CHAPTER IV

A SUBHARMONIC FUNCTION ON A CYLINDER

We study the boundary behavior of a subharmonic function on a cylinder.

Let $x = (x_1,\ldots,x_n) \in R^n$ $(n \geq 3)$. A system of cylindrical coordinates for x is given by

$$\rho^2 = x_1^2 + \ldots + x_{n-1}^2$$

$$x_i = \rho \sin\theta_1 \ldots \sin\theta_{i-1}\cos\theta_i \quad (i = 1,2,\ldots, n-2)$$

(4-1) $$x_{n-1} = \rho \sin\theta_1 \ldots \sin\theta_{n-2}$$

where $0 \leq \theta_{n-2} \leq 2\pi$; $0 \leq \theta_i \leq \pi$ $(i = 1,\ldots, n-3)$.

It can be shown, under the transformation (4-1), that

$$\rho^{n-2}\sin^{n-3}\theta_1\ldots\sin\theta_{n-3} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

$$= \frac{\partial}{\partial\rho}(\rho^{n-2}\sin^{n-3}\theta_1\ldots\sin\theta_{n-3}\frac{\partial}{\partial\rho}) + \frac{\partial}{\partial\theta_1}(\rho^{n-4}\sin^{n-3}\theta_1\ldots\sin\theta_{n-3}\frac{\partial}{\partial\theta_1})$$

$$+ \sum_{i=2}^{n-3} \frac{\partial}{\partial\theta_i}(\rho^{n-4}\sin^{n-5}\theta_1\ldots\sin^{n-i-3}\theta_{i-1}\sin^{n-i-2}\theta_i\ldots\sin\theta_{n-3}\frac{\partial}{\partial\theta_i})$$

$$+ \frac{\partial}{\partial\theta_{n-2}}(\rho^{n-4}\sin^{n-5}\theta_1\ldots\sin\theta_{n-5}(\sin\theta_{n-3})^{-1}\frac{\partial}{\partial\theta_{n-2}})$$

$$+ \frac{\partial}{\partial x_n}(\rho^{n-2}\sin^{n-3}\theta_1\ldots\sin\theta_{n-3}\frac{\partial}{\partial x_n}) \qquad (4\text{-}2).$$

If a is positive real number and $n \geq 3$, we let $C(a)$ denote the upper half cylinder

$$C(a) = \{x \in R^n \mid 0 \leq \rho < \frac{\pi}{2a} \text{ and } x_n > 0\}.$$

If $n = 2$, we let $C(a)$ be the upper half strip.

$$C(a) = \{x \in R^2 \mid -\frac{\pi}{2a} < x_1 < \frac{\pi}{2a} \text{ and } x_2 > 0\}.$$

Let S be a function defined on $C(a)$, we define

$$M(b) = \sup \{S(x) \mid x_n = b, \ x \in C(a)\} \qquad (b > 0)$$

and

$$M(b)^+ = \max \{M(b), 0\}.$$

We first study a special form of the Bessel function. The Bessel's differential equation

$$(4\text{-}3) \qquad x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

has a particular solution which is denoted by $J_\nu(x)$

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+\nu}}{2^{2m+\nu}\, m!\, \Gamma(\nu+m+1)}$$

where ν is real and

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt \qquad (\alpha > 0).$$

By integration by parts we obtain

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha}\, dt = -e^{-t} t^{\alpha} \Big|_0^\infty + \alpha \int_0^\infty e^{-t} t^{\alpha-1}\, dt = \alpha\, \Gamma(\alpha).$$

$J_\nu(x)$ is called the Bessel function of the first kind of order ν. We consider the differential equation

$$(4\text{-}4) \qquad xy'' + (1+2k)y' + \lambda^2 xy = 0.$$

It can be reduced to Bessel's defferential equation by letting

$y = \frac{u}{x^k}$ $(x \neq 0)$.

