CHAPTER II

ON HYPERGEODESIC DIFFERENTIAL EQUATIONS

<u>Notations</u> :

2.1 ψ is an n-vector valued function of a real variable t $\psi^1(t), \psi^2(t), \ldots, \psi^n(t)$ denoted by $\vec{\psi}(t) = (\psi^1(t), \ldots, \psi^n(t))$.

2.2 $\dfrac{d\psi^i}{dt} = \mathring{\psi}^i$, $\dfrac{d^2\psi^i}{dt^2} = \ddot{\psi}^i$, $\dfrac{d^3\psi^i}{dt^3} = \dddot{\psi}^i$, $i = 1,2,\ldots,n$

2.3 $G^i_{j_1\cdots j_k}(\psi)\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k} = \sum\limits_{j_1=1}^{n}\cdots\sum\limits_{j_k=1}^{n} G^i_{j_1\cdots j_k}(\psi)\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k}$ where

$i = 1,\ldots,n$ and $k = 1,2,\ldots$

2.4 $J^i_{k_1\cdots k_m}(\psi)\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_m} = \sum\limits_{k_1=1}^{n}\cdots\sum\limits_{k_m=1}^{n} J^i_{k_1\cdots k_m}(\psi)\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_m}$ where

$i = 1,2,\ldots,n$ and $m = 1,2,3,\ldots$

2.5 $L^i_\ell(\psi)t^\ell = \sum\limits_{\ell=1}^{\infty} L^i_\ell(\psi)t^\ell$ where $i = 1,2,\ldots,n$

2.6 $K^i_{k_1\cdots k_m}(\psi)_{j_1\cdots j_k}\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_m}\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k} = \sum\limits_{k_1=1}^{n}\cdots\sum\limits_{k_m=1}^{n}\sum\limits_{j_1=1}^{n}\cdots$

$\sum\limits_{j_k=1}^{n} K^i_{k_1\cdots k_m}(\psi)_{j_1\cdots j_k}\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_m}\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k}$

where $i = 1,2,\ldots,n$, $m = 1,2,\ldots$, $k = 1,2,\ldots$

$$2.7 \quad M^{i}_{j_1..j_k\ell}(\psi)\dot{\psi}^{j_1}..\dot{\psi}^{j_k}t^{\ell} = \sum_{j_1=1}^{n}..\sum_{j_k=1}^{n}\sum_{\ell=1}^{\infty} M^{i}_{j_1..j_k\ell}(\psi)\dot{\psi}^{j_1}..\dot{\psi}^{j_k}t^{\ell}$$

where $i = 1,\dots,n$ and $k = 1,2,\dots$

$$2.8 \quad N^{i}_{k_1..k_m\ell}(\psi)\ddot{\psi}^{k_1}..\ddot{\psi}^{k_m}t^{\ell} = \sum_{k_1=1}^{n}..\sum_{k_m=1}^{n}\sum_{\ell=1}^{\infty} N^{i}_{k_1..k_m\ell}(\psi)\ddot{\psi}^{k_1}..\ddot{\psi}^{k_m}t^{\ell}$$

where $i = 1,\dots,n$ and $m = 1,2,\dots$

$$2.9 \quad R^{i}_{k_1..k_m j_1..j_k\ell}(\psi)\ddot{\psi}^{k_1}..\ddot{\psi}^{k_m}\dot{\psi}^{j_1}..\dot{\psi}^{j_k}t^{\ell} = \sum_{k_1=1}^{n}..\sum_{k_m=1}^{n}\sum_{j_1=1}^{n}..$$

$$..\sum_{j_k=1}^{n}\sum_{\ell=1}^{\infty} R^{i}_{k_1..k_m j_1..j_k\ell}(\psi)\ddot{\psi}^{k_1}..\ddot{\psi}^{k_m}\dot{\psi}^{j_1}..\dot{\psi}^{j_k}t^{\ell}$$

where $i = 1,\dots,n$ and $m = 1,2,\dots$ and $k = 1,2,\dots$

$$2.10 \quad u^{p_1}u^{p_2}...u^{p_k}v^{q_1}...v^{q_\ell}\frac{\partial^{k+\ell}\psi^{i}}{\partial u^{p_1}..\partial u^{p_k}\partial v^{q_1}..\partial v^{q_\ell}}(p,\alpha u,\alpha v,t)$$

$$= \sum_{p_1=1}^{n}..\sum_{p_k=1}^{n}\sum_{q_1=1}^{n}..\sum_{q_\ell=1}^{n} u^{p_1}..u^{p_k}v^{q_1}..v^{q_\ell}\frac{\partial^{k+\ell}}{\partial u^{p_1}..\partial u^{p_k}\partial v^{q_1}..\partial v^{q_\ell}}\psi^{i}(p,\alpha u,\alpha v,t)$$

where $k = 1,2,\dots$ and $\ell = 1,2,\dots$

Notations in 2.3 to 2.10 are called Einstein Summation convention.

Introduction to Theorem

$$\Pi_1(x^1,\ldots,x^n,x^{n+1},\ldots,x^{2n},x^{2n+1},\ldots,x^{3n},x^{3n+1}) = (x^1,\ldots,x^n)$$

$$\Pi_2(x^1,\ldots,x^n,x^{n+1},\ldots,x^{2n},x^{2n+1},\ldots,x^{3n},x^{3n+1}) = (x^{n+1},\ldots,x^{2n})$$

$$\Pi_3(x^1,\ldots,x^n,x^{n+1},\ldots,x^{2n},x^{2n+1},\ldots,x^{3n},x^{3n+1}) = (x^{2n+1},\ldots,x^{3n})$$

$$\Pi_4(x^1,\ldots,x^n,x^{n+1},\ldots,x^{2n},x^{2n+1},\ldots,x^{3n},x^{3n+1}) = x^{3n+1} .$$

Let Ω be a connected open set of R^{3n+1} such that $\forall\vec{p} \in \Pi_1(\Omega) \quad (\vec{p},\vec{u},\vec{v},0) \in \Omega \quad \forall\vec{u},\vec{v} \in R^n$.

Let $H : \Omega \to R^n$ be analytic. H determines a third order differential equation $\dddot{\psi}^i = H^i(\vec{\psi},\dot{\vec{\psi}},\ddot{\vec{\psi}},t)$ where $\vec{\psi} = (\psi^1,\ldots,\psi^n)$, $\dot{\vec{\psi}} = (\dot{\psi}^1,\ldots,\dot{\psi}^n)$, and $\ddot{\vec{\psi}} = (\ddot{\psi}^1,\ldots,\ddot{\psi}^n)$.

Given $\vec{p} \in \Pi_1(\Omega)$, $\vec{u}$, $\vec{v} \in R^n$ we take the initial conditions to be

$$\vec{\psi}(0) = \vec{p} = (p^1,\ldots,p^n) \in \Pi_1(\Omega)$$
$$\dot{\vec{\psi}}(0) = \vec{u} = (u^1,\ldots,u^n) \in R^n$$
$$\ddot{\vec{\psi}}(0) = \vec{v} = (v^1,\ldots,v^n) \in R^n.$$

By the Fundamental Theorem of Ordinary Differential Equations, there exists an open subset U of 0 in $\Pi_4(\Omega)$ and a unique function $\Phi_{\vec{p},\vec{u},\vec{v}} : U \to \Pi_1(\Omega)$ which is a solution to the differential equation $\dddot{\psi}^i = H^i(\vec{\psi},\dot{\vec{\psi}},\ddot{\vec{\psi}},t)$ with the above given initial conditions.

Write $\Phi_{\vec{p},\vec{u},\vec{v}}(t) = \psi(\vec{p},\vec{u},\vec{v},t)$ for all $t \in U$.

Let $\alpha \in R$, define $\alpha\vec{u} = (\alpha u^1,\dots,\alpha u^n)$ and $\alpha\vec{v} = (\alpha^2 v^1,\dots,\alpha^2 v^n)$ as was done in chapter I.

<u>Theorem 2.11</u> Suppose we have an analytic third order differential equation $\dddot{\psi}^i = H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t)$ for $i = 1,\dots, n$ with initial conditions $\psi^i(\vec{p},\vec{u},\vec{v},0) = p^i$, $\mathring{\psi}^i(\vec{p},\vec{u},\vec{v},0) = u^i$, $\ddot{\psi}^i(\vec{p},\vec{u},\vec{v},0) = v^i$ whose solution $\psi^i(\vec{p},\vec{u},\vec{v},t)$ satisfies the functional equation

$$\psi^i(\vec{p},\alpha\vec{u},\alpha\vec{v},t) = \psi^i(\vec{p},\vec{u},\vec{v},f(\alpha,t))$$

where $f: W \to R$ is an analytic function on W a neighbourhood of $(0,0)$ in R^2, $(\vec{p},\alpha\vec{u},\alpha\vec{v},t) \in \Omega \subseteq R^{3n+1}$, $(\alpha,t) \in W$.

Furthermore, assume that $f(\alpha,0) = f(0,t) = 0 \quad \forall(\alpha,t) \in W$. Then the differential equation must be of the form

$$\dddot{\psi}^i = G^i_{j_1 j_2 j_3}(\vec{\psi})\mathring{\psi}^{j_1}\mathring{\psi}^{j_2}\mathring{\psi}^{j_3} + K^i_{k_1 j_1}(\vec{\psi})\ddot{\psi}^{k_1}\mathring{\psi}^{j_1} \quad \text{and } f(\alpha,t)$$

must equal αt.

<u>Proof</u> Since $\dddot{\psi}^i = H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t)$ is analytic, by hypothesis we can write $H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t)$ as a power series.

$$H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t) = G^i(\vec{\psi}) + G^i_{j_1}(\vec{\psi})\mathring{\psi}^{j_1} + G^i_{j_1 j_2}(\vec{\psi})\mathring{\psi}^{j_1}\mathring{\psi}^{j_2} + \dots$$

$$+ J^i_{k_1}(\vec{\psi})\ddot{\psi}^{k_1} + J^i_{k_1 k_2}(\vec{\psi})\ddot{\psi}^{k_1}\ddot{\psi}^{k_2} + \dots + L^i_\ell(\vec{\psi})t^\ell$$

$$+ K^i_{k_1 j_1}(\vec{\psi})\ddot{\psi}^{k_1}\mathring{\psi}^{j_1} + K^i_{k_1 j_1 j_2}(\vec{\psi})\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2} + \dots$$

$$+ K^i_{k_1k_2j_1}(\vec{\psi})\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1} + K^i_{k_1k_2j_1j_2}(\vec{\psi})\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1}\dot{\psi}^{j_2} + \ldots$$

$$+ M^i_{j_1\ell}(\vec{\psi})\dot{\psi}^{j_1}t^\ell + M^i_{j_1j_2\ell}(\vec{\psi})\dot{\psi}^{j_1}\dot{\psi}^{j_2}\,t^\ell + \ldots$$

$$+ N^i_{k_1\ell}(\vec{\psi})\ddot{\psi}^{k_1}t^\ell + N^i_{k_1k_2\ell}(\vec{\psi})\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}t^\ell + \ldots$$

$$+ R^i_{k_1j_1\ell}(\vec{\psi})\ddot{\psi}^{k_1}\dot{\psi}^{j_1}t^\ell + R^i_{k_1j_1j_2\ell}(\vec{\psi})\ddot{\psi}^{k_1}\dot{\psi}^{j_1}\dot{\psi}^{j_2}t^\ell$$

$$+ \;\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ R^i_{k_1k_2j_1\ell}(\vec{\psi})\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1}t^\ell + R^i_{k_1k_2j_1j_2\ell}(\psi)\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1}\dot{\psi}^{j_2}\,t^\ell$$
$$+ \ldots\ldots$$

For convenience, we will replace $\vec{p}$ by p, $\vec{u}$ by u, and $\vec{v}$ by v and $f(\alpha,t)$ by f .

