Chapter IV

EXISTENCE THEOREMS

The main purpose of this chapter is to prove that the necessary condition (2.3.1) is also sufficient for the existence of n -STS. This proof which is due to Moore can be found in [3]. First it will be proved in Theorem 3.1 that n - STS exists for all n with $n \equiv 1$ or 3 (mod 6) and $n \leq 321$. This theorem also shows that (n - 7) - STS exists except for finite number of n. Finally we shall show in Theorem 3.3 that (n,7) - STS exists for all n with $n \equiv 1$ or 3 (mod 6) and $n \geq 321$.

3.1 <u>Theorem.</u> If $n \equiv 1$ or 3(mod 6) and $n \leq 321$, then n - STS exists. Furthermore if $n \neq 1,3,9,13,25,27,33,37,67,69,75,81,97, 109,201,289,321$, then (n,7) - STS exists. The above seventeen numbers will be called the <u>exceptional numbers</u>.

<u>Proof</u> : We shall show that n - STS exists, where $n \equiv 1$ or 3 (mod 6) and $n \leq 321$, by using Method I or Method II to construct n - STS from STS of smaller orders whose existence is known. We note that the existence of 1 - STS, 3 - STS, 7 - STS is already exhibited in Example (i),(ii),(iii) of Chapter II. The existence of 13 - STS will be shown by direct construction in Section 5.2 of Chapter V. The following table shows how n - STS can be constructed for $n \equiv 1$ or 3 (mod 6), $n \neq 1,3,7,13$ and $n \leq 321$. Column (1) gives the complete enumeration of values of n for all $n \equiv 1$ or 3 (mod 6), $n \neq 1,3,7,13$ and $n \leq 321$. Column (2) gives the method of constructing n - STS from STS in Column (3).

Column (4) gives the reason of existence of (n,7) - STS.

Table VI

(1) Value of n	(2) n-STS can be constructed by Method	(3) from	(4) Existence of a sub-system of order 7 can be seen from the fact that
9	I	3-STS and 3-STS	-
15	II	7-STS,3-STS and 1-STS	(7,7)-STS exists
19	LL	3-STS,7-STS and 1-STS	(7,7)-STS exists
21	I	3-STS and 7-STS	(7,7)-STS exists
25	II	3-STS,9-STS and 1-STS	-
27	I	3-STS and 9-STS	-
31	II	3-STS,15-STS and 7-STS	(7,7)-STS exists
33	II	3-STS,13-STS and 3-STS	-
37	II	3-STS,13-STS and 1-STS	-
39	II	3-STS,15-STS and 3-STS	(15,7)-STS exists
43	II	3-STS,15-STS and 1-STS	(15,7)-STS exists
45	I	3-STS and 15-STS	(15,7)-STS exists
49	I	7-STS and 7-STS	(7,7)-STS exists
51	II	3-STS,19-STS and 3-STS	(19,7)-STS exists
55	II	3-STS,19-STS and 1-STS	(19,7)-STS exists
57	I	3-STS and 19-STS	(19,7)-STS exists
61	II	3-STS,21-STS and 1-STS	(21,7)-STS exists
63	I	3-STS and 21-STS	(21,7)-STS exists
67	II	33-STS,3-STS and 1-STS	-
69	II	3-STS,25-STS and 3-STS	-

Cont.