$$\frac{dy}{dx} = \frac{x^k \frac{du}{dx} - kx^{k-1}u}{x^{2k}} = \frac{u'}{x^k} - \frac{ku}{x^{k+1}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{u'}{x^k} - \frac{ku}{x^{k+1}}\right)$$

$$= \frac{x^k u'' - kx^{k-1}u'}{x^{2k}} - \frac{ku'x^{k+1} - (k+1)x^k \cdot ku}{x^{2k+2}}$$

$$= \frac{u''}{x^k} - \frac{2ku'}{x^{k+1}} + \frac{(k+1)ku}{x^{k+2}}$$

Substitute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (4-4) we obtain

$$x\left(\frac{u''}{x^k} - \frac{2ku'}{x^{k+1}} + \frac{(k+1)ku}{x^{k+2}}\right) + (1+2k)\left(\frac{u'}{x^k} - \frac{ku}{x^{k+1}}\right) + \lambda^2 x \frac{u}{x^k} = 0$$

$$\frac{u''}{x^{k-1}} - \frac{2ku'}{x^k} + \frac{(k+1)ku}{x^{k+1}} + (1+2k)\frac{u'}{x^k} - \frac{(1+2k)ku}{x^{k+1}} + \frac{\lambda^2 u}{x^{k-1}} = 0$$

$$\frac{u''}{x^{k-1}} + \frac{u'}{x^k} - \frac{k^2u}{x^{k+1}} + \frac{\lambda^2 u}{x^{k-1}} = 0$$

$$x^2u'' + xu' - k^2u + \lambda^2x^2u = 0$$

$$x^2u'' + xu' + (\lambda^2x^2 - k^2)u = 0$$

This equation has a particular solution $J_k(\lambda x)$. Therefore

$$u = J_k(\lambda x)$$

$$yx^k = J_k(\lambda x)$$

$$y = \frac{J_k(\lambda x)}{x^k}$$

From this we can conclude that $\frac{J_k(\lambda x)}{x^k}$ satisfies

$$(4\text{-}5) \qquad x\left(\frac{J_k(\lambda x)}{x^k}\right)'' + (1+2k)\left(\frac{J_k(\lambda x)}{x^k}\right)' + \lambda^2 x\left(\frac{J_k(\lambda x)}{x^k}\right) = 0.$$

4.2 Lemma. Let $-\infty < a_0 < b_0 < +\infty$. Suppose $\phi \in C^1([a_0,b_0]$ such that $\phi(a_0) = 0$ and $\phi > 0$ in (a_0, b_0). Assume that u and v belong to $C^2([a_0,b_0])$, $v(a_0) > 0$, $u(a_0) \neq 0$ and $u(c) = 0$ for some $c \in (a_0,b_0)$. If u and v satisfy

$$(4\text{-}6) \qquad (\phi(t)u'(t))' + p\phi(t)u(t) = 0$$

$$(4\text{-}7) \qquad (\phi(t)v'(t))' + p\phi(t)v(t) < 0$$

whenever $a_0 < t < b_0$ where p is a real number, then v vanishes at least once in (a_0,c).

Proof : Take c to be the smallest number larger than a_0 such that $u(c) = 0$. Suppose that $v(t) \neq 0$ for any $t \in (a_0,c)$. Then by the hypothesis that $v(a_0) > 0$ and v is continuous, we get $v > 0$ on (a_0,c). We may assume that $u > 0$ on (a_0,c), by multiplying, if necessary u and the equation (4-6) by -1, and we then have that $u'(c) \leq 0$. By multiplying (4-6) by v, (4-7) by u and subtracting, we get

$$[\emptyset(t)(u'(t)v(t)-v'(t)u(t))]' > 0.$$

Integration yields

$$(4\text{-}8) \qquad [\emptyset(t)(u'(t)v(t)-v'(t)u(t))]_{a_0}^{c} > 0.$$

But

$$[\emptyset(t)(u'(t)v(t)-v'(t)u\ (t))]_{a_0}^{c} = \emptyset(c)u'(c)v(c)$$

$$\leq 0$$

since $\emptyset(c) > 0$, $u'(c) \leq 0$ and $v(c) \geq 0$. This contradicts (4-8). Therefore v vanishes at least once in (a_0,c).