Since $\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,f)$ $\quad i = 1,\ldots,n\ \forall p \in \Pi_1(\Omega)$ $u, v \in R^n$, α, $t \in R$ differentiate this equation with respect to t, we get

(2.11-1) $\quad \dot{\psi}^i(p,\alpha u,\alpha v,t) \quad = \dot{\psi}^i(p,u,v,f)f_t$ where $f_t = \frac{\partial f}{\partial t}$.

let t = 0 then (2.11-1) becomes

$$\alpha u^i \quad = u^i f_t(\alpha,0) \text{ implying that } f_t(\alpha,0) = \alpha$$

differentiate (2.11-1) with respect to t, we get

(2.11-2) $\quad \ddot{\psi}^i(p,\alpha u,\alpha v,t) \quad = \ddot{\psi}^i(p,u,v,f)f_t^2 + \dot{\psi}^i(p,u,v,f)f_{tt}$

let t in (2.11-2) be zero,

$\alpha^2 v^i = v^i \alpha^2 + u^i f_{tt}(\alpha,0)$ implying that $f_{tt}(\alpha,0) = 0.$

Again, differentiate (2.11-2) with respect to t.

(2.11-3) $\dddot{\psi}^i(p,\alpha u,\alpha v,t) = \dddot{\psi}^i(p,u,v,f)f_t^3 + \mathring{\psi}^i(p,u,v,f)f_{ttt}$

$$+ 3f_t f_{tt}\ \ddot{\psi}^i(p,u,v,f)$$

Substitue (2.11-3) by the power series expansion, this gives

$$G^i(\psi(p,\alpha u,\alpha v,t)) + G^i_{j_1}(\psi(p,\alpha u,\alpha v,t))\mathring{\psi}^{j_1}(p,\alpha u,\alpha v,t)$$

$$+ G^i_{j_1 j_2}(\psi(p,\alpha u,\alpha v,t))(\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t) + \ldots.$$

$$+ J^i_{k_1}(\psi(p,\alpha u,\alpha v,t))\ddot{\psi}^{k_1}(p,\alpha u,\alpha v,t)$$

$$+ J^i_{k_1 k_2}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2})(p,\alpha u,\alpha v,t) + \ldots.$$

$$+ L^i_{\ell}(\psi(p,\alpha u,\alpha v,t))t^{\ell} + K^i_{k_1 j_1}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1})(p,\alpha u,\alpha v,t)$$

$$+ K^i_{k_1 j_1 j_2}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t) + \ldots.$$

$$+ K^i_{k_1 k_2 j_1}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1})(p,\alpha u,\alpha v,t)$$

$$+ K^i_{k_1 k_2 j_1 j_2}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t) + \ldots.$$

$$+ M^i_{j_1 \ell}(\psi(p,\alpha u,\alpha v,t))\mathring{\psi}^{j_1}(p,\alpha u,\alpha v,t)t^{\ell}$$

$$+ M^i_{j_1 j_2 \ell}(\psi(p,\alpha u,\alpha v,t))(\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t)t^{\ell} + \ldots.$$

$$+ N^i_{k_1 \ell}(\psi(p,\alpha u,\alpha v,t))\ \ddot{\psi}^{k_1}(p,\alpha u,\alpha v,t)t^{\ell}$$

$$+ N^i_{k_1k_2\ell}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2})(p,\alpha u,\alpha v,t)t^\ell + \ldots.$$

$$+ R^i_{k_1j_1\ell}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1})(p,\alpha u,\alpha v,t)t^\ell$$

$$+ R^i_{k_1j_1j_2\ell}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t)t^\ell + \ldots.$$

$$+ R^i_{k_1k_2j_1\ell}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1})(p,\alpha u,\alpha v,t)t^\ell$$

$$+ R^i_{k_1k_2j_1j_2\ell}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,\alpha u,\alpha v,t)t^\ell + \ldots.$$

$$= f_t^3\ G^i(\psi(p,u,v,f)) + f_t^3\ G^i_{j_1}(\psi(p,u,v,f))\mathring{\psi}^{j_1}(p,u,v,f)$$

$$+ f_t^3\ G^i_{j_1j_2}(\psi(p,u,v,f))(\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,u,v,f) + \ldots.$$

$$+ f_t^3\ J^i_{k_1}(\psi(p,u,v,f))\ \ddot{\psi}^{i_1}(p,u,v,f)$$

$$+ f_t^3\ J^i_{k_1k_2}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2})(p,u,v,f) + \ldots.$$

$$+ L^i_\ell(\psi(p,u,v,f))f_t^3\ t^\ell + f_t^3\ K^i_{k_1j_1}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1})(p,u,v,f)$$

$$+ f_t^3\ K^i_{k_1j_1j_2}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,u,v,f) + \ldots.$$

$$+ f_t^3\ K^i_{k_1k_2j_1}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1})(p,u,v,f)$$

$$+ f_t^3\ K^i_{k_1k_2j_1j_2}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2})(p,u,v,f)$$

$$+ \ldots + f_t^3\ M^i_{j_1\ell}(\psi(p,u,v,f))\mathring{\psi}^{j_1}(p,u,v,f)t^\ell$$

$$+ f_t^3 M^i_{j_1j_2\ell}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2})(p,u,v,f)t^\ell + \ldots.$$

$$+ f_t^3 N^i_{k_1\ell}(\psi(p,u,v,f))\ddot{\psi}^{k_1}(p,u,v,f)t^\ell$$

$$+ f_t^3 N^i_{k_1k_2\ell}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2})(p,u,v,f)t^\ell + \ldots.$$

$$+ f_t^3 R^i_{k_1j_1\ell}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1})(p,u,v,f)t^\ell$$

$$+ f_t^3 R^i_{k_1j_1j_2\ell}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1}\dot{\psi}^{j_2})(p,u,v,f)t^\ell + \ldots.$$

$$+ f_t^3 R^i_{k_1k_2j_1\ell}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1})(p,u,v,f)t^\ell$$

$$+ f_t^3 R^i_{k_1k_2j_1j_2\ell}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dot{\psi}^{j_1}\dot{\psi}^{j_2})(p,u,v,f)t^\ell$$

$$+ \ldots + \dot{\psi}^i(p,u,v,f)f_{ttt} + 3f_tf_{tt}\ddot{\psi}^i(p,u,v,f).$$

let t = 0, then

$$G^i(p) + G^i_{j_1}(p)\alpha u^{j_1} + G^i_{j_1j_2}(p)\alpha^2u^{j_1}u^{j_2} + \ldots + J^i_{k_1}(p)\alpha^2v^{k_1}$$

$$+ J^i_{k_1k_2}(p)\alpha^4v^{k_1}v^{k_2} + \ldots + K^i_{k_1j_1}(p)\alpha^3v^{k_1}u^{j_1} + K^i_{k_1j_1j_2}(p)\alpha^4v^{k_1}u^{j_1}u^{j_2} + \ldots$$

$$+ K^i_{k_1k_2j_1}(p)\alpha^5v^{k_1}v^{k_2}u^{j_1} + K^i_{k_1k_2j_1j_2}(p)\alpha^6v^{k_1}v^{k_2}u^{j_1}u^{j_2} + \ldots$$

$$= \alpha^3G^i(p) + \alpha^3G^i_{j_1}(p)u^{j_1} + \alpha^3G^i_{j_1j_2}(p)u^{j_1}u^{j_2} + \ldots + \alpha^3J^i_{k_1}(p)v^{k_1}$$

$$+ \alpha^3J^i_{k_1k_2}(p)v^{k_1}v^{k_2} + \ldots + \alpha^3K^i_{k_1j_1}(p)v^{k_1}u^{j_1} + \alpha^3K^i_{k_1j_1j_2}(p)v^{k_1}u^{j_1}u^{j_2} + \ldots$$

$$+ \alpha^3K^i_{k_1k_2j_1}(p)v^{k_1}v^{k_2}u^{j_1} + \alpha^3K^i_{k_1k_2j_1j_2}(p)v^{k_1}v^{k_2}u^{j_1}u^{j_2} + \ldots + u^if_{ttt}$$

we get

$$(1-\alpha^3)G^i(p)+(\alpha-\alpha^3)G^i_{j_1}(p)u^{j_1}+(\alpha^2-\alpha^3)G^i_{j_1j_2}(p)u^{j_1}u^{j_2}$$

$$+(\alpha^3-\alpha^3)G^i_{j_1j_2j_3}(p)u^{j_1}u^{j_2}u^{j_3}+\ldots+(\alpha^2-\alpha^3)J^i_{k_1}(p)v^{k_1}$$

$$+(\alpha^4-\alpha^3)J^i_{k_1k_2}(p)v^{k_1}v^{k_2}+\ldots+(\alpha^3-\alpha^3)K^i_{k_1j_1}(p)v^{k_1}u^{j_1}$$

$$+(\alpha^4-\alpha^3)K^i_{k_1j_1j_2}(p)v^{k_1}u^{j_1}u^{j_2}+(\alpha^5-\alpha^3)K^i_{k_1k_2j_1}(p)v^{k_1}v^{k_2}u^{j_1}$$

$$+(\alpha^6-\alpha^3)K^i_{k_1k_2j_1j_2}(p)v^{k_1}v^{k_2}u^{j_1}u^{j_2}+\ldots$$

$$= u^i f_{ttt}(\alpha,0)$$

Choose $\alpha = \alpha_0$, $\alpha_0 \neq \pm 1, 0$ then

$$G^i(p) = 0, \quad G^i_{j_1}(p) = \frac{f_{ttt}(\alpha_0,0)\delta^i_{j_1}}{\alpha_0-\alpha_0^3} = c \quad \text{constant independent of } \alpha_0$$

$$\text{where} \quad \delta^i_{j_1} = \begin{cases} 0, & i \neq j_1 \\ 1, & i = j_1 \end{cases}$$

$G^i_{j_1j_2j_3}(p)$ arbitraty $G^i_{j_1\cdots j_n}(p) = 0 \quad \forall n \geq 2$, all J's = 0

$K^i_{k_1j_1}(p)$ arbitrary other K's = 0 .

So now we have

$$(2.11\text{-}4)\quad \dddot{\psi}^i = c\mathring{\psi}^i+ G^i_{j_1j_2j_3}(\psi)\mathring{\psi}^{j_1}\mathring{\psi}^{j_2}\mathring{\psi}^{j_3}+ L^i_\ell(\psi)t^\ell+ K^i_{k_1j_1}(\psi)\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}$$

$$+ M^i_{j_1\ell}(\psi)\mathring{\psi}^{j_1}t^\ell+ M^i_{j_1j_2\ell}(\psi)\mathring{\psi}^{j_1}\mathring{\psi}^{j_2}t^\ell+\ldots+ N^i_{k_1\ell}(\psi)\ddot{\psi}^{k_1}t^\ell$$

$$+ N^i_{k_1k_2\ell}(\psi)\dot{\psi}^{k_1}\dot{\psi}^{k_2}t^\ell + \ldots + R^i_{k_1j_1\ell}(\psi)\dot{\psi}^{k_1}\dot{\psi}^{j_1}t^\ell$$

$$+ R^i_{k_1j_1j_2\ell}(\psi)\dot{\psi}^{k_1}\dot{\psi}^{j_1}\dot{\psi}^{j_2}t^\ell + \ldots + R^i_{k_1k_2j_1\ell}(\psi)\dot{\psi}^{k_1}\dot{\psi}^{k_2}\dot{\psi}^{j_1}t^\ell$$

$$+ R^i_{k_1k_2j_1j_2\ell}(\psi)\dot{\psi}^{k_1}\dot{\psi}^{k_2}\dot{\psi}^{j_1}\dot{\psi}^{j_2}t^\ell + \ldots .$$

Next we want to consider the L's, M's, N's, and R's and prove that they are zero.

