Chapter III

CONSTRUCTION OF STEINER TRIPLES SYSTEMS FROM THE GIVEN STEINER TRIPLE SYSTEMS OF SMALLER ORDERS.

3.0 Introduction

This chapter deals with various general methods for constructing STS from STS of smaller orders. In [2] Doyen gave methods for constructing STS of orders 2n + 3, 2n + 7, from STS of orders n which were constructed from cyclic groups. It appears that Doyen's methods can be applied to STS of orders n which are not constructed from groups also. In Section 3.3 and Section 3.4 we apply these methods to certain classes of STS of orders n.

3.1 Two Useful Lemmas

3.1.1 Lemma. Let A be a set of n elements and S(A) be a family of 3-subsets of A. If S(A) has the following properties :

(i) S(A) contains at most $\frac{1}{6}n(n-1)$ 3-subsets of A

(ii) for any 2-subset T of A there exists a 3-subset H in S(A) such that $T \subset H$.

Then (A,S(A)) is a STS of order n.

Proof : We have to show that if T is any 2-subset of A, then there exists one and only one H in S(A) such that $T \subset H$. Suppose that there exists a 2-subset T_1 of A such that there exist distinct 3-subsets S_i, S_j in S(A) with $T_1 \subset S_i$, $T_1 \subset S_j$. Assume that S(A) contains exactly r 3-subsets. Therefore $r \leq \frac{1}{6} n(n-1)$. Let $\mathcal{H} = \{ R / R$ is a 2-subset of some 3-subsets in $S(A) \}$. Let T(A) be the set of all 2-subsets of A. By (ii) we see that $T(A) \subset \mathcal{H}$. From $|T(A)| = \frac{1}{2} n(n-1)$ we have $|\mathcal{H}| \geq \frac{1}{2} n(n-1)$. Since S_i and S_j contribute at most 5 2-subsets in $\mathcal{H}$ and since each of element in $S(A) - \{ S_i, S_j \}$ contributes at most 3 2-subsets in $\mathcal{H}$, hence $|\mathcal{H}| \leq 5 + 3(r-2)$. But $r \leq \frac{1}{6} n(n-1)$. Thus $5 + 3(r-2) = 3r - 1 \leq 3(\frac{1}{6} n(n-1)) - 1 = \frac{1}{2} n(n-1) - 1$. Hence $|\mathcal{H}| < \frac{1}{2} n(n-1)$. This contradicts the preceding remark that $|\mathcal{H}| \geq \frac{1}{2} n(n-1)$. Hence for each 2-subset T of A there exists one and only one 3-subset H in S(A) such that $T \subset H$. Therefore (A,S(A)) is a STS of order n.

3.1.2 Lemma. Let f be a one to one mapping on a set A onto B. If (A,S(A)) is a STS of order n and

$$S(B) = \{ \{ f(a), f(b), f(c) \} / \{ a,b,c \} \in S(A) \}.$$

Then (B,S(B)) is a STS of order n which is isomorphic to (A,S(A)).

Proof : Since f is a one to one mapping on A onto B, it follows that B contains n elements and that S(B) is a family of 3-subsets of B. Let F be a mapping on S(A) into S(B) defined by $F(\{a,b,c\}) = \{ f(a), f(b), f(c) \}$. Since f is a one to one, hence F is well-defined. For any $H' \in S(B)$ we see by defini-tion of S(B) that there exists $\{a,b,c\} \in S(A)$ such that $H' = \{ f(a), f(b), f(c) \}$. Hence $F(\{a,b,c\}) = H'$. Thus F is a mapping on S(A) onto S(B). We shall show that F is one to one. Let $H_1 = \{a,b,c\}$ and $H_2 = \{r,s,t\}$ be any members of