4.3 Theorem. Let $a > 0$. Then the function $t \to \frac{J_\nu(\lambda t)}{t^\nu}$ never vanishes in $(0, \frac{\pi}{2a}]$. Moreover, $\inf_{0<t\leq\frac{\pi}{2a}} \frac{J_\nu(\lambda t)}{t^\nu} > 0$ where $\nu = \frac{n-3}{2}$, $\lambda = \sqrt{n-1}\ a$. $(n \geq 3)$.

Proof : We can show that the function $\frac{a}{t}\tan(at)$ is creasing in $(0,\frac{\pi}{2a})$ and

$$\lim_{t \to 0^+} \frac{a}{t}\tan(at) = a^2$$

Therefore $\frac{a}{t}\tan(at) > a^2$ $(0 < t < \frac{\pi}{2a})$. We consider that

$$\begin{aligned}(t^{n-2}(\cos(at))')' + t^{n-2}\lambda^2\cos(at) &= -t^{n-2}a^2\cos(at) - (n-2)t^{n-3}a\sin(at) \\ &\quad + \lambda^2 t^{n-2}\cos(at) \\ &= t^{n-2}\cos(at)\{\lambda^2 - a^2 - (n-2)\tfrac{a}{t}\tan(at)\} \\ &= t^{n-2}\cos(at)\{(n-2)a^2 - (n-2)\tfrac{a}{t}\tan(at)\} \\ &< 0\end{aligned}$$

(4-9) $$(t^{n-2}(\cos(at)')' + t^{n-2}\lambda^2\cos(at) < 0$$

where $\lambda^2 = (n-1)a^2$ and the last inequality follows from the fact that $\frac{a}{t}\tan(at) > a^2$. The function $\frac{J_\nu(\lambda t)}{t^\nu}$ satisfies (4-5) and for $\nu = \frac{n-3}{2}$ we get

(4-10) $$\left(\frac{J_\nu(\lambda t)}{t^\nu}\right)'' + \frac{n-2}{t}\left(\frac{J_\nu(\lambda t)}{t^\nu}\right)' + \lambda^2\left(\frac{J_\nu(\lambda t)}{t^\nu}\right) = 0.$$

Multiplying (4-10) by t^{n-2} and rearranging we obtain

(4-11) $$\left(t^{n-2}\left(\frac{J_\nu(\lambda t)}{t^\nu}\right)'\right)' + t^{n-2}\lambda^2\left(\frac{J_\nu(\lambda t)}{t^\nu}\right) = 0.$$

By (4-9),(4-11) and Lemma 4.2 for $a_0 = 0$, $b_0 = \frac{\pi}{2a}$, $\phi(t) = t^{n-2}$, $p = \lambda^2$ and $v(t) = \cos(at)$ we get that $\frac{J_\nu(\lambda t)}{t^\nu} \neq 0$ for any t in $(0,\frac{\pi}{2a}]$ since $v(t)$ never vanishes in $(0,\frac{\pi}{2a})$. Since

$$\lim_{t \to 0^+} \frac{J_\nu(\lambda t)}{t^\nu} = \frac{\lambda^\nu}{2^\nu \Gamma(\nu+1)},$$

it follows that $\inf\limits_{0<t\leq\frac{\pi}{2a}} \frac{J_\nu(\lambda t)}{t^\nu} > 0$. This completes the proof.

For each $b > 0$ and $n \geq 2$, we introduce

$$D(b) = C(a) \cap \{x \in R^n | 0 < x_n < b\}$$

and

$$E(b) = C(a) \cap \{x \in R^n | x_n = b > 0\}.$$

Therefore by Theorem 3.14, there is a generalized Dirichlet solution, denoted by h_b, for the indicator function $\chi_{E(b)}$ of a subset E(b) of $\partial D(b)$. We note that $0 \leq h_b \leq 1$. The following theorem indicates the growth of $h_b(x)$ with respect to b.

4.4 Theorem. Let $a > 0$. If we fix each point x in $C(a)$, then there exist positive numbers A and B such that

$$h_b(x) \leq A\, e^{-\lambda b} \qquad (b > B)$$

where $\lambda = \sqrt{n-1}\, a$, $(n \geq 2)$.