Now $\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,f)\ \forall\, p\epsilon\Pi_1(\Omega), \forall u, \forall v \in R^n,\ \alpha,\ t \in R$

so we can differentiate this equation with respect to α. This gives

$$(2.11\text{-}5)\quad u^{p_1}\frac{\partial\psi^i}{\partial u^{p_1}}(p,\alpha u,\alpha v,t) + 2\alpha v^{q_1}\frac{\partial\psi^i}{\partial v^{q_1}}(p,\alpha u,\alpha v,t) = \dot{\psi}^i(p,u,v,f)f_\alpha$$

$$\text{where } f_\alpha = \frac{\partial f}{\partial\alpha},\quad p_1 = 1,\ldots,n$$
$$q_1 = 1,\ldots,n\ .$$

Let $\alpha = 0$ we get

$$u^{p_1}\frac{\partial\psi^i}{\partial u^{p_1}}(p,0,0,t) = u^i f_\alpha(0,t) \text{ implying that } \frac{\partial\psi^i}{\partial u^{p_1}}(p,0,0,t) = f_\alpha(0,t)\delta^i_{p_1}$$

$$\text{where } \delta^i_{p_1} = \begin{cases} 0\ , & i \neq p_1 \\ 1\ , & i = p_1\ . \end{cases}$$

Differentiate (2.11-5) with respect to α

$$(2.11\text{-}6)\quad u^{p_1}u^{p_2}\frac{\partial^2\psi^i}{\partial u^{p_1}\partial u^{p_2}}(p,\alpha u,\alpha v,t)+2\alpha u^{p_1}v^{q_2}\frac{\partial^2\psi^i}{\partial u^{p_1}\partial v^{q_2}}(p,\alpha u,\alpha v,t)$$

$$+\,2v^{q_1}\frac{\partial\psi^i}{\partial v^{q_1}}(p,\alpha u,\alpha v,t)+2\alpha u^{p_2}v^{q_1}\frac{\partial^2\psi^i}{\partial u^{p_2}\partial v^{q_1}}(p,\alpha u,\alpha v,t)$$

$$+\,4\alpha^2 v^{q_1}v^{q_2}\frac{\partial^2\psi^i}{\partial v^{q_1}\partial v^{q_2}}(p,\alpha u,\alpha v,t)$$

$$=\dot{\psi}^i(p,u,v,f)f_{\alpha\alpha}+f_\alpha^2\ddot{\psi}^i(p,u,v,f)\quad\text{where } p_2,\ q_2=1,2,\ldots,n.$$

Let $\alpha = 0$ we get

$$u^{p_1}u^{p_2}\frac{\partial^2\psi^i}{\partial u^{p_1}\partial u^{p_2}}(p,0,0,t)+2v^{q_1}\frac{\partial\psi^i}{\partial v^{q_1}}(p,0,0,t)=u^i f_{\alpha\alpha}(0,t)+f_\alpha^2(0,t)v^i$$

implying that $f_{\alpha\alpha}(0,t)=0$, $\dfrac{\partial^2\psi^i(p,0,0,t)}{\partial u^{p_1}\partial u^{p_2}}=0$ and

$$\frac{\partial\psi^i}{\partial v^{q_1}}(p,0,0,t)=\frac{f_\alpha^2(0,t)}{2}\delta^i_{q_1}\quad\text{where}\quad \delta^i_{q_1}=\begin{cases}0, & i\neq q_1\\ 1, & i=q_1.\end{cases}$$

Differentiate (2.11-6) with respect to α

$$(2.11\text{-}7)\quad u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,\alpha u,\alpha v,t)+6\alpha u^{r_1}u^{r_2}v^{r_3}\frac{\partial^3\psi^i}{\partial u^{r_1}\partial u^{r_2}\partial v^{r_3}}(p,\alpha u,\alpha v,t)$$

$$+\,6u^{r_4}v^{r_5}\frac{\partial^2\psi^i}{\partial u^{r_4}\partial v^{r_5}}(p,\alpha u,\alpha v,t)+12\alpha^2u^{r_6}v^{r_7}v^{r_8}\frac{\partial^3\psi^i}{\partial u^{r_6}\partial v^{r_7}\partial v^{r_8}}(p,\alpha u,\alpha v,t)$$

$$+\,12\alpha v^{r_9}v^{r_{10}}\frac{\partial^2\psi^i}{\partial v^{r_9}\partial v^{r_{10}}}(p,\alpha u,\alpha v,t)+8\alpha^3v^{q_1}v^{q_2}v^{q_3}\frac{\partial^3\psi^i}{\partial v^{q_1}\partial v^{q_2}\partial v^{q_3}}(p,\alpha u,\alpha v,t)$$

$$= \dot{\psi}^i(p,u,v,f)f_{\alpha\alpha\alpha} + 3f_\alpha f_{\alpha\alpha}\ddot{\psi}^i(p,u,v,f) + f_\alpha^3\dddot{\psi}^i(p,u,v,f)$$

where $p_1, p_2, p_3, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, q_1, q_2,$

$q_3 = 1,2,\ldots,n$

Replace $\dddot{\psi}^i(p,u,v,f)$ in (2.11-7) by the power series expansion, this gives

$$u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,\alpha u,\alpha v,t) + 6\alpha u^{r_1}u^{r_2}v^{r_3}\frac{\partial^3\psi^i}{\partial u^{r_1}\partial u^{r_2}\partial u^{r_3}}(p,\alpha u,\alpha v,t)$$

$$+ 6u^{r_4}v^{r_5}\frac{\partial^2\psi^i}{\partial u^{r_4}\partial v^{r_5}}(p,\alpha u,\alpha v,t) + 12\alpha^2 u^{r_6}v^{r_7}v^{r_8}\frac{\partial^3\psi^i}{\partial u^{r_6}\partial v^{r_7}\partial v^{r_8}}(p,\alpha u,\alpha v,t)$$

$$+ 12\alpha v^{r_9}v^{r_{10}}\frac{\partial^2\psi^i}{\partial v^{r_9}\partial v^{r_{10}}}(p,\alpha u,\alpha v,t) + 8\alpha^3 v^{q_1}v^{q_2}v^{q_3}\frac{\partial^3\psi^i}{\partial v^{q_1}\partial v^{q_2}\partial v^{q_3}}(p,\alpha u,\alpha v,t)$$

$$= \dot{\psi}^i(p,u,v,f)f_{\alpha\alpha\alpha} + 3f_\alpha f_{\alpha\alpha}\ddot{\psi}^i(p,u,v,f) + f_\alpha^3 c\dot{\psi}^i(p,u,v,f)$$

$$+ f_\alpha^3\, G^i_{j_1j_2j_3}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3})(p,u,v,f)$$

$$+ f_\alpha^3\, L^i_\ell(\psi(p,u,v,f))t^\ell + f_\alpha^3\, K^i_{k_1j_1}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1})(p,u,v,f)$$

$$+ f_\alpha^3\, M^i_{j_1\ell}(\psi(p,u,v,f))\dot{\psi}^{j_1}(p,u,v,f)t^\ell$$

$$+ f_\alpha^3\, M^i_{j_1j_2\ell}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2})(p,u,v,f)t^\ell + \ldots.$$

$$+ f_\alpha^3\, N^i_{k_1\ell}(\psi(p,u,v,f))\ddot{\psi}^{k_1}(p,u,v,f)t^\ell$$

$$+ f_\alpha^3 N^i_{k_1k_2\ell}(\psi(p,u,v,f))(\ddot\psi^{k_1}\ddot\psi^{k_2})(p,u,v,f)t^\ell + \dots$$

$$+ f_\alpha^3 R^i_{k_1j_1\ell}(\psi(p,u,v,f))(\ddot\psi^{k_1}\dot\psi^{j_1})(p,u,v,f)t^\ell$$

$$+ f_\alpha^3 R^i_{k_1j_1j_2\ell}(\psi(p,u,v,f))(\ddot\psi^{k_1}\dot\psi^{j_1}\dot\psi^{j_2})(p,u,v,f)t^\ell + \dots$$

$$+ f_\alpha^3 R^i_{k_1k_2j_1\ell}(\psi(p,u,v,f))(\ddot\psi^{k_1}\ddot\psi^{k_2}\dot\psi^{j_1})(p,u,v,f)t^\ell$$

$$+ f_\alpha^3 R^i_{k_1k_2j_1j_2\ell}(\psi(p,u,v,f))(\ddot\psi^{k_1}\ddot\psi^{k_2}\dot\psi^{j_1}\dot\psi^{j_2})(p,u,v,f)t^\ell$$

$$+ \dots$$

Let $\alpha = 0$, we have

$$u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,0,0,t) + 6u^{r_4}v^{r_5}\frac{\partial^2\psi^i}{\partial u^{r_4}\partial v^{r_5}}(p,0,0,t) = u^i f_{\alpha\alpha\alpha}(0,t)$$

$$+ cu^i f_\alpha^3(0,t) + f_\alpha^3(0,t)G^i_{j_1j_2j_3}(p)u^{j_1}u^{j_2}u^{j_3} + f_\alpha^3(0,t)L^i_\ell(p)t^\ell$$

$$+ f_\alpha^3(0,t)K^i_{k_1j_1}(p)v^{k_1}u^{j_1} + f_\alpha^3(0,t)M^i_{j_1\ell}(p)\,u^{j_1}t^\ell$$

$$+ f_\alpha^3(0,t)M^i_{j_1j_2\ell}(p)u^{j_1}u^{j_2}t^\ell + \dots + f_\alpha^3(0,t)N^i_{k_1\ell}(p)v^{k_1}t^\ell$$

$$+ f_\alpha^3(0,t)N^i_{k_1k_2\ell}(p)v^{k_1}v^{k_2}t^\ell + \dots + f_\alpha^3(0,t)R^i_{k_1j_1\ell}(p)v^{k_1}u^{j_1}t^\ell$$

$$+ f_\alpha^3(0,t)R^i_{k_1j_1j_2\ell}(p)v^{k_1}u^{j_1}u^{j_2}t^\ell + \dots + f_\alpha^3(0,t)R^i_{k_1k_2j_1\ell}(p)v^{k_1}v^{k_2}u^{j_1}t^\ell$$

$$+ f_\alpha^3(0,t)R^i_{k_1k_2j_1j_2\ell}(p)v^{k_1}v^{k_2}u^{j_1}u^{j_2}t^\ell + \dots$$

$$(2.11\text{-}8)\quad u^{p_1}u^{p_2}u^{p_3}\,\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,0,0,t) + 6u^{r_4}v^{r_5}\,\frac{\partial^2\psi^i}{\partial v^{r_4}\partial v^{r_5}}(p,0,0,t)$$

$$= (f_\alpha^3(0,t)G^i_{j_1j_2j_3}(p) + f_\alpha^3(0,t)M^i_{j_1j_2j_3\ell}(p)t^\ell)u^{j_1}u^{j_2}u^{j_3}$$

$$+ (f_\alpha^3(0,t)K^i_{j_1k_1}(p) + f_\alpha^3(0,t)t^\ell R^i_{k_1j_1\ell}(p))u^{j_1}v^{k_1}$$

$$+ (f_{\alpha\alpha\alpha}(0,t) + cf_\alpha^3(0,t) + f_\alpha^3(0,t)M^i_{j_1\ell}(p)t^\ell\delta_i^{j_1})u^i$$

