$$\gamma_n(C)=i\oint_C \langle n(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle d\mathbf{R} \tag{A.7}$$

Applying the Stoke's theorem, we obtain

$$\gamma_n(C)=-\iint_{S(C)} \mathbf{V}_n(\mathbf{R})\cdot ds \tag{A.8}$$

Where S(C) is a surface integral enclosed by the closed loop C, and

$$\begin{aligned}\mathbf{V}_n(\mathbf{R}) &= \mathrm{Im}\ \nabla_{\mathbf{R}}\times\langle n(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle\\ &= \mathrm{Im}\langle\nabla_R n(\mathbf{R})|\times|\nabla_R n(\mathbf{R})\rangle\\ &= \mathrm{Im}\sum\langle\nabla_R n(\mathbf{R})|m(\mathbf{R})\rangle\times\langle m(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle\end{aligned} \tag{A.9}$$

Where we have used Eq.(A.7) and the vector identity:

$$\nabla\times[f(\mathbf{x})\nabla g(\mathbf{x})]=[\nabla f(\mathbf{x})]\times[\nabla g(\mathbf{x})] \tag{A.10}$$

Next, if we multiply some phase factor $\exp[i\chi(\mathbf{R})]$ to $|n(\mathbf{R})\rangle$ it is easy to see that

$$\langle n(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle\rightarrow\langle n(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle+i\nabla\chi(\mathbf{R}) \tag{A.11}$$

Therefore, because of the identity $\nabla\times\nabla\chi=0$ we can conclude that $\mathbf{V}_n(\mathbf{R})$ does not change under such transformation. This means that the geometric phase is independent of choice of phase factor of eigenstates.

Multiplying $\langle m|$ form the left to the equation [35]

$$[\nabla_R H(\mathbf{R})]n(\mathbf{R})\rangle+H(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle=[\nabla_R E_n(\mathbf{R})]n(\mathbf{R})\rangle$$
$$+E_n(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle$$

(A.12)

obtained by differentiating Eq. (A.1) with respect to the external parameter $\mathbf{R}$, we have

$$\langle m(\mathbf{R})|\nabla_R n(\mathbf{R})\rangle = \frac{\langle m(\mathbf{R})|\nabla_R H(\mathbf{R})|n(\mathbf{R})\rangle}{E_n - E_m} \quad ; \ n \neq m \tag{A.13}$$

Hence, the integrand of the surface integral in Eq.(A.8) is expressed as

$$\mathbf{V}_n(\mathbf{R}) = \text{Im} \sum_{m \neq n} \frac{\langle n(\mathbf{R})|\nabla_R H(\mathbf{R})|m(\mathbf{R})\rangle \times \langle m(\mathbf{R})|\nabla_R H(\mathbf{R})|n(\mathbf{R})\rangle}{[E_m(\mathbf{R}) - E_n(\mathbf{R})]^2} \tag{A.14}$$

From Eq. (A.14) we can notice that $\mathbf{V}_n(\mathbf{R})$ passing through the surface S(C) is 2-form, and this 2-form is obtained from (A.14) by substituting ∇ by the exterior d and $\times$ by the wedge produce $\wedge$[32]. The validity of this generalization is congruous with the observation that in the three-dimensional version the geometric phase factor is independent of the choice [14]. Moreover we find that the knowledge of the state ket $|n(\mathbf{R})\rangle$ itself is usually unnecessary to evaluate $\mathbf{V}_n(\mathbf{R})$. Finally, let us go back to Eq.(A.3) and rewrite it as

$$|n(\mathbf{R}_0), t_0 = 0; T\rangle = \exp(i\gamma_n(C)) \exp\left\{-\frac{i}{\hbar}\int_0^T E_n(\mathbf{R}(t))dt\right\}|n(\mathbf{R}_0)\rangle \tag{A.15}$$

where

$$\gamma_n(C) = -\iint_{S(C)} \mathbf{V}_n(\mathbf{R}) \cdot d\mathbf{s} \tag{A.16}$$

In conclusion, $\gamma_n(C)$ is called the Berry's phase taking the form of Eq. (A.16) and coming from the adiabatic change of the external parameter.