Proof : First we prove for the case $n = 2$. We represent a point in R^2 as a complex number in $\mathbb{C}$ and recall that an analytic function f such that $f' \neq 0$ on a domain $D \subset \mathbb{C}$ is called a conformal mapping. It is well known that every harmonic function of u, v transforms into a harmonic function of x, y under the change of variables

$$u + iv = f(x + iy)$$

where f is analytic function. To see this, let H denote any harmonic function of u and v. By this transformation $H(u,v)$ is transformed into a function $H(u(x,y), v(x,y))$. Let

$$H^*(x,y) = H(u(x,y), v(x,y)).$$

By differentiation we have

$$\frac{\partial H^*}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial H^*}{\partial y} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 H^*}{\partial x^2} = \frac{\partial H}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[\frac{\partial^2 H}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 H}{\partial v \partial u}\frac{\partial v}{\partial x}\right] + \frac{\partial H}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\left[\frac{\partial^2 H}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 H}{\partial v^2}\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2 H^*}{\partial y^2} = \frac{\partial H}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\left[\frac{\partial^2 H}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 H}{\partial v \partial u}\frac{\partial v}{\partial y}\right] + \frac{\partial H}{\partial v}\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}\left[\frac{\partial^2 H}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 H}{\partial v^2}\frac{\partial v}{\partial y}\right].$$

Adding,

$$\frac{\partial^2 H^*}{\partial x^2} + \frac{\partial^2 H^*}{\partial y^2} = \frac{\partial H}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial H}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 H}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$+ 2\frac{\partial^2 H}{\partial u \partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] + \frac{\partial^2 H}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right].$$

Since u, v and H are harmonic, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and

$\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2} = 0$. Also, by the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Therefore

$$\frac{\partial^2 H^*}{\partial x^2} + \frac{\partial^2 H^*}{\partial y^2} = 0$$

and H* is a harmonic function of x and y.