$S(A)$ such that $F(H_1) = F(H_2)$. Therefore $\{f(a), f(b), f(c)\} = \{f(r), f(s), f(t)\}$. Without loss of generality assume that $f(a) = f(r)$, $f(b) = f(s)$, $f(c) = f(t)$. But f is one to one, there--fore $a = r$, $b = s$, $c = t$. Hence $H_1 = H_2$. Thus F is a one to one mapping on $S(A)$ onto $S(B)$. Since $S(A)$ contains $\frac{1}{6}n(n-1)$ elements, hence $S(B)$ contains $\frac{1}{6}n(n-1)$ elements. Thus to prove that $(B, S(B))$ is a STS of order n, it suffices to show that for any 2-subset T' of B there exists a 3-subset H' in $S(B)$ such that $T' \subset H'$. Let $T' = \{x, y\}$ be any 2-subset of B. Since $f(A) = B$, it follows that there exists a, b in A such that $f(a) = x$, $f(b) = y$. But f is one to one and $x \neq y$, therefore $a \neq b$. Hence a and b are distinct elements of A so that there exists a unique element c in A such that $\{a, b, c\} \in S(A)$. Let $H' = \{f(a), f(b), f(c)\}$. Then $H' \in S(B)$ and $T' \subset H'$. Hence $(B, S(B))$ is a STS of order n.

By definition of $S(B)$ we see that f is an isomorphism from $(A, S(A))$ onto $(B, S(B))$. Hence $(B, S(B))$ is a STS of order n which is isomorphic to $(A, S(A))$.

3.1.3 <u>Corollary</u>. Let $(A, S(A))$ be a STS of order n. For any element x, let ${}^xA = \{x\} \times A$, $A^x = A \times \{x\}$,

$$S({}^xA) = \{\{x\} \times \{a,b,c\} \;/\; \{a,b,c\} \in S(A)\},$$

$$S(A^x) = \{\{a,b,c\} \times \{x\} \;/\; \{a,b,c\} \in S(A)\}.$$

Then $({}^xA, S({}^xA))$ and $(A^x, S(A^x))$ are STS of orders n which are isomorphic to $(A, S(A))$.

<u>Proof</u> : Define a mapping f on a onto ${}^xA$ and a mapping g on A onto A^x by $f(a) = (x, a)$ and $g(a) = (a, x)$ for all a in A. We see that

f and g are one to one mappings on A onto ${}^{x}A$ and A^{x} respectively. Furthermore we have

$S({}^{x}A) = \{ \{ f(a), f(b), f(c) \} / \{ a,b,c \} \in S(A) \}$ and $S(A^{x}) = \{ \{ g(a), g(b), g(c) \} / \{ a,b,c \} \in S(A) \}$. Thus by Lemma 3.1.2, $({}^{x}A, S({}^{x}A))$ and $(A^{x}, S(a^{x}))$ are STS of orders n, which are isomorphic to (A,S(A)).

In what follows any STS of order n will be called an n-STS and any STS of order n that has a subsystem of order k will be called an (n,k) - STS.

3.2 Construction of 2n + 1 - STS from n - STS

In various construction of STS to be considered in this chapter and also Chapter V we shall come across the cartesian product of ah arbitrary set A and a set of non-negative integers of the form $\{ 0,1,\ldots, \ell - 1 \}$, where ℓ is a positive integer. For convenience in the sequel we shall denote any element (a, i) of $A \times \{ 0,1,\ldots, \ell - 1 \}$ by a_i. Furthermore if $x_0, x_1, \ldots, x_{\ell - 1}$ are under consideration, the symbol x_i with $i \geq \ell$ will also be used to denote the $x_{i'}$, where $0 \leq i' < \ell$ and $i \equiv i' \pmod{\ell}$.

3.2.1 Theorem. Let (A,S(A)) be n - STS. Let $B = (A \times \{0,1\}) \cup \{\infty\}$, where $\infty \notin A \times \{0,1\}$, be the set of m = 2n + 1 elements. If h is a mapping on S(A) into $\{ 0,1 \}$. Let S(B) be the family of the following 3-subsets of B :

(i) $\{ \infty, x_0, x_1 \}$, where $x \in A$,

(ii) $\{x_i, y_i, z_i\}$, $\{x_i, y_{i+1}, z_{i+1}\}$, $\{x_{i+1}, y_i, z_{i+1}\}$, $\{x_{i+1}, y_{i+1}, z_i\}$,

where $\{x,y,z\} \in S(A)$ and $h(\{x,y,z\}) = i$, $0 \leq i \leq 1$

Then $(B,S(B))$ is m - STS.

