CHAPTER III

THE GENERALIZED DIRICHLET PROBLEM

Now we shall study the Dirichlet problem for a bounded open subset G of R^n and for an indicator function on an open subset of ∂G, we show that an associated harmonic function can be constructed by a method which is known as the Perron-Wiener-Brelot method.

3.1 <u>Definition</u>. Let G be any nonempty open subset of R^n and let f be any extended real-valued function defined on ∂G. The <u>Generalized Dirichlet problem</u> is that of constructing a harmonic function h on G corresponding to the boundary function f.

3.2 <u>Definition</u>. An extended real-valued function u is <u>hyperharmonic</u> on G if it is superharmonic or identically $+\infty$ on each component of G and u is <u>hypoharmonic</u> on G if it is subharmonic or identically $-\infty$ on each component of G.

3.3 <u>Definition</u>. The <u>upper class</u> of functions $\mathcal{U}_f$ determined by the function f on ∂G is given by

$$\mathcal{U}_f = \{u \mid u \text{ is hyperharmonic on } G, \liminf_{y \to x} u(y) \geq f(x) \text{ for all } x \in \partial G, \; u \text{ is bounded below on } G\}.$$

The <u>lower class</u> of function $\mathcal{L}_f$ determined by f is given by

$$\mathcal{L}_f = \{u \mid u \text{ is hypoharmonic on } G, \limsup_{y \to x} u(y) \leq f(x) \text{ for all } x \in \partial G, \; u \text{ is bounded above on } G\}.$$

Note that $\mathcal{U}_f$ contains the function that is identically $+\infty$ on G and that $\mathcal{L}_f$ contains the function which is identically $-\infty$ on G.

3.4 Definition. $\bar{H}_f = \inf\{u \mid u \in \mathcal{U}_f\}$ is the upper solution for the generalized Dirichlet problem for the boundary function f.
$\underline{H}_f = \sup\{u \mid u \in \mathcal{L}_f\}$ is the corresponding lower solution.

Before proving the next lemma a few words should be said about the upper and lower solutions relative to the components of G. Let W be a component and let $\mathcal{U}_f^W$ and $\bar{H}_f^W$ be the upper class and upper solution, respectively, relative to the component W. We first remark that $\mathcal{U}^W_{f|_{\partial W}} = \mathcal{U}_{f|_W}$ where

$$\mathcal{U}^W_{f|_{\partial W}} = \{u \mid u \text{ is hyperharmonic on } W, \liminf_{y \to x} u(y) \geq f(x) \text{ for all } x \in \partial W, \; u \text{ is bounded below on } W\},$$

$$\mathcal{U}_{f|_W} = \{u|W \mid u \in \mathcal{U}_f\}.$$

to verify this suppose $u \in \mathcal{U}^W_{f|_{\partial W}}$ and let $u^* = u$ on W and $u^* = +\infty$ on $G \setminus W$ Clearly $u^* \in \mathcal{U}_f$ and $u = u^*|_W$ so $u \in \mathcal{U}_{f|_W}$. This shows that $\mathcal{U}^W_{f|_{\partial W}} \subset \mathcal{U}_{f|_W}$. Suppose $u \in \mathcal{U}_f$ and $x \in \partial W$. Then $x \in \partial G$ and

$$\liminf_{\substack{y \to x \\ x \in W}} u(y) \geq \liminf_{\substack{y \to x \\ y \in G}} u(y) \geq f(x).$$

Therefore $u|_W \in \mathcal{U}^W_{f|_{\partial W}}$ and $\mathcal{U}_{f|_W} \subset \mathcal{U}^W_{f|_{\partial W}}$. It follows that $\bar{H}^W_{f|_{\partial W}} = \bar{H}_{f|_W}$.
Because of this it usually suffices to consider components of G.

3.5 Lemma. $\bar{H}_f$ and $\underline{H}_f$ are identically $+\infty$, identically $-\infty$ or harmonic on each component of G.