APPENDIX B

HOMOTOPY THEORY: A BRIEF INTRODUCTION

This addendum is dedicated to the theory of the first homotopy group or the fundamental group, yet, unfortunately, because of the limitation of space to write it is sole a brief introductory level that is presented here. However, there are many books showing about this subject, especially the books involving the algebraic topology. Massey's book [61] and Nakahara's book [32] are highly recommended to read additionally to more details and advanced subjects. Furthermore, if solitary the basic topics concerning with path integrals are required, the Schulman's book [30] will be recommended. Finally, since this section states about mathematical subject involving to definitions and theorems, thus to preserve their meanings most of them are recited from the Nakahara's book [32]. Nevertheless, some proofs are neglected here.

Historically, the fundamental group was introduced by the great French mathematician Henri Poincare in 1895 while the higher-dimensional analogs of the fundamental group, called the homotopy groups, were presented in a series of four papers by Witold Hurewicz in 1935-1936 [61]. Although it was first proposed over a century, the fundamental group has been applied in physics, especially quantum theory, for few decades merely.

Because the fundamental group involves to loops, α, in the topological space, X [32], let us now start at the definition of paths and loops.

Definition B.1 Let X be a topological space and let $I = [0,1]$. A continuous map $\alpha : I \to X$ is called path with an initial point x_0 and an end point x_1, if α

(0) = x_0 and $\alpha(1) = x_1$. If $\alpha(0) = \alpha(1) = x_0$, the path is called a loop with base point x_0.

The crucial point is that the set of paths or loops in the topological space X may be endowed with an algebraic structure as follows. Before going beyond the special path, a constant path $c_x : I \to X$ defined by $c_x(s) = x$, $s \in I$, must be introduced first. Since $c_x(0) = c_x(1) = x$, thus it is loop also.

Definition B.2 Let $\alpha, \beta : I \to X$ be paths such that $\alpha(1) = \beta(0)$. The product of α and β, denoted by $\alpha * \beta$ is a path in X defined by

$$\begin{aligned} \alpha * \beta(s) &= \alpha(2s) \quad , 0 \le s \le 1/2 \\ &= \beta(2s-1) \quad , 1/2 \le s \le 1 \end{aligned} \tag{B.1}$$

This definition is seen clearly in Fig.14

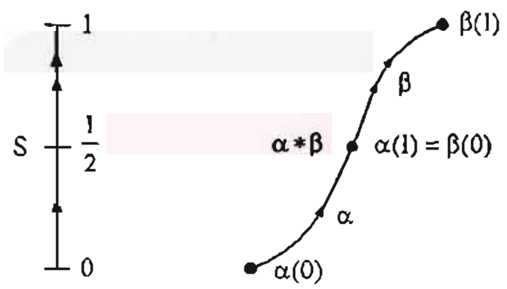

Fig. 14 The product $\alpha * \beta$ of paths α and β with a common end point [32].

Definition B.3 Let $\alpha : I \to X$ be a path from x_0 to x_1. The inverse path α^{-1} of α is defined by

$$\alpha^{-1}(s) = \alpha(1-s) \tag{B.2}$$

Let us now consider

$$\alpha * \alpha^{-1}(s) = \alpha(2s) \qquad ,0 \le s \le \tfrac{1}{2}$$
$$= \alpha^{-1}(2s-1) = \alpha(2-2s) \;, \tfrac{1}{2} \le s \le 1 \qquad \text{(B.3)}$$

It is clear that $\alpha*\alpha^{-1}$ does not equal to c_x. This means that a constant map c_x cannot be the unit element. In other words, we need a concept of homotopy to define a group operation in the space of loops. To form a group the equivalence relation, homotopic, is required. Therefore, let us now present the definition of homotopic relation as follows.