$$+ (f_\alpha^3(0,t)L^i_\ell(p)t^\ell) + (f_\alpha^3(0,t)M^i_{j_1j_2\ell}(p)t^\ell)u^{j_1}u^{j_2}$$

$$+ (f_\alpha^3(0,t)M^i_{j_1j_2j_3j_4\ell}(p)t^\ell)u^{j_1}u^{j_2}u^{j_3}u^{j_4}$$

$$+ (f_\alpha^3(0,t)M^i_{j_1j_2j_3j_4j_5\ell}(p)t^\ell)u^{j_1}u^{j_2}u^{j_3}u^{j_4}u^{j_5} + \ldots$$

$$+ f_\alpha^3(0,t)N^i_{k_1\ell}(p)t^\ell v^{k_1} + f_\alpha^3(0,t)N^i_{k_1k_2\ell}(p)t^\ell v^{k_1}v^{k_2} + \ldots$$

$$+ f_\alpha^3(0,t)R^i_{k_1j_1j_2\ell}(p)t^\ell\, u^{j_1}u^{j_2}v^{k_1} + \ldots$$

$$+ f_\alpha^3(0,t)R^i_{k_1k_2j_1\ell}(p)t^\ell\, u^{j_1}v^{k_1}v^{k_2} + f_\alpha^3(0,t)R^i_{k_1k_2j_1j_2\ell}(p)t^\ell v^{k_1}v^{k_2}u^{j_1}u^{j_2}$$

$$+ \ldots.$$

We want to show that

i) $$\frac{\partial^3\psi^i}{\partial u^{j_1}\partial u^{j_2}\partial u^{j_3}}(p,0,0,t) = f_\alpha^3(0,t)G^i_{j_1j_2j_3}(p)$$

ii) $$\frac{6\partial^2\psi^i}{\partial u^{j_1}\partial v^{k_1}}(p,0,0,t) = f_\alpha^3(0,t)K^i_{j_1k_1}(p)$$

iii) $f_{\alpha\alpha\alpha}(0,t) + cf_{\alpha}^{3}(0,t) = 0$

iv) there exists $t_0 \neq 0$ in W s.t. $f_{\alpha}(0,t_0) \neq 0$

let $\alpha = 0$ then (2.11-7) becomes

$$(2.11\text{-}9)\quad u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,0,0,t) + 6u^{r_4}v^{r_5}\frac{\partial^2\psi^i}{\partial u^{r_4}\partial v^{r_5}}(p,0,0,t)$$

$$= u^i f_{\alpha\alpha\alpha}(0,t) + f_{\alpha}^{3}(0,t)\dddot{\psi}^i(p,u,v,0)$$

From (2.11-4) let t = 0, we get

$$\dddot{\psi}^i(p,u,v,0) = cu^i + G^i_{j_1j_2j_3}(p)u^{j_1}u^{j_2}u^{j_3} + K^i_{k_1j_1}(p)v^{k_1}u^{j_1}$$

substitute this equation into (2.11-9)

$$u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}(p,0,0,t) + 6u^{r_4}v^{r_5}\frac{\partial^2\psi^i}{\partial u^{r_4}\partial v^{r_5}}(p,0,0,t)$$

$$= u^i f_{\alpha\alpha\alpha}(0,t) + f_{\alpha}^{3}(0,t)cu^i + f_{\alpha}^{3}(0,t)G^i_{j_1j_2j_3}(p)u^{j_1}u^{j_2}u^{j_3} + f_{\alpha}^{3}(0,t)K^i_{k_1j_1}(p)v^{k_1}u^{j_1}$$

this equation implies that

$f_{\alpha\alpha\alpha}(0,t) + f_{\alpha}^{3}(0,t)c = 0$ so iii) is proved

$\dfrac{\partial^3\psi^i}{\partial u^{j_1}\partial u^{j_2}\partial u^{j_3}}(p,0,0,t) = f_{\alpha}^{3}(0,t)G^i_{j_1j_2j_3}(p)$ so i) is proved

$\dfrac{6\partial^2\psi^i}{\partial u^{j_1}\partial v^{k_1}}(p,0,0,t) = f_{\alpha}^{3}(0,t)K^i_{j_1k_1}(p)$ so ii) is proved.

To prove iv), suppose $f_\alpha(0,t) = 0$ in W, we claim that

$$\frac{\partial^n}{\partial\alpha^n} f(0,t) = 0 \quad \forall n = 2,3,\ldots$$

We shall use induction on n to prove this.

But first, we shall prove the following equation by induction.

$$(2.11\text{-}10) \quad \frac{\partial^k}{\partial\alpha^k}\psi^i(p,\alpha u,\alpha v,t) = f_\alpha \frac{\partial^{k-1}}{\partial\alpha^{k-1}}\mathring{\psi}^i(p,u,v,v,f)$$

$$+ \binom{k-1}{1}\frac{\partial^{k-2}}{\partial\alpha^{k-2}}\mathring{\psi}^i(p,u,v,f)\frac{\partial f_\alpha}{\partial\alpha} + \binom{k-1}{2}\frac{\partial^{k-3}}{\partial\alpha^{k-3}}\mathring{\psi}^i(p,u,v,f)\frac{\partial^2 f_\alpha}{\partial\alpha^2}$$

$$+ \binom{k-1}{3}\frac{\partial^{k-4}}{\partial\alpha^{k-4}}\mathring{\psi}^i(p,u,v,f)\frac{\partial^3 f_\alpha}{\partial\alpha^3} + \ldots + \binom{k-1}{k-1}\mathring{\psi}^i(p,u,v,f)\frac{\partial^{k-1}}{\partial\alpha^{k-1}} f_\alpha$$

$$\forall k = 2,3,\ldots .$$

The proof goes as follows :

for k = 2, $\quad \dfrac{\partial^2\psi^i}{\partial\alpha^2}(p,\alpha u,\alpha v,t) = f_\alpha \dfrac{\partial\mathring{\psi}^i}{\partial\alpha}(p,u,v,f) + \binom{1}{1}\mathring{\psi}^i(p,u,v,f)f_{\alpha\alpha}$

Assume (2.11-10) is true for all $k = 2,3,\ldots,n$

for $k = n+1$,

$$\frac{\partial^{n+1}}{\partial\alpha^{n+1}}\psi^i(p,\alpha u,\alpha v,t) = \frac{\partial}{\partial\alpha}\left(\frac{\partial^n}{\partial\alpha^n}\psi^i(p,\alpha u,\alpha v,t)\right)$$

$$= \frac{\partial}{\partial\alpha}\left(f_\alpha \frac{\partial^{n-1}}{\partial\alpha^{n-1}}\mathring{\psi}^i(p,u,v,f)\right.$$

$$+ \binom{n-1}{1}\frac{\partial^{n-2}}{\partial\alpha^{n-2}}\mathring{\psi}^i(p,u,v,f)\frac{\partial f_\alpha}{\partial\alpha}$$

$$+ \binom{n-1}{2}\frac{\partial^{n-3}}{\partial\alpha^{n-3}}\mathring{\psi}^i(p,u,v,f)\frac{\partial^2 f_\alpha}{\partial\alpha^2}$$

$$+ \binom{n-1}{3}\frac{\partial^{n-4}}{\partial\alpha^{n-4}}\mathring{\psi}^i(p,u,v,f)\frac{\partial^3 f_\alpha}{\partial\alpha^3}$$

$$\left. + \ldots + \binom{n-1}{n-1}\mathring{\psi}^i(p,u,v,f)\frac{\partial^{n-1} f_\alpha}{\partial\alpha^{n-1}}\right)$$

so $\frac{\partial^{n+1}}{\partial\alpha^{n+1}} \psi^i(p,\alpha u,\alpha v,t) = f_\alpha \frac{\partial^n}{\partial\alpha^n} \mathring{\psi}^i(p,u,v,f)$

$$+ [\binom{n-1}{0}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\mathring{\psi}^i(p,u,v,f)\frac{\partial f_\alpha}{\partial\alpha} + \binom{n-1}{1}\frac{\partial^{n-1}}{\partial\alpha^{n-1}} \mathring{\psi}^i(p,u,v,f \frac{\partial f_\alpha}{\partial\alpha}]$$

$$+ [\binom{n-1}{1}\frac{\partial^{n-2}}{\partial\alpha^{n-2}} \mathring{\psi}^i(p,u,v,f) \frac{\partial^2 f_\alpha}{\partial\alpha^2}$$

$$+ \binom{n-1}{2} \frac{\partial^{n-2}}{\partial\alpha^{n-2}} \mathring{\psi}^i(p,u,v,f) \frac{\partial^2 f_\alpha}{\partial\alpha^2}]$$

$$+ [\binom{n-1}{2} \frac{\partial^{n-3}}{\partial\alpha^{n-3}}\mathring{\psi}^i(p,u,v,f) \frac{\partial^3 f_\alpha}{\partial\alpha^3}$$

$$+ \binom{n-1}{3} \frac{\partial^{n-3}}{\partial\alpha^{n-3}} \mathring{\psi}^i(p,u,v,f) \frac{\partial^3 f_\alpha}{\partial\alpha^3}] + \ldots$$

$$+ [\binom{n-1}{n-2} \frac{\partial\mathring{\psi}^i}{\partial\alpha}(p,u,v,f) \frac{\partial^{n-1} f_\alpha}{\partial\alpha^{n-1}}$$

$$+ \binom{n-1}{n-1} \frac{\partial\mathring{\psi}^i}{\partial\alpha}(p,u,v,f) \frac{\partial^{n-1}}{\partial\alpha^{n-1}} f_\alpha]$$

$$+ \binom{n-1}{n-1} \mathring{\psi}^i(p,u,v,f) \frac{\partial^n f_\alpha}{\partial\alpha^n}$$

Since $\binom{n}{k-1}+\binom{n}{k} = \binom{n+1}{k}$ so we have

$$(2.11\text{-}11) \quad \frac{\partial^{n+1}}{\partial\alpha^{n+1}}\psi^i(p,\alpha u,\alpha v,t) = f_\alpha\frac{\partial^n}{\partial\alpha^n}\mathring{\psi}^i(p,u,v,f)+\binom{n}{1}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\mathring{\psi}^i(p,u,v,f)\frac{\partial f_\alpha}{\partial\alpha}$$

$$+\binom{n}{2}\frac{\partial^{n-2}}{\partial\alpha^{n-2}}\mathring{\psi}^i(p,u,v,f)\frac{\partial^2 f_\alpha}{\partial\alpha^2} + \ldots.$$

$$+\binom{n}{n-1}\frac{\partial\mathring{\psi}^i}{\partial\alpha}(p,u,v,f)\frac{\partial^{n-1} f_\alpha}{\partial\alpha^{n-1}}$$

$$+\binom{n}{n}\mathring{\psi}^i(p,u,v,f) \frac{\partial^n f_\alpha}{\partial\alpha^n}$$

so (2.11-10) is true for all $k = 2,3,\dots$

Now let us turn back to the proof that if $f_\alpha(0,t) \equiv 0$, then

$$\frac{\partial^n f(0,t)}{\partial\alpha^n} \equiv 0 \qquad \forall\, n = 2,3,\dots$$

For $n = 2$ we proved that $f_{\alpha\alpha}(0,t) = 0$.

Assume $$\frac{\partial^k f(0,t)}{\partial\alpha^k} \equiv 0 \qquad k = 2,3,\dots, n$$

Before we continue our proof let us prove the following lemma.