Proof : We can assume that G is connected. If $\mathcal{U}_f$ contains only the identically $+\infty$ functions, then $\bar{H}_f = +\infty$ and we are through. Suppose $\mathcal{U}_f$ contains a hyperharmonic function that is not identically $+\infty$ on G. Then

$$\bar{H}_f = \inf \{u | u \in \mathcal{U}_f, \text{ u is superharmonic on G}\}.$$

To show that $\bar{H}_f$ is either identically $-\infty$ or harmonic on G, we define

$$u^* = \begin{cases} PI(u,B) & \text{on } B \\ u & \text{on } G \setminus B \end{cases}$$

where u is a superharmonic member of $\mathcal{U}_f$ and B is a ball with $\bar{B} \subset G$. Then u^* is harmonic on B, $u^* \le u$ on G, and u^* is superharmonic on G by Theorem 2.12. For $x \in \partial G$

$$\liminf_{y \to x} u^*(y) = \liminf_{y \to x} u(y) \ge f(x)$$

and u^* is bounded below by the same constant bounding u, then $u^* \in \mathcal{U}_f$. Since $u^* \le u$ for every $u \in \mathcal{U}_f$, $\inf \{u^* | u \in \mathcal{U}_f\} \le \inf \{u | u \in \mathcal{U}_f\}$. But $u^* \in \mathcal{U}_f$ so $u^* \ge \inf \{u | u \in \mathcal{U}_f\}$ and hence $\inf \{u^* | u \in \mathcal{U}_f\}$ $\ge \inf \{u | u \in \mathcal{U}_f\} = \bar{H}_f$. Therefore $\bar{H}_f = \inf\{u^* | u \in \mathcal{U}_f\}$. To show that $\mathcal{U}_f$ is left-directed family, let $u_1, u_2 \in \mathcal{U}_f$. For $x \in \partial G$, $\liminf_{y \to x} u_i(y) \ge f(x)$, $i = 1,2$. This implies that $\liminf_{y \to x} (\min(u_1(y), u_2(y))$ $= \min (\liminf_{y \to x} u_1(y), \liminf_{y \to x} u_2(y)) \ge f(x)$. Since u_1 and u_2 are bounded below, $\min (u_1,u_2)$ is bounded below on G and and $\min(u_1,u_2) \in \mathcal{U}_f$. Therefore, given u_1 and $u_2 \in \mathcal{U}_f$ there is $u_3 = \min(u_1,u_2) \in \mathcal{U}_f$ such that

$u_3 \le u_1$ and $u_3 \le u_2$, and hence $\mathcal{U}_f$ is left directed-family. $\{u^* | u \in \mathcal{U}_f\}$ is left-directed family of harmonic functions. Then by Theorem 1.16 $\bar{H}_f$ is either $-\infty$ or harmonic on G.

3.6 Definition. If $\bar{H}_f = \underline{H}_f$ and both harmonic on G, then f is called a resolutive boundary function and $H_f = \bar{H}_f = \underline{H}_f$ is called the generalized Dirichlet solution for f.

The above method of obtaining a harmonic function corresponding to the boundary function f is called the Perron-Wiener-Brelot method.

3.7 Lemma. Let G be a bounded open subset of R^n. If $u \in \mathcal{L}_f$ and $v \in \mathcal{U}_f$, then $u \le v$ and $\underline{H}_f \le \bar{H}_f$ on G.

Proof : We can assume that G is connected. If either v is identically $+\infty$ or u is identically $-\infty$, then $u \le v$ on G trivially. It suffices to show that $v-u \ge 0$ on G where v-u is superharmonic. If $x \in \partial G$ and f(x) is finite, then

$$\lim_{y \to x} \inf(v-u)(y) \ge \lim_{y \to x} \inf v(y) - \lim_{y \to x} \sup u(y) \ge f(x)-f(x) = 0;$$

if $f(x) = +\infty$, $\lim_{y \to x} \inf(v-u)(y) \ge 0$, since $\lim_{y \to x} \inf v(y) = +\infty$ and $\lim_{y \to x} \sup u(y) < +\infty$ with a similar result holding if $f(x) = -\infty$.