<u>Proof</u>: By Proposition 2.2.9 the total number of triples in S(A) is $\frac{1}{6}n(n-1)$ so that the total number of 3-subsets in S(B) of the form (ii) is at most $(4)(\frac{1}{6}n(n-1)) = \frac{2}{3}n(n-1)$. Moreover the total number of 3-subsets in S(B) of the form (i) is at most n. Hence the total number of 3-subsets in S(B) is at most $\frac{2}{3}n(n-1) + n = \frac{1}{6}2n(2n+1) = \frac{1}{6}m(m-1)$. Thus to prove that (B,S(B)) is m - STS, it suffices to show that for any 2-subset T of B there exists a 3-subset H in S(B) such that $T \subset H$. To show this let T be any 2-subset of B. We shall show by cases that there exists a 3-subset H in S(B) such that $T \subset H$.

case 1. $T = \{\infty, x_i\}$.

Let $H = \{\infty, x_i, x_{i+1}\}$. Then $H \in S(B)$ and $T \subset H$.

case 2. $T = \{x_i, y_i\}$, $x \neq y$.

Since x,y are distinct elements of A and (A,S(A)) is a STS, hence there exists a unique element z in A such that $\{x,y,z\} \in S(A)$. Then $h(\{x,y,z\}) = j$ for some $j \in \{0,1\}$. Let $H = \{x_i, y_i, z_j\}$. Then $H \in S(B)$ and $T \subset H$.

case 3. $T = \{x_i, x_j\}$, $i \neq j$.

Let $H = \{\infty, x_i, x_j\}$. Then $H \in S(B)$ and $T \subset H$.

case 4. $T = \{x_i, y_j\}$, $x \neq y$, $i \neq j$.

Since x,y are distinct elements of A and (A,S(A)) is a STS, hence there exists a unique element z in A such that $\{x,y,z\} \in S(A)$. By definition of h we have either $h(\{x,y,z\}) = i$ or $h(\{x,y,z\}) = j$. If $h(\{x,y,z\}) = i$, let $H = \{x_i, y_j, z_j\}$. Then $H \in S(B)$ and $T \subset H$. In case $h(\{x,y,z\}) = j$, let $H = \{x_i, y_j, z_i\}$. Then $H \in S(B)$ and $T \subset H$.

Therefore (B,S(B)) is m - STS.

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3.3 Construction of 2n + 3 - STS from n - STS

3.3.1 Definition. Any triples H_1 and H_2 of a STS are said to be non-intersecting triples if $H_1 \cap H_2 = \Phi$.

3.3.2 Definition. Let (A,S(A)) be a STS of order $n \equiv 3 \pmod 6$. Let A_0, A_1, A_2 be disjoint subsets of A and Let $\mathcal{F}$ be a family of non-intersecting triples in S(A) of size $\frac{n}{3}$ such that for each $F \in \mathcal{F}$, $|F \cap A_i| = 1$, $i = 0,1,2$. We say that A_0, A_1, A_2 and $\mathcal{F}$ satisfy Property I with respect to (A,S(A)).

Let (A,S(A)) be a STS. If there exist A_0, A_1, A_2 and $\mathcal{F}$ such that A_0, A_1, A_2 and $\mathcal{F}$ satisfy Property I with respect to (A,S(A)). We say that (A,S(A)) is a STS with Property I[1].

3.3.3 Lemma. Let A_0, A_1, A_2 and $\mathcal{F}$ satisfy Property I with respect to a STS (A,S(A)). Then

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(1) A STS of order $n \equiv 3 \pmod 6$ with Property I exists. This will be proved in Section 5.4 of Chapter V.