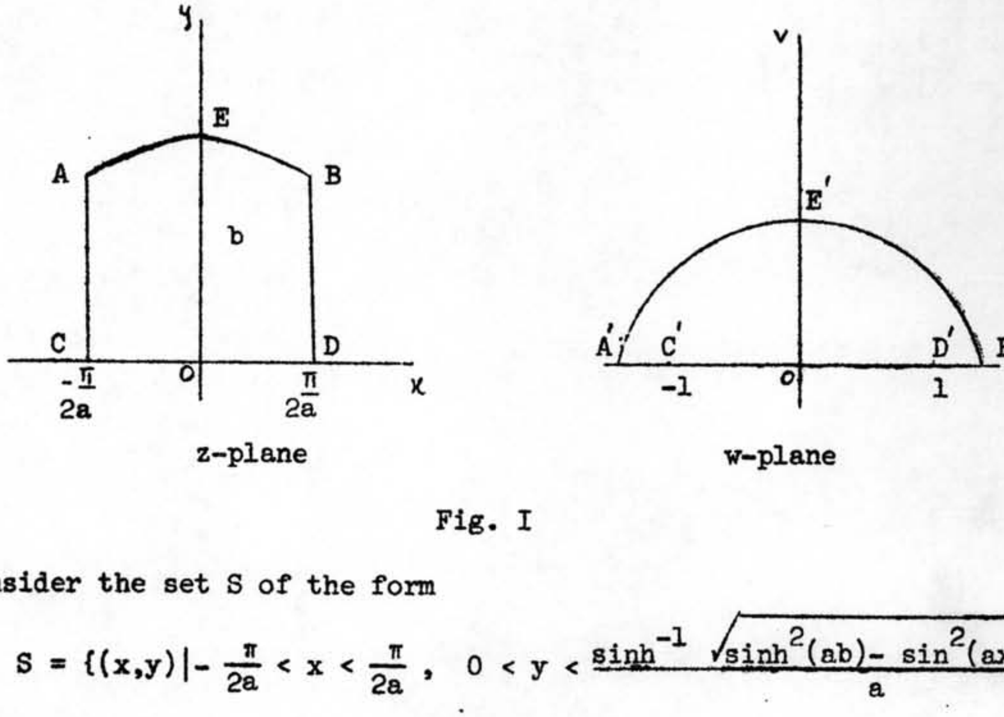

Fig. I

Consider the set S of the form

$$S = \{(x,y) \mid -\frac{\pi}{2a} < x < \frac{\pi}{2a},\ 0 < y < \frac{\sinh^{-1}\sqrt{\sinh^2(ab) - \sin^2(ax)}}{a}\}.$$

We want to find the harmonic function H which satisfies the following boundary conditions :

$$H(-\frac{\pi}{2a}, y) = H(\frac{\pi}{2a}, y) = 0 \qquad (0<y<\frac{\sinh^{-1}\sqrt{\sinh^2(ab)-\sin^2(ax)}}{a})$$

$$H(x, \frac{\sinh^{-1}\sqrt{\sinh^2(ab)-\sin^2(ax)}}{a}) = 1 \quad (-\frac{\pi}{2a} < x < \frac{\pi}{2a}).$$

and $H(x,0) = 0 \qquad (-\frac{\pi}{2a} < x < \frac{\pi}{2a})$.

To do this we transform S on the z-plane into the w-plane by the conformal mapping $w = \sin(az)$. As indicated in Fig.I, the image of the base of S is the segment of the u-axis between the point $u = -1$ and $u = 1$, the image of the side AC and BD are the segments A'C' and B'D' on the u-axis respectively, and the image of the curve AEB is

$A'E'B' = \{(u,v) \in \text{w-plane} \mid u^2 + v^2 = \sin h^2(ab)\}$. By interchanging the variables u and v to polar coordinates, we can write

$$u = r\cos\theta, \quad v = r\sin\theta$$

and $A'E'B' = \{(u,v) \mid r = \sin h\,(ab),\ 0 \le \theta \le \pi\}$. A harmonic function H of r and θ on the semicircle center at 0 with radius $R = \sinh(ab) > 1$ is

$$(4\text{-}12) \qquad H(r, \theta) = \frac{2}{\pi}\tan^{-1}\frac{2Rr\sin\theta}{R^2 - r^2} .$$

H is zero on A'B' and unity on the curve A'E'B'. Changing to the coordinates x and y by means of the transformation

$$(4\text{-}13) \qquad w = \sin(az) = \sin(ax+iay)$$

$$= \sin(ax)\cosh(ay) + i\cos(ax)\sinh(ay),$$

we find that $u = \sin(ax)\cosh(ay)$, $v = \cos(ax)\sinh(ay)$. Therefore

$$r\cos\theta = \sin(ax)\cosh(ay)$$

$$r\sin\theta = \cos(ax)\sinh(ay)$$

and $r = \sqrt{\sinh^2(ay)+\sin^2(ax)}$, $\theta = \tan^{-1}(\frac{\cos(ax)\sinh(ay)}{\sin(ax)\cosh(ay)}) = \tan^{-1}(\cot(ax)\tanh(ay))$.

The function H given by equation (4-12) therefore becomes

$$H^*(x,y) = \frac{2}{\pi}\tan^{-1}\frac{2R\sqrt{\sinh^2(ay)+\sin^2(ax)}\sin(\tan^{-1}(\cot(ax)\tanh(ay))}{R^2-(\sinh^2(ay)+\sin^2(ax))}.$$