Lemma 2.12 For $n > 2$, $$\frac{\partial^n}{\partial\alpha^n}\psi^i(p,\alpha u,\alpha v,t) = u^{p_1}\cdots u^{p_n}\frac{\partial^n}{\partial u^{p_1}\cdots\partial u^{p_n}}\psi^i(p,\alpha u,\alpha v,t)$$

$$+\sum_{r=0}^{n-1}\sum_{s=1}^{n} X^i_{p_1\cdots p_r q_1\cdots q_s}(\alpha,u,v,p,t)\,u^{p_1}\cdots u^{p_r}v^{q_1}\cdots v^{q_s}$$

where $2 \leq r + s \leq n$, where $X^i_{p_1\cdots p_r q_1\cdots q_s}(\alpha,u,v,p,t)$ is a function of α,p,u,v,t.

Proof We will prove this by induction on n.

For $n = 3$; $$\frac{\partial^3}{\partial\alpha^3}\psi^i(p,\alpha u,\alpha v,t) = u^{p_1}u^{p_2}u^{p_3}\frac{\partial^3\psi^i(p,\alpha u,\alpha v,t)}{\partial u^{p_1}\partial u^{p_2}\partial u^{p_3}}$$

$$+ 6\alpha u^{p_1}u^{p_2}v^{q_1}\frac{\partial^3\psi^i}{\partial u^{p_1}\partial u^{p_2}\partial v^{q_1}}(p,\alpha u,\alpha v,t) + 6u^{p_1}v^{q_1}\frac{\partial^2\psi^i(p,\alpha u,\alpha v,t)}{\partial u^{p_1}\partial v^{q_1}}$$

$$+ 12\alpha^2 u^{p_1}v^{q_1}v^{q_2}\frac{\partial^3\psi^i(p,\alpha u,\alpha v,t)}{\partial u^{p_1}\partial v^{q_1}\partial v^{q_2}}$$

$$+ 12\alpha\, v^{q_1}v^{q_2}\frac{\partial^2\psi^i(p,\alpha u,\alpha v,t)}{\partial v^{q_1}\partial v^{q_2}}$$

$$+ 8\alpha^3 v^{q_1}v^{q_2}v^{q_3}\frac{\partial^3\psi^i(p,\alpha u,\alpha v,t)}{\partial v^{q_1}\partial v^{q_2}\partial v^{q_3}}$$

Hence Lemma 2.12. is true for n = 3.

Assume Lemma 2.12. is true for n = k, that is

$$\frac{\partial^k}{\partial\alpha^k}\psi^i(p,\alpha u,\alpha v,t) = u^{p_1}\ldots u^{p_k}\frac{\partial^k}{\partial u^{p_1}\ldots\partial u^{p_k}}\psi^i(p,\alpha u,\alpha v,t)$$

$$+\sum_{r=0}^{k-1}\sum_{s=1}^{k} X_{p_1\ldots p_r q_1\ldots q_s}(\alpha,u,v,p,t)u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}$$

such that $2 \leq r + s \leq k$.

$$\frac{\partial^{k+1}}{\partial\alpha^{k+1}}\psi^i(p,\alpha u,\alpha v,t) = u^{p_1}\ldots u^{p_{k+1}}\frac{\partial^{k+1}}{\partial u^{p_1}\ldots\partial u^{p_{k+1}}}\psi^i(p,\alpha u,\alpha v,t)$$

$$+ 2\alpha u^{p_1}\ldots u^{p_k}v^{q_1}\frac{\partial^{k+1}}{\partial u^{p_1}\ldots\partial u^{p_k}\partial v^{q_1}}\psi^i(p,\alpha u,\alpha v,t)$$

$$+\sum_{r=0}^{k-1}\sum_{s=1}^{k} u^{p_1}\ldots u^{p_r}u^{p_{r+1}}v^{q_1}\ldots v^{q_s}\frac{\partial}{\partial u^{p_{r+1}}}X_{p_1\ldots p_r q_1\ldots q_s}$$

$$+\sum_{r=0}^{k-1}\sum_{s=1}^{k} 2\alpha u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}v^{q_{s+1}}\frac{\partial}{\partial v^{q_{s+1}}}X_{p_1\ldots p_r q_1\ldots q_s}$$

$$+\sum_{r=0}^{k-1}\sum_{s=1}^{k} Y_{p_1\ldots p_r q_1\ldots q_s}(\alpha,u,v,p,t)u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}$$

so $$\frac{\partial^{k+1}}{\partial\alpha^{k+1}}\psi^i(p,\alpha u,\alpha v,t) = u^{p_1}\ldots u^{p_{k+1}}\frac{\partial^{k+1}}{\partial u^{p_1}\ldots\partial u^{p_k}}\psi^i(p,\alpha u,\alpha v,t)$$

$$+\sum_{r=0}^{k}\sum_{s=1}^{k+1} Z(\alpha,u,v,p,t)u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}\,.$$

So Lemma 2.12 is proved.

Substitute $\alpha = 0$ into equation (2.11-11) we obtain

$$u^{p_1}\ldots u^{p_{n+1}} \frac{\partial^{n+1}}{\partial u^{p_1}\ldots\partial u^{p_{n+1}}}\psi^i(p,0,0,t)+\sum_{r=0}^{n}\sum_{s=1}^{n+1} X_{p_1\ldots p_r q_1\ldots q_s}(0,u,v,p,t)u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}$$

$$= f_\alpha(0,t)\frac{\partial^n}{\partial\alpha^n}\mathring{\psi}^i(p,u,v,0)+\binom{n}{1}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\mathring{\psi}^i(p,u,v,0)\frac{\partial}{\partial\alpha}f_\alpha(0,t)$$

$$+\binom{n}{2}\frac{\partial^{n-2}}{\partial\alpha^{n-2}}\mathring{\psi}^i(p,u,v,0)\frac{\partial^2}{\partial\alpha^2}f_\alpha(0,t)+\ldots\ldots$$

$$+\binom{n}{n-1}\frac{\partial}{\partial\alpha}\mathring{\psi}^i(p,u,v,0)\frac{\partial^{n-1}}{\partial\alpha^{n-1}}f_\alpha(0,t)+\binom{n}{n}\mathring{\psi}^i(p,u,v,0)\frac{\partial^n}{\partial\alpha^n}f_\alpha(0,t)$$

where $2 \leqslant r + s \leqslant n$.

$$u^{p_1}\ldots u^{p_{n+1}} \frac{\partial^{n+1}}{\partial u^{p_1}\ldots\partial u^{p_{n+1}}}\psi^i(p,0,0,t)+\sum_{r=0}^{n}\sum_{s=1}^{n+1} X_{p_1\ldots p_r q_1\ldots q_s}(0,u,v,p,t)u^{p_1}\ldots u^{p_r}v^{q_1}\ldots v^{q_s}$$

$$= u^i\frac{\partial^n}{\partial\alpha^n}f_\alpha(0,t) \quad = \quad u^i\frac{\partial^{n+1}}{\partial\alpha^{n+1}}f(0,t).$$

So $\frac{\partial^{n+1}}{\partial\alpha^{n+1}}f(0,t) = 0$, that is $\frac{\partial^n}{\partial\alpha^n}f(0,t) \equiv 0, \quad \forall n = 2,3,\ldots$

To prove $f_\alpha(0,t) \neq 0$ for some t, all we need to show is the following.

<u>Lemma 2.13</u> If $f(\alpha,t)$ is analytic in a neighbourhood of (0,0) and $f(0,t) = 0$, $\frac{\partial^n}{\partial\alpha^n}f(0,t) = 0$ for $n = 1,2,\ldots$, then $f(\alpha,t) \equiv 0$.

<u>Proof</u> Fix t_0 such that $f(\alpha,t_0)$ is defined in a neighbourhood of $\alpha = 0$ denote $f(\alpha,t_0)$ by $F_{t_0}(\alpha)$.

Since $F_{t_0}(\alpha)$ is analytic

$$F_{t_0}(\alpha) = F_{t_0}(0)+F'_{t_0}(0)\frac{\alpha}{1!}+ F''_{t_0}(0)\frac{\alpha^2}{2!} +\dots+ F^{(n)}_{t_0}(0)\ \frac{\alpha^n}{n!} +\dots$$

$$f(\alpha,t_0)= f(0,t_0)+ \frac{\partial}{\partial\alpha}f(0,t_0)\alpha+ \frac{\partial^2}{\partial\alpha^2}f(0,t_0)\frac{\alpha^2}{2!} +\dots+ \frac{\partial^n}{\partial\alpha^n}f(0,t_0)\frac{\alpha^n}{n!} +\dots$$

$$= 0 \ .$$

Since t_0 is arbitary, we must have that $f(\alpha,t) \equiv 0$.

Thus Lemma 2.13 is proved.

So we have that $f(\alpha,t) \equiv 0 \quad \forall \alpha, t \in R$

and
$$\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,f(\alpha,t))$$
$$= \psi^i(p,u,v,0)$$
$$= p^i$$

hence
$$\mathring{\psi}^i(p,\alpha u,\alpha v,t) = 0$$

implying that
$$\mathring{\psi}^i(p,\alpha u,\alpha v,0) = 0$$

but
$$\mathring{\psi}^i(p,\alpha u,\alpha v,0) = \alpha u^i$$

so
$$\alpha u^i = 0 \qquad \forall \alpha \ , \forall u.$$

This is a contradiction, so the supposition on the top of page 42 that $f_\alpha(0,t) = 0$ is false, hence $t_0 \neq 0$ such that $f_\alpha(0,t_0) \neq 0$ so iv) is proved.

From (2.11-8) and by knowing that $\exists t_0 \neq 0$ such that $f_\alpha(0,t_0) \neq 0$ we can conclude ,from equation (2.11-8)page 40, that $L^i_\ell(p) = 0,\ \ell,\ i$

all N^i's = 0 , all R^i's = 0, all M^i's = 0 .

Finally we see that from (2.11-4) all we have left is

$$\dddot{\psi}^i = c\mathring{\psi}^i + G^i_{j_1 j_2 j_3}(\psi)\mathring{\psi}^{j_1}\mathring{\psi}^{j_2}\mathring{\psi}^{j_3} + K^i_{k_1 j_1}(\psi)\ddot{\psi}^{k_1}\mathring{\psi}^{j_1}$$

Next, we want to show that c = 0.

Into equation (2.11-3) on page 32 substitute

$$\dddot{\psi}^i = G^i_{j_1j_2j_3}(\vec{\psi})\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1k_1}(\vec{\psi})\dot{\psi}^{j_1}\ddot{\psi}^{k_1} + c\dot{\psi}^i .$$

This gives

$$c\dot{\psi}^i(p,\alpha u,\alpha v,t) + G^i_{j_1j_2j_3}(\psi(p,\alpha u,\alpha v,t))(\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3})(p,\alpha u,\alpha v,t)$$

$$+ K^i_{k_1j_1}(\psi(p,\alpha u,\alpha v,t))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1})(p,\alpha u,\alpha v,t)$$

$$= G^i_{j_1j_2j_3}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3})(p,u,v,f)f_t^3 + c\dot{\psi}^i(p,u,v,f)f_t^3$$

$$+ f_t^3 K^i_{k_1j_1}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1})(p,u,v,f) + \dot{\psi}^i(p,u,v,f)f_{ttt}$$

$$+ 3f_tf_{tt}\ddot{\psi}^i(p,u,v,f).$$

Using (2.11-1) and (2.11-2) we get

$$G^i_{j_1j_2j_3}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3})(p,u,v,f)f_t^3 + c\dot{\psi}^i(p,u,v,f)f_t$$

$$+ K^i_{k_1j_1}(\psi(p,u,v,f))[\ddot{\psi}^{k_1}(p,u,v,f)f_t^2 + \dot{\psi}^{k_1}(p,u,v,f)f_{tt}]\dot{\psi}^{j_1}(p,u,v,f)f_t$$

$$= G^i_{j_1j_2j_3}(\psi(p,u,v,f))(\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3})(p,u,v,f)f_t^3 + c\dot{\psi}^i(p,u,v,f)f_t^3$$

$$+ f_t^3 K^i_{k_1j_1}(\psi(p,u,v,f))(\ddot{\psi}^{k_1}\dot{\psi}^{j_1})(p,u,v,f) + \dot{\psi}^i(p,u,v,f)f_{ttt}$$

$$+ 3f_tf_{tt}\ddot{\psi}^i(p,u,v,f)$$

so $[K^i_{j_1k_1}(\psi(p,u,v,f))(\mathring{\psi}^{j_1}\mathring{\psi}^{k_1})(p,u,v,f)-3\ddot{\psi}^i(p,u,v,f)]f_t f_{tt}$

$= \mathring{\psi}^i(p,u,v,f)[f_{ttt}- cf_t+ cf_t^3] \quad \forall p \in \Pi_1(\Omega), \forall u \in R^n$

$\forall v \in R^n, \forall \alpha \ , \ t \in R.$

Fix u_0 such that $u_0^i \neq 0$ i, choose a small neighbourhood U of u_0 such that $\dot{\psi}^i(p,u,v,t) \neq 0$ $i = 1,\ldots,n$ $u \in U$ (we can do this by continuity). Then

$$\frac{1}{\mathring{\psi}^i(p,u,v,f)}[K^i_{j_1k_1}(\psi(p,u,v,f))(\mathring{\psi}^{j_1}\mathring{\psi}^{k_1})(p,u,v,f)-3\ddot{\psi}^i(p,u,v,f)]f_t f_{tt}$$

$= f_{ttt}- cf_t+ cf_t^3$ for small t.