Therefore $\lim_{y \to x} \inf(v-u)(y) \ge 0$ for all $x \in \partial G$ and $v-u \ge 0$ on G by Theorem 2.6. We have $u \le v$ on G and this gives $\underline{H}_f \le \bar{H}_f$ on G.

3.8 Theorem. Let G be a bounded open subset of R^n. If f is bounded on ∂G and there is a harmonic function h on G such that $\lim_{y \to x} h(y) = f(x)$ for all $x \in \partial G$, then f is resolutive and $H_f = h$.

Proof : Since f is bounded and $\lim_{y \to x} h(y) = f(x)$ for all $x \in \partial G$, h is bounded by Theorem 2.6. h belongs to both $\mathcal{U}_f$ and $\mathcal{L}_f$. Therefore $\bar{H}_f \le h \le \underline{H}_f$, but $\underline{H}_f \le \bar{H}_f$ then $\bar{H}_f = h = \underline{H}_f$. Since h is harmonic, f is a resolutive boundary function.

3.9 Lemma. Let G be a bounded open subset of R^n, let f and g be extended real-valued functions on ∂G, and let c be any real number

(i) if $f = c$ on ∂G, then f is resolutive and $H_f = c$ on G,

(ii) $\bar{H}_{f+c} = \bar{H}_f + c$ and $\underline{H}_{f+c} = \underline{H}_f + c$. If f is resolutive, then f+c is resolutive and $H_{f+c} = H_f + c$

(iii) if $c > 0$, then $\bar{H}_{cf} = c\bar{H}_f$ and $\underline{H}_{cf} = c\underline{H}_f$. If f is resolutive, then cf is resolutive and $H_{cf} = cH_f$; $c > 0$

(iv) $\bar{H}_{-f} = -\underline{H}_f$. If f is resolutive, then -f is resolutive $H_{-f} = -H_f$.

(v) If $f \le g$, then $\bar{H}_f \le \bar{H}_g$ and $\underline{H}_f \le \underline{H}_g$.

(vi) $\bar{H}_{f+g} \le \bar{H}_f + \bar{H}_g$ and $\underline{H}_{f+g} \ge \underline{H}_f + \underline{H}_g$ whenever the sums are defined. If f and g are resolutive and f+g is defined, then f+g is resolutive with $H_{f+g} = H_f + H_g$.

Proof : (i) Suppose $f = c$ on ∂G. It is easily seen that $c \in \mathcal{U}_f$ and $c \in \mathcal{L}_f$ and $\overline{H}_f \leq c \leq \underline{H}_f$. Therefore $\overline{H}_f = c = \underline{H}_f$ and since c is harmonic f is resolutive and $H_f = c$ on G

(ii),(iii),(iv),(v) and (vi) are easily proved directly from the definitions.

3.10 Lemma. Let G be a bounded open subset of R^n. If $\{f_j\}$ is an increasing sequence of boundary functions, $\lim_{j \to \infty} f_j = f$ and $\overline{H}_{f_1} > -\infty$ on G, then $\lim_{j \to \infty} \overline{H}_{f_j} = \overline{H}_f$; if, in addition, $\{f_j\}$ is a sequence of resolutive boundary functions, then $\overline{H}_f = \underline{H}_f$ and f is resolutive if either $\overline{H}_f$ or $\underline{H}_f$ is finite.

Proof : Suppose that G is connected. Since $f_j \leq f$, $\overline{H}_{f_j} \leq \overline{H}_f$. If $\lim_{j \to \infty} \overline{H}_{f_j} = +\infty$, then there is nothing to prove so we assume that $\lim_{j \to \infty} \overline{H}_{f_j}(x_0) < +\infty$ for some $x_0 \in G$. Since $-\infty < \overline{H}_{f_1}(x_0) \leq \overline{H}_{f_j}(x_0) < +\infty$, $\overline{H}_{f_j}$ is not identically $+\infty$ and $\overline{H}_{f_j}$ is harmonic by Lemma 3.5. Since the limit of an increasing harmonic functions is harmonic or identically $+\infty$ and $\lim_{j \to \infty} \overline{H}_{f_j}(x_0) < +\infty$, $\lim_{j \to \infty} \overline{H}_{f_j}$ is harmonic on G. Given $\varepsilon > 0$, for each j, choosing $v_j \in \mathcal{U}_{f_j}$ such that