Since H is harmonic on the semicircle, H* must be a harmonic function of x and y in S. The boundary conditions for the two functions must be the same on corresponding parts of the boundaries. Fixing (x,y) in S we see that

$$\lim_{R\to\infty} RH^*(x,y) = \lim_{R\to\infty} R.\frac{2}{\pi}\tan^{-1}\frac{2R\sqrt{\sinh^2(ay)+\sin^2(ax)}\sin(\tan^{-1}(\cot(ax)\tanh(ay)))}{R^2-(\sinh^2(ay)+\sin^2(ax))}$$

$$= \lim_{t\to 0}\frac{1}{t}.\frac{2}{\pi}\tan^{-1}\frac{2t\sqrt{\sinh^2(ay)+\sin^2(ax)}\sin(\tan^{-1}(\cot(ax)\tanh(ay)))}{1-t^2(\sinh^2(ay)+\sin^2(ax))}$$

$$= \frac{4}{\pi}\sqrt{\sinh^2(ay)+\sin^2(ax)}\ \ \sin(\tan^{-1}(\cot(ax)\tanh(ay)))$$

where the last equality is obtained by using L' Hôpital rule. This together with the fact that

$$\lim_{b\to\infty}\frac{\sinh(ab)}{e^{ab}} = \frac{1}{2},$$

we can find a constant A such that

$$H^*(x,y) \leq Ae^{-ab} \qquad (b > B)$$

Since h_b is the generalized Dirichlet solution for $\chi_{E(b)}$, $h_b - H^*$ is harmonic on S and $h_b - H^* \leq 0$ on ∂S. Since $h_b - H^*$ satisfies the maximum principle on S, $h_b - H^*$ cannot attain its supremum on S. Since $\bar{S}$ is compact, $h_b - H^*$ attains its supremum on $\bar{S}$, in fact, on ∂S. Therefore $h_b - H^* \leq 0$ on S. Fixing $(x,y) \in S \subset D(b)$, we have

$$h_b(x,y) \leq H^*(x,y).$$

Therefore for sufficiently large b we obtain

$$h_b(x,y) \leq A e^{-ab}$$

where A is a constant. This completes the proof for the case n = 2.

Next we prove for the case $n \geq 3$, let

$$F_\nu(\lambda\rho) = \begin{cases} \dfrac{\lambda^\nu}{2^\nu \Gamma(\nu+1)} & \text{if } \rho = 0 \\[2ex] \dfrac{J_\nu(\lambda\rho)}{\rho^\nu} & \text{if } 0 < \rho \leq \dfrac{\pi}{2a} \end{cases}$$

where $\nu = \frac{n-3}{2}$, $\lambda = \sqrt{n-1}\, a$. Thus by Theorem 4.3, $F_\nu(\lambda\rho)$ is positive in $[0, \frac{\pi}{2a}]$ and let

$$K = \min_{0 \leq \rho \leq \frac{\pi}{2a}} F_\nu(\lambda\rho) > 0 .$$

We define a function H_b by

$$H_b(x) = \frac{\sinh(\lambda x_n)\, F_\nu(\lambda\rho)}{K \sinh(\lambda b)} .$$

For $\rho > 0$, we consider that

$$\Delta H_b(x) = \frac{1}{K\sinh(\lambda b)}\left[\frac{n-2}{\rho}\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)'\sinh(\lambda x_n)+\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)''\sinh(\lambda x_n)\right.$$

$$\left.+\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)(\sinh(\lambda x_n))''\right]$$

$$= \frac{\sinh(\lambda x_n)J_\nu(\lambda\rho)}{K\rho^\nu\sinh(\lambda b)}\left[\frac{\rho^\nu}{J_\nu(\lambda\rho)}\left(\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)''+\frac{n-2}{\rho}\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)'\right)+\left(\frac{\sinh(\lambda x_n)''}{\sinh(\lambda x_n)}\right)\right]$$

$$= H_b(x)\left[\frac{\rho^\nu}{J_\nu(\lambda\rho)}\left(\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)''+\frac{n-2}{\rho}\left(\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)'+\lambda^2\frac{J_\nu(\lambda\rho)}{\rho^\nu}-\lambda^2\frac{J_\nu(\lambda\rho)}{\rho^\nu}\right)+\lambda^2\right]$$

$$= 0$$

where the first equality is obtained by using (4-2) and the last equality follows from the fact that $\frac{J_\nu(\lambda\rho)}{\rho^\nu}$ satisfies