<u>case 1</u> $f_t f_{tt} \neq 0$. Since RHS of the above equation is independent of p, u and v, hence

$$\frac{1}{\dot{\psi}^i(p,u,v,f)}[K^i_{j_1k_1}(\mathring{\psi}(p,u,v,f))(\mathring{\psi}^{j_1}\mathring{\psi}^{k_1})(p,u,v,f)- 3\ddot{\psi}^i(p,u,v,f)]$$

must be independent of p, u and v, so

$$\frac{1}{\dot{\psi}^i(p,u,v,f)}[K^i_{j_1k_1}(\psi(p,u,v,f)(\mathring{\psi}^{j_1}\mathring{\psi}^{k_1})(p,u,v,f)- 3\ddot{\psi}^i(p,u,v,f)]$$

$= k(\alpha,t).$

Let $t = 0$, fix α_0 , we get

$$v^i = \frac{1}{3} K^i_{k_1j_1}(p)\, u^{j_1} u^{k_1} - \frac{1}{3} k\,(\alpha_0,0)u^i$$

But u and v are independent variables, therefore we get a contradiction for the case $f_t f_{tt} \not\equiv 0$. Thus we must have

<u>case 2</u> $f_t f_{tt} \equiv 0$, since $f_t(\alpha,0) = \alpha \neq 0$. And since f is an analytic function, by Theorem 1.15, we have

$$f_{tt}(\alpha,t) \equiv 0 \text{ for t sufficiently small}$$

so $f_{ttt}(\alpha,t) = 0$, and since

$$c = \frac{f_{ttt}(\alpha,0)}{\alpha-\alpha^3}\,\delta^i_j \quad \text{we get} \quad c = 0$$

also we get $f_t(\alpha,t) = g(\alpha)$, for some function g

so $f(\alpha,t) = g(\alpha)t + A(\alpha)$, for some function A

$f(\alpha,0) = A(\alpha)$ implying $A(\alpha) = 0$

then $f(\alpha,t) = g(\alpha)t$

but $f_t(\alpha,t) = g(\alpha)$, let $t = 0$, we have

$f_t(\alpha,0) = g(\alpha)$, but one page 31 we proved that $f_t(\alpha,0) = \alpha$

therefore $\alpha = g(\alpha)$,

So $f(\alpha,t) = \alpha t$.

So we get that our differential equation must be of the form

$$\dddot{\psi}^i = G^i_{j_1 j_2 j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1 k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1} \quad (2.11\text{-}12)$$

This finished the proof of Theorem 2.11.

To prove the converse we need Lemma 2.14 and Theorem 2.15.

<u>Lemma 2.14</u> Let $\psi^i(p,u,v,t)$ be the solution to a differential equation of the form

$$\dddot{\psi}^i = G^i_{j_1 j_2 j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1 k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1} \quad , \quad \text{where}$$

$G^i_{j_1 j_2 j_3}$, $K^i_{j_1 k_1}$ are analytic on an open subset D of R^n, with initial conditions $\psi^i(p,u,v,0) = p^i$,

$$\dot{\psi}^i(p,u,v,0) = u^i \quad , \quad \ddot{\psi}^i(p,u,v,0) = v^i \text{ . Then}$$

$$\frac{\partial^n}{\partial t^n}\psi^i(p,u,v,t) = Y^i_{j_1\cdots j_n}(\psi)\dot{\psi}^{j_1}\cdots\dot{\psi}^{j_n} + Y^i_{j_1\cdots j_{n-2}k_1}(\psi)\dot{\psi}^{j_1}\cdots\dot{\psi}^{j_{n-2}}\ddot{\psi}^{k_1} + \cdots$$

$$+ \begin{cases} Y^i_{k_1\cdots k_{\frac{n}{2}}}(\psi)\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_{\frac{n}{2}}} & \text{if n is even} \\ Y^i_{j_1 k_1\cdots k_{\frac{n-1}{2}}}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1}\cdots\ddot{\psi}^{k_{\frac{n-1}{2}}} & \text{if n is odd.} \end{cases}$$

for all $n \geq 3$. That is,

$\frac{\partial^n}{\partial t^n}\psi^i(p,u,v,t)$ = polynomial in $\dot{\psi}$ and $\ddot{\psi}$ such that each term has exactly n dots.

<u>Proof</u> for n = 3, Lemma 2.14 is true.

Assume Lemma 2.14 is true for $n = m \geq 3$, that is

$$\frac{\partial^m}{\partial t^m}\psi^i(p,u,v,t) = Y^i_{j_1\cdot\cdot j_m}(\psi)\dot\psi^{j_1}\ldots\dot\psi^{j_m} + Y^i_{k_1 j_1\cdot\cdot j_{m-2}}(\psi)\dot\psi^{j_1}\ldots\dot\psi^{j_{m-2}}\ddot\psi^{k_1} + \ldots$$

$$+\begin{cases} Y^i_{k_1\ldots k_{\frac{m}{2}}}(\psi)\ddot\psi^{k_1}\ldots\ddot\psi^{k_{\frac{m}{2}}} & \text{if m is even} \\ Y^i_{j_1 k_1\cdot\cdot k_{\frac{m-1}{2}}}(\psi)\dot\psi^{j_1}\ddot\psi^{k_1}\ldots\ddot\psi^{k_{\frac{m-1}{2}}} & \text{if m is odd.}\end{cases}$$

That is, each term has exactly m dots by the induction hypothesis.

For n = m+1 we get

$$\frac{\partial^{m+1}}{\partial t^{m+1}}\psi^i(p,u,v,t) = \frac{\partial Y^i_{j_1\cdot\cdot j_m}(\psi)}{\partial x^{j_{m+1}}}\dot\psi^{j_1}\ldots\dot\psi^{j_m}\dot\psi^{j_{m+1}}$$

$$+ Y^i_{j_1\ldots j_m}(\psi)\ddot\psi^{j_1}\dot\psi^{j_2}\ldots\dot\psi^{j_m} + \ldots + Y^i_{j_1\cdot\cdot j_m}(\psi)\dot\psi^{j_1}\ldots\dot\psi^{j_{m-1}}\ddot\psi^{j_m}$$

$$+ \frac{\partial Y^i_{j_1\cdot\cdot j_{m-2}}}{\partial x^{j_{m-1}}}(\psi)\dot\psi^{j_1}\ldots\dot\psi^{j_{m-1}}\ddot\psi^{k_1}$$

$$+ Y^i_{j_1\ldots j_{m-2}k_1}(\psi)\ddot\psi^{j_1}\dot\psi^{j_2}\ldots\dot\psi^{j_{m-2}}\ddot\psi^{k_1} + \ldots + Y^i_{j_1\cdot\cdot j_{m-2}k_1}(\psi)\dot\psi^{j_1}\ldots\ddot\psi^{j_{m-2}}\ddot\psi^{k_1}$$

$$+ Y^i_{j_1\ldots j_{m-2}k_1}(\psi)\dot\psi^{j_1}\ldots\dot\psi^{j_{m-2}}\dddot\psi^{k_1} + \ldots$$

$$
+\begin{cases}
\dfrac{\partial Y^i_{k_1..k_{\frac{m}{2}}}}{\partial x^{j_1}}(\psi)\mathring{\psi}^{j_1}\ddot{\psi}^{k_1}\dots\ddot{\psi}^{k_{\frac{m}{2}}} + Y^i_{k_1..k_{\frac{m}{2}}}(\psi)\dddot{\psi}^{k_1}\ddot{\psi}^{k_2}..\ddot{\psi}^{k_{\frac{m}{2}}}+\dots \\[2ex]
+\, Y^i_{k_1..k_{\frac{m}{2}}}(\psi)\,\ddot{\psi}^{k_1}\dots\dddot{\psi}^{k_{\frac{m}{2}}} \qquad \text{if } m \text{ is even} \\[2ex]
+\, \dfrac{\partial Y^i_{k_1..k_{\frac{m-1}{2}}j_1}}{\partial x^{j_2}}(\psi)\ddot{\psi}^{k_1}\dots\ddot{\psi}^{k_{\frac{m-1}{2}}}\mathring{\psi}^{j_1}\mathring{\psi}^{j_2} \\[2ex]
+\, Y^i_{j_1k_1..k_{\frac{m-1}{2}}}(\psi)\,\ddot{\psi}^{j_1}\ddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dots\ddot{\psi}^{k_{\frac{m-1}{2}}}+\dots \\[2ex]
+\, Y^i_{j_1k_1..k_{\frac{m-1}{2}}}(\psi)\,\mathring{\psi}^{j_1}\dddot{\psi}^{k_1}\ddot{\psi}^{k_2}\dots\ddot{\psi}^{k_{\frac{m-1}{2}}}+\dots. \\[2ex]
+\, Y^i_{j_1k_1..k_{\frac{m-1}{2}}}(\psi)\,\mathring{\psi}^{j_1}\ddot{\psi}^{k_1}\dots\dddot{\psi}^{k_{\frac{m-1}{2}}} \qquad \text{if } m \text{ is odd.}
\end{cases}
$$