Definition B.4 Let $\alpha, \beta : I \rightarrow X$ be loops at x_0. They are said to be homotopic, written as $\alpha \sim \beta$, if there exists a continuous map $F : I \times I \rightarrow X$ such that

$$F(s,0) = \alpha(s)\ ,\ F(s,1) = \beta(s) \quad \text{for all } s \in I \qquad \text{(B.4a)}$$

$$F(0,t) = F(1,t) = x_0 \qquad \text{for all } t \in I \qquad \text{(B.4b)}$$

The connecting map F is called a homotopy between α and β.

The significant aspect is that the homotopic relation is an equivalence relation. To see more clearly, its proof should be shown here. Since an equivalence relation, ~, must satisfy the reflective, symmetric, and transitive requirements [32], thus we have to shown now that

(1) Reflectivity: $\alpha \sim \alpha$.

Let $F(s,t) \equiv \alpha(s)$ for any $t \in I$, hence

$F(s,0) = F(s,1) = \alpha(s)$ and

$F(0,t) = \alpha(0) = \alpha(1) = F(1,t)$.

Therefore, it is easy to see that $\alpha \sim \alpha$.

(2) Symmetry: If $\alpha \sim \beta$, then $\beta \sim \alpha$.

Define $F(s,t)$ such that $\alpha \sim \beta$, and then consider $F(s,1-t)$ that we can find that

$$F(s,0) = \beta(s) \quad , \quad F(s,1) = \alpha(s) \text{ and}$$
$$F(0,1-t) = F(1,1-t) = x_0 .$$

Thus, we can deduce that if $\alpha \sim \beta$, then $\beta \sim \alpha$.

(3) Transitivity: If $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

Let $F(s,t)$ is a homotopy between α and β and $G(s,t)$ is a homotopy between β and γ as shown in Fig. 15. From the picture it can be seen that a homotopy between α and γ may be (Fig. 15)

$$\begin{aligned} H(s,t) &= F(s,2t) \quad , \quad 0 \le t \le 1/2 \\ &= G(s,2t-1) \quad , \quad 1/2 \le t \le 1 \end{aligned} \tag{B.5}$$

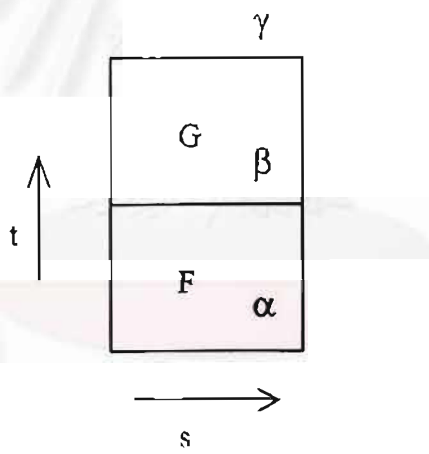

Fig. 15 A homotopy H between α and γ via β [32].

Next, the equivalence class of loops via the homotopic relation, $\sim$, is introduced and it is denoted by $[\alpha]$. Moreover, since it concerns with the homotopic, it is called the homotopy class of α.

Definition B.5 Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$ is denoted by $\pi_1(X, x_0)$ and called the fundamental group (or the first homotopy group) of X at x_0. The product of homotopy classes $[\alpha]$ and $[\beta]$ is defined by

$$[\alpha] * [\beta] = [\alpha * \beta] \tag{B.6}$$

The important consequence of the definition of the product is that it is independent of the representative, that is, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha * \beta \sim \alpha' * \beta'$.

Theorem The fundamental group is a group. Specifically, if α, β,... are loops at $x \in X$, the following group properties are satisfied.

(1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$

(2) $[\alpha] * [c_x] = [\alpha]$ and $[c_x] * [\alpha] = [\alpha]$

(3) $[\alpha] * [\alpha^{-1}] = [c_x]$, hence $[\alpha]^{-1} = [\alpha^{-1}]$

This means that $\pi_1(X, x)$ is a group whose unit element is the homotopy class of the constant map c_X. Moreover, the inverse of the homotopy class $[\alpha]$ is defined by $[\alpha]^{-1} = [\alpha^{-1}]$. Unfortunately, the proof of the theorem is not presented here.