$$v_j(x_0) < \overline{H}_{f_j}(x_0) + \varepsilon\, 2^{-j}$$

and define

$$v = \lim_{i \to \infty} \overline{H}_{f_i} + \sum_{j=1}^{\infty} (v_j - \overline{H}_{f_j}) .$$

For each j, $v_j - \bar{H}_{f_j}$ is superharmonic on G, since $v_j \geq \bar{H}_{f_j}$, $v_j - \bar{H}_{f_j}$ is non-negative and smaller than $\varepsilon 2^{-j}$ at x_0. From the inequality

$$v \geq \lim_{i \to \infty} \bar{H}_{f_i} + v_j - \bar{H}_{f_j} = \lim_{i \to \infty} (\bar{H}_{f_i} - \bar{H}_{f_j}) + v_j$$

and the fact that $\lim_{i \to \infty} \bar{H}_{f_i} \geq \bar{H}_{f_j}$, we obtain $v \geq v_j$ for each j. Since v_j is bounded below, v is bounded below. It is easily seen that v is superharmonic on G. Moreover, for $x \in \partial G$ and for each j,

$$\lim_{y \to x} \inf v(y) \geq \lim_{y \to x} \inf v_j(y) \geq f_j(x)$$

and

$$\lim_{j \to \infty} (\lim_{y \to x} \inf v(y)) \geq \lim_{j \to \infty} f_j(x)$$

$$\lim_{y \to x} \inf v(y) \geq f(x)$$

Therefore $v \in \mathcal{U}_f$ and $v(x_0) \geq \bar{H}_f(x_0)$. Since $\lim_{j \to \infty} \bar{H}_{f_j}(x_0) \leq \bar{H}_f(x_0) \leq v(x_0)$ $= \lim_{i \to \infty} \bar{H}_{f_i}(x_0) + \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \lim_{j \to \infty} \bar{H}_{f_j}(x_0) + \varepsilon$, ε is **arbitrary**, $\lim_{j \to \infty} \bar{H}_{f_j}(x_0)$ $= \bar{H}_f(x_0)$. Since $\lim_{j \to \infty} \bar{H}_{f_j}(x_0)$ is finite, $\bar{H}_f(x_0)$ is finite and $\bar{H}_f$ is harmonic on G. Since $\bar{H}_f - \lim_{j \to \infty} \bar{H}_{f_j}$ is harmonic on G, $\bar{H}_f - \lim_{j \to \infty} \bar{H}_{f_j}$ satisfies minimum principle on G and $\bar{H}_f - \lim_{j \to \infty} \bar{H}_{f_j}$ is a non-negative function which vanishes at $x_0 \in G$. Therefore $\bar{H}_f - \lim_{j \to \infty} \bar{H}_{f_j}$ attains its infimum on G and $\bar{H}_f - \lim_{j \to \infty} \bar{H}_{f_j}$ is identically zero on G. If the f_j's are resolutive, then $\bar{H}_{f_j} = \underline{H}_{f_j}$ and $\bar{H}_f = \lim_{j \to \infty} \bar{H}_{f_j} = \lim_{j \to \infty} \underline{H}_{f_j} = \underline{H}_f$; it follows that $\bar{H}_f = \underline{H}_f$ and that f is resolutive if f either is finite.

3.11 __Lemma__. If u is bounded subharmonic function on the bounded open set G such that $f(x) = \lim_{y \to x} u(y)$ exists for all $x \in \partial G$, then f is resolutive boundary function.