(i) for each x in A there exists a 3-subsets H in $\mathcal{F}$ such that $x \in F$

(ii) $A = A_0 \cup A_1 \cup A_2$

Proof : Since $\mathcal{F}$ consists of $\frac{n}{3}$ disjoint 3-subsets, hence $\left|\bigcup_{F \in \mathcal{F}} F\right| = (\frac{n}{3})(3) = n$. Suppose that there exists an x in A such that for any 3-subsets F in $\mathcal{F}$, $x \notin F$. Hence $\bigcup_{F \in \mathcal{F}} F$ is a proper subset of A. Therefore $\left|\bigcup_{F \in \mathcal{F}} F\right| < n$ which is a contradiction. Thus (i) is proved. To prove (ii), note that it suffices to show that $A \subset A_0 \cup A_1 \cup A_2$. Let x be any element of A. By (i) there exists $F \in \mathcal{F}$ such that $x \in F$. It follows from definition of $\mathcal{F}$ that $x \in A_i$ for some $i \in \{0,1,2\}$. Therefore $x \in A_0 \cup A_1 \cup A_2$ so that $A \subset A_0 \cup A_1 \cup A_2$.

3.3.4 Theorem. Let (A,S(A)) be a STS of order n with A_0, A_1, A_2 and $\mathcal{F}$ satisfying Property I with respect to (A,S(A)). Let g be a mapping on $S(A) - \mathcal{F}$ into $\{0,1\}$ and h be a mapping on $\mathcal{F}$ into $\{0,1,\ldots,7\}$. Let $B = (A \times \{0,1\}) \cup \{\infty, \infty_0, \infty_1\}$, where $\infty, \infty_0, \infty_1$ are distinct elements which are not in $A \times \{0,1\}$. Let S(B) be the family of the following 3-subsets of B :

(i) $\{\infty, \infty_0, \infty_1\}$,

(ii) $\{\infty, x_0, x_1\}$, where $x \in A$,

(iii) $\{x_i, y_i, z_i\}$, $\{x_i, y_{i+1}, z_{i+1}\}$, $\{x_{i+1}, y_i, z_{i+1}\}$, $\{x_{i+1}, y_{i+1}, z_i\}$, where $\{x,y,z\} \in S(A) - \mathcal{F}$ and $g(\{x,y,z\}) = i$, $i \in \{0,1\}$,

(iv) for any $\{x,y,z\} \in \mathcal{F}$ such that $x \in A_0$, $y \in A_1$, $z \in A_2$