(1) Value of n	(2) n-STS can be constructed by Method	(3) from	(4) Existence of a sub-system of order 7 can be seen from the fact that
73	II	7-STS,13-STS and 3-STS	(7,7)-STS exists
75	I	3-STS and 25-STS	-
79	II	3-STS,31-STS and 7-STS	(7,7)-STS exists
81	I	9-STS and 9-STS	-
85	II	3-STS,33-STS and 7-STS	(7,7)-STS exists
87	II	3-STS,31-STS and 3-STS	(31,7)-STS exists
91	II	15-STS,7-STS and 1-STS	(7,7)-STS exists
93	I	3-STS and 31-STS	(31,7)-STS exists
97	II	3-STS,33-STS and 1-STS	-
99	II	49-STS,3-STS and 1-STS	(49,7)-STS exists
103	II	25-STS,7-STS and 3-STS	(7,7)-STS exists
105	I	7-STS and 15-STS	(7,7)-STS exists
109	II	9-STS,13-STS and 1-STS	-
111	II	55-STS,3-STS and 1-STS	(55,7)-STS exists
115	II	3-STS,39-STS and 1-STS	(39,7)-STS exists
117	I	3-STS and 39-STS	(39,7)-STS exists
121	II	3-STS,45-STS and 7-STS	(7,7)-STS exists
123	II	61-STS,3-STS and 1-STS	(61,7)-STS exists
127	II	7-STS,19-STS and 1-STS	(7,7)-STS exists
129	I	3-STS and 43-STS	(43,7)-STS exists
133	II	3-STS,45-STS and 1-STS	(45,7)-STS exists
135	I	15-STS and 9-STS	(15,7)-STS exist
139	II	3-STS,51-STS and 7-STS	(7,7)-STS exists

Cont.

(1) Value of n	(2) n-STS can be constructed by Method	(3) from	(4) Existence of a sub-system of order 7 can be seen from the fact that
141	II	7-STS,21-STS and 1-STS	(7,7)-STS exists
145	II	3-STS,49-STS and 1-STS	(49,7)-STS exists
147	I	3-STS and 49-STS	(49,7)-STS exists
151	II	3-STS,51-STS and 1-STS	(51,7)-STS exists
153	I	3-STS and 51-STS	(51,7)-STS exists
157	II	3-STS,57-STS and 7-STS	(7,7)-STS exists
159	II	79-STS,3-STS and 1-STS	(79,7)-STS exists
163	II	27-STS,7-STS and 1-STS	(7,7)-STS exists
165	II	27-STS,9-STS and 3-STS	(27,7)-STS exists
169	II	3-STS,57-STS and 1-STS	(57,7)-STS exists
171	I	3-STS and 57-STS	(57,7)-STS exists
175	I	7-STS and 25-STS	(7,7)-STS exists
177	II	3-STS,61-STS and 3-STS	(61,7)-STS exists
181	II	9-STS,21-STS and 1-STS	(21,7)-STS exists
183	I	3-STS and 61-STS	(61,7)-STS exists
189	I	3-STS and 63-STS	(63,7)-STS exists
193	II	19-STS,13-STS and 3-STS	(19,7)-STS exists
195	I	15-STS and 13-STS	(15,7)-STS exists
199	II	33-STS,7-STS and 1-STS	(7,7)-STS exists
201	I	3-STS and 67-STS	-
205	II	3-STS,73-STS and 7-STS	(7,7)-STS exists
207	II	103-STS,3-STS and 1-STS	(103,7)-STS exists

Cont.

(1) Value of n	(2) n-STS can be constructed by Method	(3) from	(4) Existence of a sub-system of order 7 can be seen from the fact that
211	II	7-STS,31-STS and 1-STS	(7,7)-STS exists
213	II	7-STS,33-STS and 3-STS	(7,7)-STS exists
217	I	7-STS and 31-STS	(7,7)-STS exists
219	I	3-STS and 73-STS	(73,7)-STS exists
223	II	111-STS,3-STS and 1-STS	(111,7)-STS exists
225	I	15-STS and 15-STS	(15,7)-STS exists
229	II	19-STS,13-STS and 1-STS	(19,7)-STS exists
231	II	19-STS,15-STS and 3-STS	(15,7)-STS exists
235	II	39-STS,7-STS and 1-STS	(7,7)-STS exists
237	I	3-STS and 79-STS	(79,7)-STS exists
241	II	3-STS,85-STS and 7-STS	(7,7)-STS exists
243	I	3-STS and 81-STS	(81,7)-STS exists
247	II	123-STS,3-STS and 1-STS	(123,7)-STS exists
249	II	3-STS,85-STS and 3-STS	(85,7)-STS exists
253	II	3-STS,85-STS and 1-STS	(85,7)-STS exists
255	I	3-STS and 85-STS	(85,7)-STS exists
259	I	7-STS and 37-STS	(7,7)-STS exists
261	I	3-STS and 87-STS	(87,7)-STS exists
265	II	3-STS,93-STS and 7-STS	(7,7)-STS exists
267	II	133-STS,3-STS and 1-STS	(133,7)-STS exists
271	II	45-STS,7-STS and 1-STS	(7,7)-STS exists
273	II	45-STS,9-STS and 3-STS	(45,7)-STS exists

Cont.