__Proof__ : Clearly $u \in \mathcal{L}_f$ and $u \le \underline{H}_f$. Since u is bounded, $\underline{H}_f$ is harmonic on G. Then $\lim_{y \to x} \inf \underline{H}_f(y) \ge \lim_{y \to x} \inf u(y) = f(x)$ for all $x \in \partial G$ and therefore $\underline{H}_f \in \mathcal{U}_f$. It follows that $\underline{H}_f \ge \overline{H}_f$; but since we always have $\underline{H}_f \le \overline{H}_f$, $\underline{H}_f = \overline{H}_f$ and f is resolutive.

3.12 __Lemma__. Let K be a compact subset of R^n and let f be a continuous function on K. Then, given $\varepsilon > 0$, there is a function u which is the difference of two continuous subharmonic functions defined on a ball containing K such that $\sup_{x \in K} |f(x)-u(x)| < \varepsilon$.

__Proof__ : It follows from the Stone-Weierstrass theorem that for each $\varepsilon > 0$ there is a function u polynomial in the n coordinate variables such that $\sup_{x \in K} |f(x)-u(x)| < \varepsilon$. We must show that u can be expressed as the difference of two continuous subharmonic functions defined on a ball $B \supset K$. Consider $v(y) = \|y\|^2$, $\Delta v = 2n \ge 0$. Then v is continuous subharmonic on R^n and for $\lambda \ge 0$, λv is a continuous subharmonic function on B. Choosing $\lambda_0 > 0$ such that $\Delta(u + \lambda_0 v) \ge 0$ on B, then $u = (u + \lambda_0 v) - \lambda_0 v$ with $u + \lambda_0 v$ and $\lambda_0 v$ are continuous subharmonic functionson B.

3.13 Theorem (Wiener). If f is a continuous real-valued function on the boundary ∂G of the bounded open set G, then f is resolutive.

Proof : Since ∂G is compact, let B be a ball containing ∂G, there is a function $u = v-w$, where v and w are continuous subharmonic functions defined on B such that $\sup_{x \in G} |f(x)-u(x)| < \varepsilon$. Since $v(x) = \lim_{y \to x} v(y)$ and $w(x) = \lim_{y \to x} w(y)$ for all $x \in \partial G$ and v, w are bounded subharmonic functions, $v_{|\partial G}$ and $w_{|\partial G}$ are resolutive boundary functions and $u_{|\partial G} = v_{|\partial G} - w_{|\partial G}$ is resolutive. From $\sup_{x \in G} |f(x)-u_{|\partial G}(x)| < \varepsilon$ we get $f(x) < u_{|\partial G}(x) + \varepsilon$. By Lemma 3.9 $\bar{H}_f \le \bar{H}_{u_{|\partial G}} + \varepsilon$. Likewise $\bar{H}_f \geq \bar{H}_{u_{|\partial G}} - \varepsilon$. Therefore

$$|\bar{H}_f(x) - \bar{H}_{u_{|\partial G}}(x)| = |\bar{H}_f(x) - H_{u_{|\partial G}}(x)| < \varepsilon$$

for all $x \in \partial G$ and then $\sup_{x \in \partial G} |\bar{H}_f(x) - H_{u_{|\partial G}}(x)| < \varepsilon$ that is, $H_{u_{|\partial G}}$ approximates uniformly to $\bar{H}_f$. In the same way $H_{u_{|\partial G}}$ approximates uniformly to $\underline{H}_f$. Therefore $\bar{H}_f = \underline{H}_f$, but since the above inequalities also show that $\bar{H}_f$ is finite, f is resolutive.

3.14 Theorem. If C is a subset of R^n, then χ_C , the indicator function of C, is a resolutive boundary function.

<u>Proof</u> : If C is open, then χ_C is lower semicontinuous on R^n. If C is closed, then χ_C is upper semicontinuous on R^n. Therefore, there is a sequence of continuous functions $\{f_j\}$ on R^n such that $\lim_{j \to \infty} f_j = \chi_C$ on C and $0 \le f_j \le 1$ for each j. By Theorem 3.13 f_j is resolutive for each j. It follows from Lemma 3.10 that χ_C is a resolutive boundary function.