- $\{x_0,y_0,z_0\}$, $\{x_1,y_1,z_1\}$, $\{\infty_0,x_0,z_1\}$, $\{\infty_0,x_1,y_0\}$, $\{\infty_0,y_1,z_0\}$, $\{\infty_1,x_0,y_1\}$, $\{\infty_1,y_0,z_1\}$, $\{\infty_1,x_1,z_0\}$ if $h(\{x,y,z\}) = 0$
- $\{x_0,y_0,z_0\}$, $\{x_1,y_1,z_1\}$, $\{\infty_0,x_0,y_1\}$, $\{\infty_0,y_0,z_1\}$, $\{\infty_0,x_1,z_0\}$, $\{\infty_1,x_0,z_1\}$, $\{\infty_1,x_1,y_0\}$, $\{\infty_1,y_1,z_0\}$ if $h(\{x,y,z\}) = 1$
- $\{x_0,y_0,z_1\}$, $\{x_1,y_1,z_0\}$, $\{\infty_0,x_0,z_0\}$, $\{\infty_0,x_1,y_0\}$, $\{\infty_0,y_1,z_1\}$, $\{\infty_1,x_0,y_1\}$, $\{\infty_1,y_0,z_0\}$, $\{\infty_1,x_1,z_1\}$ if $h(\{x,y,z\}) = 2$
- $\{x_0,y_0,z_1\}$, $\{x_1,y_1,z_0\}$, $\{\infty_0,x_0,y_1\}$, $\{\infty_0,y_0,z_0\}$, $\{\infty_0,x_1,z_1\}$, $\{\infty_1,x_0,z_0\}$, $\{\infty_1,x_1,y_0\}$, $\{\infty_1,y_1,z_1\}$ if $h(\{x,y,z\}) = 3$
- $\{x_0,y_1,z_0\}$, $\{x_1,y_0,z_1\}$, $\{\infty_0,x_0,z_1\}$, $\{\infty_0,y_0,z_0\}$, $\{\infty_0,x_1,y_1\}$, $\{\infty_1,x_0,y_0\}$, $\{\infty_1,x_1,z_0\}$, $\{\infty_1,y_1,z_1\}$ if $h(\{x,y,z\}) = 4$
- $\{x_0,y_1,z_0\}$, $\{x_1,y_0,z_1\}$, $\{\infty_0,x_0,y_0\}$, $\{\infty_0,x_1,z_0\}$, $\{\infty_0,y_1,z_1\}$, $\{\infty_1,x_0,z_1\}$, $\{\infty_1,y_0,z_0\}$, $\{\infty_1,x_1,y_1\}$ if $h(\{x,y,z\}) = 5$
- $\{x_0,y_1,z_1\}$, $\{x_1,y_0,z_0\}$, $\{\infty_0,x_0,z_0\}$, $\{\infty_0,y_0,z_1\}$, $\{\infty_0,x_1,y_1\}$, $\{\infty_1,x_0,y_0\}$, $\{\infty_1,y_1,z_0\}$, $\{\infty_1,x_1,z_1\}$ if $h(\{x,y,z\}) = 6$
- $\{x_0,y_1,z_1\}$, $\{x_1,y_0,z_0\}$, $\{\infty_0,x_0,y_0\}$, $\{\infty_0,y_1,z_0\}$, $\{\infty_0,x_1,z_1\}$, $\{\infty_1,x_0,z_0\}$, $\{\infty_1,y_0,z_1\}$, $\{\infty_1,x_1,y_1\}$ if $h(\{x,y,z\}) = 7$.

Then $(B,S(B))$ is $2n + 3$ - STS

Proof : It follows from the construction of B that $|B| = 2n + 3$. We shall count the total number of 3-subsets in S(B). It can be seen that the total number of 3-subsets in S(B) of the form (i) - (ii) is $n + 1$. For each of the $(\frac{1}{6}n(n-1) - \frac{n}{3})$ triples in $S(A) - \mathcal{F}$ we can form exactly 4 3-subsets in S(B) of the form (iii), hence the total number of 3-subsets in S(B) of the form (iii) is at most $(4)(\frac{1}{6}n(n-1) - \frac{n}{3})$. Each triple in $\mathcal{F}$ contributes exactly 8 3-subsets in S(B) of the form (iv) so that the total number of 3-subsets in S(B) of the form (iv) is at most $(8)(\frac{n}{3})$. Therefore the total number of 3-subsets in S(B) of the form (i)-(iv) is at most $n + 1 + (4)(\frac{1}{6}n(n-1) - \frac{n}{3}) + (8)(\frac{n}{3}) = \frac{1}{6}(2n+3)(2n+2) = \frac{1}{6}m(m-1)$, where $m = 2n + 3$. Thus to prove that (B,S(B)) is m - STS, it suffices to show that for any 2-subset T of B there exists a 3-subset H in S(B) such that $T \subset H$. To show this let T be any 2-subset of B. We shall show by cases that there exists a 3-subset H in S(B) such that $T \subset H$.

case 1. $T \subset \{\infty, \infty_0, \infty_1\}$.

In this case we let $H = \{\infty, \infty_0, \infty_1\}$. Then $H \in S(B)$ and $T \subset H$.

case 2. $T = \{\infty, x_i\}$, $i \in \{0, 1\}$.

Let $H = \{\infty, x_i, x_{i+1}\}$. Then $H \in S(B)$ and $T \subset H$.

case 3. $T = \{\infty_0, x_0\}$.