$$u''(\rho) + \frac{n-2}{\rho}u'(\rho) + \lambda^2 u(\rho) = 0.$$

Thus H_b is harmonic on $D(b) \setminus Z^*$ where $Z^* = \{x \in D(b) | \rho = 0\}$ is a polar set. Since H_b is continuous on Z^*, H_b is harmonic on $D(b)$. h_b and H_b are bounded harmonic functions that satisfy

$$\lim_{x \to y} (H_b(x) - h_b(x) \geq 0 \qquad (y \in \partial D(b) \setminus Z)$$

where Z is the intersection of the sphere center 0 radius $\frac{\pi}{2a}$ and the hyperplane $x_n = b$ which is a polar set in R^n $(n \geq 3)$. It follows from Theorem 2.24 that $h_b \leq H_b$. Therefore

$$h_b(x) \le \frac{\sinh(\lambda x_n) F_\nu(\lambda\rho)}{K \sinh(ab)}$$

$$= \frac{\sinh(\lambda x_n) F_\nu(\lambda\rho)}{K} \left(\frac{2}{e^{\lambda b} - e^{-\lambda b}}\right)$$

$$= \frac{2 \sinh(\lambda x_n) F_\nu(\lambda\rho)}{K} \left(\frac{1}{e^{\lambda b}(1-e^{-2\lambda b})}\right) .$$

This shows that for each x fixed in C(a), there exists a positive number A which is independent of b such that

$$h_b(x) \le A e^{-\lambda b}$$

for all sufficiently large b.

4.5 Definition The limit inferior $\liminf_{b\to\infty} F(b)$ is defined by

$$\liminf_{b\to\infty} F(b) = \sup_b \{\inf_{x\ge b} F(x)\}.$$

4.6 Theorem. Let a > 0 and let S be a subharmonic in C(a). Assume that

$$\limsup_{x \to y} S(x) \le 0 \qquad (y \in \partial C(a))$$

and suppose that

$$\liminf_{b \to \infty} e^{-\lambda b} M(b)^+ \le 0$$

where $\lambda = \sqrt{n-1}\, a$. Then $S \le 0$ in C(a).

We note that an analogous result for a subharmonic function on a cone has been given in [10].

<u>Proof</u> : For any fixed b, we consider S_b defined by

$$S_b(x) = S(x) - M(b)^+ h_b(x) \qquad (x \in D(b)).$$

Since h_b is harmonic and S is subharmonic in D(b), S_b is subharmonic in D(b). By the hypothesis that

$$\lim_{x \to y} \sup S(x) \le 0 \qquad (y \in \partial C(a))$$

and the property of h_b we get S_b satisfying

$$\lim_{x \to y} \sup S_b(x) \le 0 \qquad (y \in \partial D(b)).$$

Therefore by Theorem 2.6, $S_b(x) \le 0$. This gives

$$S(x) \le M(b)^+ h_b(x).$$

Keeping x fixed we have that

$$\lim_{b \to \infty} \inf e^{-\lambda b} M(b)^+ \le 0.$$

Let $\varepsilon > 0$ be given. By Theorem 4.4 we can find positive numbers A and B such that

$$h_b(x) \le A e^{-\lambda b} \qquad (b > B).$$

We can find a sequence of positive numbers b_j, j = 1,2,... tending to infinity, with the property that for all sufficiently large j

$$e^{-\lambda b_j} M(b_j)^+ < \varepsilon/_A .$$

Since x is in $D(b_j)$ and $b_j > B$, when j is sufficiently large, we conclude that $S(x) < \varepsilon$. By letting $\varepsilon \to 0$ we get $S(x) \leq 0$ for $x \in C(a)$

4.7 Corollary. Let $a > 0$, let h be harmonic in the cylinder C(a) such that it vanishes on the boundary $\partial C(a)$. If $\max \{|h(x)| \mid |x_n| = b\} = O(e^{-\lambda b})$ as $b \to \infty$ where $\lambda = \sqrt{n-1}\, a$, then $h = 0$ in C(a).

Proof : Since h is harmonic, h is subharmonic and superharmonic on C(a). By Theorem 4.6, $h \leq 0$ and $h \geq 0$ in C(a). Therefore $h = 0$ in C(a).