(1) Value of n	(2) n-STS can be constructed by Method	(3) from	(4) Existence of a sub-system of order 7 can be seen from the fact that
277	II	3-STS,93-STS and 1-STS	(93,7)-STS exists
279	I	3-STS and 93-STS	(93,7)-STS exists
283	II	141-STS,3-STS and 1-STS	(141,7)-STS exists
285	I	15-STS and 19-STS	(15,7)-STS exists
289	II	3-STS,97-STS and 1-STS	-
291	II	145-STS,3-STS and 1-STS	(145,7)-STS exists
295	II	147-STS,3-STS and 1-STS	(147,7)-STS exists
297	II	7-STS,45-STS and 3-STS	(7,7)-STS exists
301	I	7-STS and 43-STS	(7,7)-STS exists
303	II	75-STS,7-STS and 3-STS	(7,7)-STS exists
307	II	51-STS,7-STS and 1-STS	(7,7)-STS exists
309	I	3-STS and 103-STS	(103,7)-STS exists
313	II	3-STS,105-STS and 1-STS	(105,7)-STS exists
315	I	15-STS and 21-STS	(15,7)-STS exists
319	II	159-STS,3-STS and 1-STS	(159,7)-STS exists
321	II	3-STS,109-STS and 3-STS	-

3.2 Lemma. Let m be any positive integer such that $m \equiv 1$ or $3 \pmod 6$. Then m is congruent modulo 36 to one of 1,3,7,9,13,15,19,21,25,27, 31,33.

Proof : For $m = 6k + i$, $i = 1$ or 3 we have $m = 6(6\ell + j) + i$ where $j \in \{0,1,2,3,4,5\}$. Hence $m = 36\ell + 6j + i$ where $i = 1$ or 3 and $j \in \{0,1,2,3,4,5\}$ so that $6j + i \in \{1,3,7,9,13,15,19,21,25,27,31,33\}$. Thus m is congruent modulo 36 to one of 1,3,7, 9,13,15,19,21,25,27,31,33.

3.3 Theorem. If $m \equiv 1$ or $3 \pmod 6$ and $m > 321$, then (m,7) - STS exists.

Proof : We shall prove the theorem by induction. First it will be shown that (325,7) - STS exists. Since 3-STS, (7,7)-STS, (49,7)-STS exists and $325 = 3 + 7(49 - 3)$, hence by Theorem 3.6.4, (325,7)-STS exists. Let $m \equiv 1$ or $3 \pmod 6$ be such that $m > 321$. We shall show that (m,7)-STS exists. Assume that (v,7)-STS exists for all v with $v \equiv 1$ or $3 \pmod 6$, $v > 321$, $v < m$. Let us introduce Methods A,B,C,D,E,F which are certain specializations of Method II; namely some values of n_1, n_2, n_3 are fixed. Table VII gives the values of n_1, n_2, n_3 for these methods. Column (3) gives the values of n for which n-STS can be constructed by methods in column (1) from STS in column (2).

Table VII

(1) Name of method	(2) Construct n-STS from	(3) Value of n which n-STS can be constructed
Method A	n'-STS, 3-STS and 1-STS	$2n' + 1$
Method B	3-STS, n'-STS and 1-STS	$3n' - 2$
Method C	3-STS, n'-STS and 3-STS	$3n' - 6$
Method D	n'-STS, 9-STS and 3-STS	$6n' + 3$
Method E	3-STS, n'-STS and 7-STS	$3n' - 14$
Method F	n'-STS, 7-STS and 1-STS	$6n' + 1$

We use the methods listed in Table VII to construct (m,7)-STS. Due to Lemma 3.2 m is congruent modulo 36 to one of 1,3,7,9,13,15, 19,21,25,27,31,33. In Table VIII the method and the value of n' for constructing m-STS are determined according to the residue of m modulo 36. Column (2) indicates the method for constructing STS of order m in Column (1). Column (3) gives the value of n' corresponding to m.