Thus we see that each term has $m+1$ dots, but some terms have $\dddot{\psi}^i$ in them so we substitute equation (2.11-12) and get

$$
\frac{\partial^{m+1}}{\partial t^{m+1}}\psi^i(p,u,v,t) = \frac{\partial Y^i_{j_1..j_m}}{\partial x^{j_{m+1}}}(\psi)\mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_{m+1}} + Y^i_{j_1..j_m}(\psi)\ddot{\psi}^{j_1}\mathring{\psi}^{j_2}\dots\mathring{\psi}^{j_m}
$$

$$
+\dots+ Y^i_{j_1..j_m}(\psi)\mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_{m-1}}\,\ddot{\psi}^{j_m} + \frac{\partial Y^i_{j_1\dots j_{m-2}\,k_1}}{\partial x^{j_{m-2}}}(\psi)\mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_{m-1}}\ddot{\psi}^{k_1}
$$

$$
+\, Y^i_{j_1..j_{m-2}k_1}(\psi)\ddot{\psi}^{j_1}\dots\mathring{\psi}^{j_{m-2}}\ddot{\psi}^{k_1}+\dots+ Y^i_{j_1..j_{m-2}k_1}(\psi)\mathring{\psi}^{j_1}\dots\ddot{\psi}^{j_{m-2}}\,\ddot{\psi}^{k_1}
$$

$$
+\, G^{k_1}_{j_{m-1}j_mj_{m+1}}(\psi)\, Y^i_{j_1\dots j_{m-2}k_1}(\psi)\mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_{m-2}}\mathring{\psi}^{j_{m-1}}\mathring{\psi}^{j_m}\mathring{\psi}^{j_{m+1}}
$$

$$
+\, K^{k_1}_{j_{m-1}k_2}(\psi)\, Y^i_{j_1..j_{m-2}k_1}(\psi)\, \mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_{m-2}}\mathring{\psi}^{j_{m-1}}\ddot{\psi}^{k_2} + \dots
$$

$$
+\left\{
\begin{array}{l}
\dfrac{\partial Y^i_{k_1\cdots k_{\frac{m}{2}}}}{\partial x^{j_1}}(\psi)\dot\psi^{j_1}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m}{2}}} + G^{k_1}_{j_1j_2j_3}(\psi)Y^i_{k_1\cdots k_{\frac{m}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\dot\psi^{j_3}\ddot\psi^{k_2}\cdots\ddot\psi^{k_{\frac{m}{2}}} \\
+ K^{k_1}_{j_1k_{\frac{m+2}{2}}}(\psi)Y^i_{k_1\cdots k_{\frac{m}{2}}}(\psi)\dot\psi^{j_1}\ddot\psi^{k_{\frac{m+2}{2}}}\ddot\psi^{k_2}\cdots\ddot\psi^{k_{\frac{m}{2}}} + \cdots \\
+ G^{k_{\frac{m}{2}}}_{j_1j_2j_3}(\psi)Y^i_{k_1\cdots k_{\frac{m}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\dot\psi^{j_3}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m-2}{2}}} \\
+ K^{k_{\frac{m}{2}}}_{j_1\ell}(\psi)Y^i_{k_1\cdots k_{\frac{m}{2}}}(\psi)\dot\psi^{j_1}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m-2}{2}}}\ddot\psi^{\ell} \quad \text{if m is even} \\
\dfrac{\partial Y^i_{k_1\cdots k_{\frac{m-1}{2}}j_1}}{\partial x^{j_2}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m-1}{2}}} + \cdots \\
+ G^{k_1}_{j_2j_3j_4}(\psi)Y^i_{j_1k_2\cdots k_{\frac{m-1}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\dot\psi^{j_3}\dot\psi^{j_4}\ddot\psi^{k_2}\cdots\ddot\psi^{k_{\frac{m-1}{2}}} \\
+ K^{k_1}_{j_2\ell}(\psi)Y^i_{j_1k_1\cdots k_{\frac{m-1}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\ddot\psi^{\ell_1}\ddot\psi^{k_2}\cdots\ddot\psi^{k_{\frac{m-1}{2}}} + \cdots \\
+ G^{k_{\frac{m-1}{2}}}_{j_2j_3j_4}(\psi)Y^i_{j_1k_1\cdots k_{\frac{m-1}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\dot\psi^{j_3}\dot\psi^{j_4}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m-3}{2}}} \\
+ K^{k_{\frac{m-1}{2}}}_{j_2k}(\psi)Y^i_{j_1k_1\cdots k_{\frac{m-1}{2}}}(\psi)\dot\psi^{j_1}\dot\psi^{j_2}\ddot\psi^{k_1}\cdots\ddot\psi^{k_{\frac{m-3}{2}}}\ddot\psi^{k} \quad \text{if m is odd.}
\end{array}
\right.
$$

So we see that when we replace $\dddot\psi^i$ by the RHS of equation (2.11-12) we are replacing $\dddot\psi^i$ by terms which have three dots so the result is the same, that is each term of $\frac{\partial^{m+1}}{\partial t^{m+1}}\psi^i$ has m+1 dots.

Lemma 2.14 is proved. Now for the converse.

Theorem 2.15 Let $\psi^i(p,u,v,t)$ be the solution to the differential equation

$$\dddot{\psi}^i = G^i_{j_1j_2j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1} \quad \text{where}$$

$G^i_{j_1j_2j_3}$, $K^i_{j_1k_1}$ are analytic on open subset D of R^n satisfying the initial conditions $\psi^i(p,u,v,0) = p^i$, $\dot{\psi}^i(p,u,v,0) = u^i$, $\ddot{\psi}^i(p,u,v,0) = v^i$. Then $\psi^i(p,\alpha u,\alpha v,t)$ exists if and only if $\psi^i(p,u,v,\alpha t)$ exists, and

$$\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,\alpha t)$$

Proof Since ψ^i is the solution of $\dddot{\psi}^i = G^i_{j_1j_2j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1}$ where $G^i_{j_1j_2j_3}$ and $K^i_{j_1k_1}$ are analytic on D we must have that $G^i_{j_1j_2j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1}$ is analytic on $D \times R^n \times R^n \times R$.

Fix $p_0 \varepsilon D$, u_0, $v_0 \varepsilon R^n$, $\alpha_0 \varepsilon R$.

Since $(p_0,\alpha_0u_0,\alpha_0v_0,0) \varepsilon D \times R^n \times R^n \times R$, then the Fundamental Theorem says that there exists an interval I of zero in R such that $\psi^i(p_0,\alpha_0u_0,\alpha_0v_0,t)$ exists on I and satisfies initial conditions

$$\psi^i(p_0, \alpha_0 u_0, \alpha_0 v_0, 0) = p_0^i$$

$$\dot{\psi}^i(p_0, \alpha_0 u_0, \alpha_0 v_0, 0) = \alpha_0 u_0^i$$

$$\ddot{\psi}^i(p_0, \alpha_0 u_0, \alpha_0 v_0, 0) = \alpha_0^2 v_0^i$$

and also ψ^i is an analytic function in all variables p,u,v,t.

Since $\psi^i(p,u,v,t)$ is analytic, then on I we can write

$$\psi^i(p,u,v,t) = p^i + \dot{\psi}^i(p,u,v,0)t + \ddot{\psi}^i(p,u,v,0)\frac{t^2}{2!} + \dddot{\psi}^i(p,u,v,0)\frac{t^3}{3!} + \ldots + \psi^{(n)i}(p,u,v,0)\frac{t^n}{n!} + \ldots$$

therefore

$$\psi^i(p,u,v,t) = p^i + u^i t + v^i \frac{t^2}{2!} + (G^i_{j_1 j_2 j_3}(p) u^{j_1} u^{j_2} u^{j_3} + K^i_{j_1 k_1}(p) u^{j_1} v^{k_1}) \frac{t^3}{3!} + \ldots$$

$$+ \left[Y^i_{j_1 \cdot\cdot j_n}(p) u^{j_1} \ldots u^{j_n} + Y^i_{j_1 \cdot\cdot j_{n-2} k_1}(p) u^{j_1} \ldots u^{j_{n-2}} v^{k_1} + \ldots + \begin{cases} Y^i_{k_1 \ldots k_{\frac{n}{2}}}(p) v^{k_1} \ldots v^{k_{\frac{n}{2}}} & \text{if n is even} \\ Y^i_{j_1 k_1 \ldots k_{\frac{n-1}{2}}}(p) u^{j_1} v^{k_1} \ldots v^{k_{\frac{n-1}{2}}} & \text{if n is odd} \end{cases} \right] \frac{t^n}{n!} + \ldots$$

Thus, $\psi^i(p_0,\alpha_0 u_0,\alpha_0 v_0,t) = p_0^i + \alpha_0 u_0^i t + \alpha_0^2 v_0^2 \frac{t^2}{2!}$

$$+ (G^i_{j_1 j_2 j_3}(p_0)\alpha_0^3 u_0^{j_1} u_0^{j_2} u_0^{j_3} + K^i_{j_1 k_1}(p_0)\alpha_0^3 u_0^{j_1} v_0^{k_1}) \frac{t^3}{3!} + \dots$$

$$+ \Big[Y^i_{j_1 \dots j_n}(p_0)\alpha_0^n u_0^{j_1} \dots u_0^{j_n} + Y^i_{j_1 \dots j_{n-2} k_1}(p)\alpha_0^n u_0^{j_1} \dots u_0^{j_{n-2}} v_0^{k_1} + \dots$$

$$+ \begin{cases} Y^i_{k_1 \dots k_{\frac{n}{2}}}(p_0)\alpha_0^n v_0^{k_1} \dots v_0^{k_{\frac{n}{2}}} & \text{if } n \text{ is even} \\ Y^i_{j_1 k_1 \dots k_{\frac{n-1}{2}}}(p_0)\alpha_0^n u_0^{j_1} v_0^{k_1} \dots v_0^{k_{\frac{n-1}{2}}} & \text{is } n \text{ is odd} \end{cases} \Bigg] \frac{t^n}{n!} + \dots$$

$$= p_0^i + u_0^i(\alpha_0 t) + v_0^i \frac{(\alpha_0^2 t^2)}{2!}$$

$$+ (G^i_{j_1 j_2 j_3}(p_0) u_0^{j_1} u_0^{j_2} u_0^{j_3} + K^i_{j_1 k_1}(p) u_0^{j_1} v_0^{k_1}) \frac{\alpha_0^3 t^3}{3!} + \dots$$

$$+ [Y^i_{j_1 \dots j_n}(p_0) u_0^{j_1} \dots u_0^{j_n} + Y^i_{j_1 \dots j_{n-2} k_1}(p) u_0^{j_1} \dots u_0^{j_{n-2}} v_0^{k_1} + \dots$$

$$+ \begin{cases} Y^i_{k_1 \dots k_{\frac{n}{2}}}(p_0) v_0^{k_1} \dots v_0^{k_{\frac{n}{2}}} & \text{if } n \text{ is even} \\ Y^i_{j_1 k_1 \dots k_{\frac{n-1}{2}}}(p_0) u_0^{j_1} v_0^{k_1} \dots v_0^{k_{\frac{n-1}{2}}} & \text{if } n \text{ is odd} \end{cases} \Bigg] \frac{\alpha^n t^n}{n!} + \dots$$

$$= \psi^i(p_0, u_0, v_0, \alpha_0 t)$$ (Where the Y^i's are defined on page 51)

That is, $\psi^i(p_0,\alpha_0 u_0, \alpha_0 v_0, t)$ exists if and only if $\psi^i(p_0,u_0,v_0,\alpha_0 t)$ exists and they are equal. Theorem 2.15 is proved.

Properties of $\psi(p,u,v,t)$

We have already proved that if $\vec{\psi}$ satisfies

$$\dddot{\psi}^i = G^i_{j_1 j_2 j_3}(\psi)\dot{\psi}^{j_1}\dot{\psi}^{j_2}\dot{\psi}^{j_3} + K^i_{j_1 k_1}(\psi)\dot{\psi}^{j_1}\ddot{\psi}^{k_1}$$

where $G^i_{j_1 j_2 j_3}$, $K^i_{j_1 k_1}$ are analytic on open subset D of R^n then $\psi(p,\alpha u,\alpha v,t)$ exists if and only if $\psi(p,u,v,\alpha t)$ exists and they are equal.

Using this fact, we can now show that ψ also has the following properties :

Property 1 Given $t_0 \in R$, $p_0 \in D$ then there exists neighbourhood U of the zero vector at p_0 such that $\forall u, v \in U$, $\psi(p_0,u,v,t_0)$ is defined.

