CHAPTER VII

RELATIONSHIPS BETWEEN DIRECT SUMS AND IRREDUNDANT UNIONS OF GROUPS

1 Introduction.

In this final chapter, we want to know that if a group G is the irredundant union of its subgroups $\{ G_\alpha \,/\, \alpha \in I \}$, then is G the direct sum of the $\{ G_\alpha \,/\, \alpha \in I \}$? The negative answer to this question is supplied by the following theorem:

2 Relationships between Direct Sums and Irredundant Unions of Groups.

2.1 Theorem. If an additive abelian group $G = \bigcup_{\alpha \in I} G_\alpha$ is the irredundant union of its subgroups G_α, then G can not be the direct sum of G_α .

Proof. Suppose that $G = \bigoplus_{\alpha \in I} G_\alpha$. Let β be in I. Since $G'_\beta = G_\beta \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha \neq \emptyset$, there exists g'_β belonging to G'_β . Let g be in $\bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha$ such that $g \neq o$. If $g'_\beta + g$ is in $\bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha$, then $(g'_\beta + g) - g = g'_\beta$ must belong to $\left[\bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha \right]$. Then g'_β is in $G_\beta \cap \left[\bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha \right] = \{o\}$, which implies that $g'_\beta = o$, which contradicts the choice of g'_β . Similarly, if $g'_\beta + g$ is in G_β , then $-g'_\beta + (g'_\beta + g) = g$ is in G_β, so that g must be o, which contradicts the choice of g.

Hence the theorem is proved.

2.2 <u>Remark</u>. However, an abelian group $G = \bigcup_{\alpha \in I} G_\alpha$ which is the irredundant union of its subgroups G_α can be a direct sum of other subgroups. For example, the Euclidean plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, as a vector space over $\mathbb{R}$, is a disjoint irredundant union of all 1-dimensional subspaces; at the same time, $\mathbb{R}^2$ is a direct sum of some two 1-dimensional subspaces.

EPILOGUE

Consider the following two questions:

(1) For each positive integer n, does there always exist a group which is an irredundant union of n of its subgroups ?

(2) If a group G is the irredundant union of n of its subgroups G_j, how do the structures of the G_j related to that of G ?

In this thesis, the question 1 is settled completely, in case n = 2, 3 , 2^m-1 for any integer $m>2$ and p+1 for any prime p. However, we do not know the answer to the question 1 in other cases.

We did very little with the question 2 (see Chapter IV). In general, not much is known about question 2.

Finally, we hope that this thesis will stir some activities in the research for answer to both of these questions; particular question 2 and a related question in which the union is replaced by the coset multiplication.