Table VIII

Value of m	Construct m-STS by Method	Value of n'
36t + 1	B	12t + 1
36t + 3	A	18t + 1
36t + 7	F	6t + 1
36t + 9	D	6t + 1
36t + 13	E	12t + 9
36t + 15	A	18t + 7
36t + 19	F	6t + 3
36t + 21	D	6t + 3
36t + 25	B	12t + 9
36t + 27	A	18t + 13
36t + 31	A	18t + 15
36t + 33	C	12t + 13

Observe that by the assumption and by Theorem 3.1, n'-STS exists for all cases of m. Moreover m-STS constructed as in Table VIII will contain 7-STS if n' is not an exceptional number. We shall show that when n' is an exceptional number an alternate method for constructing (m,7)-STS can always be found. Since m-STS constructed by Method F always contains 7-STS, hence it suffices to look for alternate methods to construct (m,7)-STS from Method A,B,C,D,E. Observe that if $m > 321$, n' must not be less than 67. Table IX provides alternate methods for constructing (m,7)-STS when n' are exceptional numbers.

In Column (2), by critical method we mean the method that is avoided when n' given in Column (1) is an exceptional number. Column (3) gives the value of m if n' in Column (1) and the method in Column (2) are used to construct m-STS. Column (4) gives an alternate method for constructing m-STS and Column (5) gives the reason for existence of (m,7) - STS if that alternate method is used.

Table IX

(1) n′	(2) Critical method	(3) m	(4) Alternate method		(5) Existence of a subsystem of order 7 can be seen from the fact that
			m-STS can be constructed by Method	from	
67	D	405	I	15-STS and 27-STS	(15,7)-STS exists
69	D	417	C	3-STS,141-STS and 3-STS	(141,7)-STS exists
75	D	453	C	3-STS,153-STS and 3-STS	(153,7)-STS exists
81	D	489	I	3-STS and 163-STS	(163,7)-STS exists
97	D	585	I	15-STS and 39-STS	(15,7)-STS exists
109	B	325	II	7-STS,49-STS and 3-STS	(7,7)-STS exists
109	D	657	I	9-STS and 73-STS	(73,7)-STS exists
109	E	313	B	3-STS,105-STS and 1-STS	(105,7)-STS exists
201	A	403	F	67-STS,7-STS and 1-STS	(7,7)-STS exists
201	B	601	E	3-STS,205-STS and 7-STS	(205,7)-STS exists
201	C	597	D	99-STS,9-STS and 3-STS	(99,7)-STS exists
201	D	1209	C	3-STS,405-STS and 3-STS	(405,7)-STS exists
201	E	589	I	19-STS and 31-STS	(19,7)-STS exists

Cont.

(1) n′	(2) Critical method	(3) m	(4) Alternate method		(5) Existence of a subsystem of order 7 can be seen from the fact that
			m-STS can be constructed by Method	from	
289	A	579	I	3-STS and 193-STS	(193,7)-STS exists
289	B	865	II	13-STS,73-STS and 7-STS	(7,7)-STS exists
289	C	861	I	7-STS and 123-STS	(7,7)-STS exists
289	D	1725	I	15-STS and 115-STS	(15,7)-STS exists
289	E	853	B	3-STS,285-STS and 1-STS	(285,7)-STS exists
321	A	643	E	3-STS,219-STS and 7-STS	(7,7)-STS exists
321	B	961	I	31-STS and 31-STS	(31,7)-STS exists
321	C	957	I	3-STS and 319-STS	(319,7)-STS exists
321	D	1929	I	3-STS and 643-STS	(643,7)-STS exists
321	E	949	I	13-STS and 73-STS	(73,7)-STS exists