Proof. Let t_0 be any point in R. Let p_0 be any initial point in D. Since $(p_0,0,0,0)$ is a point on the domain of definition of $\vec{H}$, hence the Fundamental Theorem for Third order Ordinary Differential Equations implies that there exists a neighbourhood W of the zero vector at p_0, and there exists an interval $I = (-r,r)$ such that $\psi(p_0,u,v,t)$ exists for all u, $v \in W$ and for all $t \in I$.

For any $t_0 \in R$, choose a real number $\alpha \neq 0$ such that $|\alpha t_0| < r$. Then $\alpha t_0 \in I$ and $\psi(p_0,u,v,\alpha t_0)$ exists for all u, $v \in W$. Since for $1 \leq i \leq n$, ψ^i satisfies $\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,\alpha t)$ for all $p \in D$, u, $v \in R^n$, $t \in J$, $\alpha \in R$.

Therefore, $\psi(p_0,\alpha u,\alpha v,t_0)$ exists for all $u, v \in W$. That is, $\psi(p_0,z,w,t_0)$ exists for all $z, w \in \alpha W$, $z = \alpha u$, $w = \alpha v$, and αW is a neighbourhood of zero vector at p_0 since $\alpha \neq 0$, then we are done.

<u>Property 2</u> Given any compact neighbourhood U of the zero vector at p_0 then there exists a neighbourhood W of zero in R such that $\forall t \in W$, $\forall u, v \in U$, $\psi(p_0,u,v,t)$ exists.

<u>Proof</u> Let $B(\vec{0}_{p_0},r_1) \supsetneq U$. By the Fundamental Theorem of Ordinary Differential Equations, we know that there exists a ball $B(\vec{0}_{p_0},r_2)$ and a neighbourhood W of 0 in R such that $\psi(p_0,u,v,t)$ exists $\forall u, v \in B(\vec{0}_{p_0},r_2)$ $\forall t \in W$.

Assume that the $B(\vec{0}_{p_0},r_2)$ is a proper subset of $B(\vec{0}_{p_0},r_1)$. To see this, choose $\alpha_0 \in R-\{0\}$ such that $|\alpha_0| < \frac{r_2}{r_1}$. For any $w \in B(\vec{0}_{p_0},r_1)$, $|\alpha_0 w| \lneq |\alpha_0||r_1| < r_2$ so $\alpha_0 w \in B(\vec{0}_{p_0},r_2)$ $\forall w \in B(\vec{0}_{p_0},r_1)$. Since $\psi(p_0,u,v,t)$ exists $\forall v \in B(\vec{0}_{p_0},r_2)$, $\forall t \in W$, hence $\psi(p_0,\alpha_0 u,\alpha_0 v,t)$ exist for all $u, v \in B(\vec{0}_{p_0},r_1)$ for all $t \in W$. Since ψ satisfies $\psi^i(p,\alpha u,\alpha v,t) = \psi^i(p,u,v,\alpha t)$ $\forall p \in D$, $u, v \in R^n$, $\alpha \in R$, $t \in J$. Hence $\psi(p_0,u,v,\alpha_0 t)$ exists and $\psi(p_0,\alpha_0 u,\alpha_0 v,t) = \psi(p_0,u,v,\alpha_0 t)$.

Thus $\psi(p_0,u,v,\alpha_0 t)$ exists for all u, $v \in B(\vec{0}_{p_0},r_1)$ for all $t \in W$. That is $\psi(p_0,u,v,t^*)$ exists for all u, $v \in B(\vec{0}_{p_0},r_1)$ for all $t^* \in \alpha W$. Then property 2 is proved.

<u>Property 3</u> (Exponential property) Given initial point $p_0 \in D$ and $t_0 \in R-\{0\}$, and a neighbourhood W of $\vec{0} = (0,0,\ldots,0)$ in R^{2n} such that $\vec{\psi}(p_0,u,v,t_0)$ is defined $\forall (u,v) \in W$ (We proved already that such a W exists in property 1). Fix v = 0, then $(u,0) \mapsto \vec{\psi}(p_0,u,0,t_0)$ is a bidifferential map of some open set of the zero vector onto an open set.

<u>Proof</u> Let $\vec{h}$ be a map defined on $\Pi_1(W)$, where Π_1 is defined on pages 29 , by $\vec{h}(u,0) = \vec{\psi}(p_0,u,0,t_0)$. Then $\vec{h}$ is a c^1 function on $\Pi_1(W)$. Since $\psi^i(p_0,\alpha u,\alpha v,t_0) = \psi^i(p_0,u,v,\alpha t_0)$ differentiate with respect to α, this gives

$$u^{j_1}\frac{\partial\psi^i}{\partial u^{j_1}}(p_0,\alpha u,\alpha v,t_0)+ 2\alpha v^{j_2}\frac{\partial\psi^i}{\partial v^{j_2}} = \dot{\psi}^i(p_0,u,v,\alpha t_0)t_0 \quad .$$

Let α = 0, we get

$$u^{j_1}\frac{\partial\psi^i}{\partial u^{j_1}}(p_0,0,0,t_0) = t_0 u^i = t_0\,\delta^i_{j_1}\,u^{j_1} \quad \text{where } \delta^i_{j_1} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} .$$

Therefore, $\frac{\partial}{\partial u^{j_1}}\psi^i(p_0,0,0,t_0) = t_0\,\delta^i_{j_1}$ and

$$J_{\vec{h}}(0) = \det \begin{vmatrix} \frac{\partial\psi^1}{\partial u^1}(p_0,0,0,t_0) & \frac{\partial\psi^1}{\partial u^2}(p_0,0,0,t_0) \ldots\ldots & \frac{\partial\psi^1}{\partial u^n}(p_0,0,0,t_0) \\ \frac{\partial\psi^2}{\partial u^1}(p_0,0,0,t_0) & \frac{\partial\psi^2}{\partial u^2}(p_0,0,0,t_0) \ldots\ldots & \frac{\partial\psi^2}{\partial u^n}(p_0,0,0,t_0) \\ \vdots & & \\ \frac{\partial\psi^n}{\partial u^1}(p_0,0,0,t_0) & \frac{\partial\psi^n}{\partial u^2}(p_0,0,0,t_0) \ldots\ldots & \frac{\partial\psi^n}{\partial u^n}(p_0,0,0,t_0) \end{vmatrix}$$

$$= \det \begin{vmatrix} t_0 & & O \\ & \ddots & \\ O & & t_0 \end{vmatrix} = t_0^n \neq 0 .$$

By the Inverse Function Theorem, there exist two open sets $V \subsetneq \Pi_{\frac{1}{2}}(W)$ of the zero vector and $W^1 \subsetneq R^n$ such that $\vec{h}$ is 1-1, differentiable on V onto W^1 and $\vec{h}^{-1}$ exists and is also differentiable. Thus $\vec{h}$ is a bidifferential map of V onto W^1.

Before we have the corollary to property 3), we need the following theorem.

Theorem 2.16 Let $f : R^n \to R^m$ be continuously differentiable in an open set containing a, where $m \leq n$. If $f(a) = 0$ and the $n \times m$ matrix $(D_i f^j(a))$ has rank m, then there is an open set $A \subset R^n$ and a differentiable function $h : A \to R^n$ with differentiable inverse such that

$$f \circ h (x^1, \ldots, x^n) = (x^{n-m+1}, \ldots, x^n) .$$

For the proof, see reference [10] page 43.

<u>Corollary</u> The same hypothesis as in Property 3 except that we do not fix $v = 0$. Then the map $\vec{k}(u,v) = \vec{\psi}(p_0,u,v,t_0)$ is locally open map at $(0,0)$, that is there exist neighbourhoods W^1 of $(0,0)$ in W and U in R^n such that $\vec{k}$ restrict on W^1 is onto U and is an open map.

<u>Proof</u> The Jacobian matrix of $\vec{k}$ at $(0,0)$ is the $n \times (2n)$ matrix

$$\begin{bmatrix} \frac{\partial\psi^1}{\partial u^1}(p_0,0,0,t_0) \cdots \frac{\partial\psi^1}{\partial u^n}(p_0,0,0,t_0) & \frac{\partial\psi^1}{\partial v^1}(p_0,0,0,t_0) \cdots \frac{\partial\psi^1}{\partial v^n}(p_0,0,0,t_0) \\ \vdots & \\ \frac{\partial\psi^n}{\partial u^1}(p_0,0,0,t_0) \cdots \frac{\partial\psi^n}{\partial u^n}(p_0,0,0,t_0) & \frac{\partial\psi^n}{\partial v^1}(p_0,0,0,t_0) \cdots \frac{\partial\psi^n}{\partial v^n}(p_0,0,0,t_0) \end{bmatrix}$$

and the $n \times n$ matrix $\frac{\partial\psi^i}{\partial u^j}(p_0,0,0,t_0)$ has non-zero determinant as we already showed. Therefore the Jacobian matrix has maximal rank n. Thus $\vec{k}$ is locally open map by Theorem 2.16.

<u>Property 4</u>. Given $p_0 \in D$, v_0, u_0 at p_0 and any real number $\alpha_0 \neq 0$, then the solution curve $\psi(p_0,u_0,v_0,t)$ with initial values p_0, u_0, v_0 and the solution curve $\psi(p_0, \alpha_0u_0, \alpha_0v_0, t)$ having initial values p_0, α_0v_0, α_0u_0 agree as point sets, that is the images of the two functions coincide.

<u>Proof</u> By the Fundamental Theorem of Ordinary Differential Equations, $\psi(p_0, \alpha_0 u_0, \alpha_0 v_0, t)$ is defined on some open interval I_1 of zero in R and $\psi(p_0, u_0, v_0, t)$ is defined on some open interval I_2 of zero in R.

Let $C_1 = \{\psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t) | \psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t)$ is defined$\}$.

Let $C_2 = \{\psi(p_0,u_0,v_0,t) | \psi(p_0,u_0,v_0,t)$ is defined$\}$.

To show $C_1 \subsetneq C_2$, let Q_1 be any point in C_1. Then there exists $t_1 \varepsilon$ R such that $Q_1 = \psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t_1)$

Now $\psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t_1) = \psi(p_0,u_0,v_0,\alpha_0 t_1) \forall p_0 \varepsilon$ D

So let $t_2 = \alpha_0 t_1$. Thus we see that there exists $t_2 \varepsilon$ R such that $\psi(p_0,u_0,v_0,t_2)$ is defined, that is, there exists $t_2 \varepsilon$ R such that $Q_1 = \psi(p_0,u_0,v_0,t_2)$ so $Q_1 \varepsilon C_2$.

Let Q_2 be any point in C_2. Then there exists $t_1 \varepsilon$ R such that $Q_2 = \psi(p_0,u_0,v_0,t_1)$. Then $\psi(p_0,u_0,v_0,\alpha_0 \frac{t_1}{\alpha_0})$ is defined.

Hence $\psi(p_0,\alpha_0 u_0,\alpha_0 v_0, \frac{t_1}{\alpha_0})$ is defined and

$$\psi(p_0,\alpha_0 u_0,\alpha_0 v_0, \frac{t_1}{\alpha_0}) = \psi(p_0,u_0,v_0,t_1)$$

Therefore let $t_2 = \frac{t_1}{\alpha_0}$. So we get that $\psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t_2)$ is defined and $\psi(p_0,\alpha_0 u_0,\alpha_0 v_0,t_2) = Q_2$

i.e. $Q_2 \varepsilon C_1$. Thus $C_1 = C_2$. The proof is complete.

We conjecture that for any $n > 3$, there exists unique form of a differential equation which satisfies an analogous functional equation and whose solution curves have the above properties. The same type of proofs used above should give this result. In fact, the differential equation should be the one which in order n has exactly n dots on the RHS. Also, we should get the exact same four properties that we proved for the case $n = 3$.