Before we will go to the general properties of the fundamental group, we should go back to necessary definition in elementary topology, the connectedness.

Definition B.6

(1) A topological space X is connected if it can be written as $X = X_1 \cup X_2$ where X_1 and X_2 are both open and $X_1 \cap X_2 = \varnothing$. Otherwise X is called disconnected.

(2) A topological space X is called arcwise connected if, for any points $x, y \in X$, there exists a continuous map $f:[0,1] \to X$ such that $f(0)=x$ and $f(1)=y$. With a few pathological exceptions, arcwise connectedness is practically equivalent to connectedness.

(4) A loop in a topological space X is a continuous map $f:[0,1] \to X$ such that $f(0)=f(1)$. If any loop in X can be continuously shrunk to a point, X is called simply connected.

From the definition of the connectedness and the fundamental group, it can be proved that if X be an arcwise-connected topological space and $x_0, x_1 \in X$, then $\pi_1(X, x)$. For the definition of isomorphic term, it is found in all topology books.

As mentioned above, we define the homotopic equivalence of only paths and loops, but, however, it can be generalized to arbitrary maps as follows.

Definition B.7 Let $f, g: Y \to X$ be continuous maps. If there exists a continuous map $F: X \times I \to Y$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$, f is said to be homotopic to g, denoted by $f \sim g$. The map F is called a homotopy between f and g.

Definition B.8 Let X and Y be topological spaces. X and Y are of the same homotopy type, written as $X \approx Y$, if there exist continuous maps $f: X \to Y$ and $g: Y \to X$such that for the identity maps $id_X$and id_Y [32], $fg \sim id_Y$ and $gf \sim id_X$. F is called the homotopy equivalence andg, its homotopy inverse.

From these definitions, we will have the theorem about isomorphism between two topological spaces as follows.

Theorem Let X and Y be topological spaces of the same homotopy type, expressed as $X \approx Y$. If $f : X \to Y$ is a homotopy equivalence, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, f(x_0))$.

This means that two topological spaces of the same homotopy type have the same fundamental group is invariant under homeomorphisms, and hence is a topological invariant. Unfortunately, it is difficult to find what is meant by of the same homotopy type. Thus, we have to consider a continuous deformation from X to Y, a subspace of X.

Definition B.9 Let $R(\neq \varnothing)$ be a subspace of X. If there exists a continuous map $f : X \to R$ such that for the restriction of $R \subset X$, $f|_R = id_R$, R is called a retract of X and f a retraction.

Definition B.10 Let R be a subspace of X. If there exists a continuous map $H: X \times I \to X$ such that

$$H(x,0) = x \ , \ H(x,1) \in R \quad \text{for any } x \in X \tag{B.7a}$$

$$H(x,t) = x \quad \text{for any } t \in I \text{ and any } x \in R \tag{B.7b}$$

R is said to be a deformation retract of X, Note that H is a homotopy between id_X and a retraction $f : X \to R$, which leaves all the points in R fixed during deformation.

Finally, the definition of the contractible space will be presented here.

Definition B.11 If a point $a \in X$ is a deformation retract of X, X is said to be contractible and the corresponding homotopy H is called the contraction.

Note that if an arcwice connected space X has a trivial fundamental group, X is said to be simply connected. Moreover, it is easy to prove that the fundamental group of a contractible space is trivial, $\pi_1(X, x_0)$ is isomorphic to $\{e\}$.

CURRICULUM VITAE

Mr. Weerachart Kilenthong was born on August 5, 1976 in Ubonratchathani. He received his B. Eng. degree in civil engineering (1st class honours, gold medal) from Chulalongkorn University in 1998. He has attended to Vietnam School on Theoretical Physics in 1999 as well. During his study for a M. Sc. degree in physics at Chulalongkorn University, he has been granted partially by Thailand Research Fund (TRF).