Since $x \in A$ and A_0, A_1, A_2 and $\mathcal{F}$ satisfy Property I with respect to (A,S(A)), hence by Lemma 3.3.1 (i) there exists $F = \{x, y, z\} \in \mathcal{F}$ such that $x \in F$. Moreover by Lemma 3.3.1 (ii) there exists $i \in \{0,1,2\}$ such that $x \in A_i$. If follows from definition of $\mathcal{F}$ that $\{y,z\} \subset A_i \cup A_{i+1}$. Without loss of generality we may assume that $y \in A_{i+1}$ and $z \in A_{i+2}$. The choice of H will depend on h(F) and i. The following table shows how $H \in S(B)$ can be chosen so that $T \subsetneq H$.

Table I

$h(F)$ \ i	0	1	2
0	$H = \{\infty_0, x_0, z_1\}$	$H = \{\infty_0, x_0, z_1\}$	$H = \{\infty_0, x_0, z_1\}$
1	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$
2	$H = \{\infty_0, x_0, z_0\}$	$H = \{\infty_0, x_0, z_1\}$	$H = \{\infty_0, x_0, y_0\}$
3	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_0\}$	$H = \{\infty_0, x_0, z_0\}$
4	$H = \{\infty_0, x_0, z_1\}$	$H = \{\infty_0, x_0, y_0\}$	$H = \{\infty_0, x_0, z_0\}$
5	$H = \{\infty_0, x_0, y_0\}$	$H = \{\infty_0, x_0, z_0\}$	$H = \{\infty_0, x_0, y_1\}$
6	$H = \{\infty_0, x_0, z_0\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_0\}$
7	$H = \{\infty_0, x_0, y_0\}$	$H = \{\infty_0, x_0, z_0\}$	$H = \{\infty_0, x_0, z_1\}$

case 4. $T = \{\infty_1, x_1\}$.

Similarly to case 3 there exists $F = \{x, y, z\} \in \mathcal{F}$ and $i \in \{0,1,2\}$ such that $x \in A_i$, $y \in A_{i+1}$, $z \in A_{i+2}$. The choice of H will depend on h(F) and i. The following table shows how $H \in S(A)$ can be chosen so that $T \subset H$.

Table II

$h(F)$ \ i	0	1	2
0	$H = \{\infty_1, x_1, z_0\}$	$H = \{\infty_1, x_1, z_0\}$	$H = \{\infty_1, x_1, z_0\}$
1	$H = \{\infty_1, x_1, y_0\}$	$H = \{\infty_1, x_1, y_0\}$	$H = \{\infty_1, x_1, y_0\}$
2	$H = \{\infty_1, x_1, z_1\}$	$H = \{\infty_1, x_1, z_0\}$	$H = \{\infty_1, x_1, y_1\}$
3	$H = \{\infty_1, x_1, y_0\}$	$H = \{\infty_1, x_1, y_1\}$	$H = \{\infty_1, x_1, z_1\}$
4	$H = \{\infty_1, x_1, z_0\}$	$H = \{\infty_1, x_1, y_1\}$	$H = \{\infty_1, x_1, z_1\}$
5	$H = \{\infty_1, x_1, y_1\}$	$H = \{\infty_1, x_1, z_1\}$	$H = \{\infty_1, x_1, y_0\}$
6	$H = \{\infty_1, x_1, z_1\}$	$H = \{\infty_1, x_1, y_0\}$	$H = \{\infty_1, x_1, y_1\}$
7	$H = \{\infty_1, x_1, y_1\}$	$H = \{\infty_1, x_1, z_1\}$	$H = \{\infty_1, x_1, z_0\}$

case 5. $T = \{\infty_0, x_1\}$.

Similarly to case 3 there exists $F = \{x, y, z\} \in \mathcal{F}$ and $i \in \{0,1,2\}$ such that $x \in A_i$, $y \in A_{i+1}$, $z \in A_{i+2}$. The choice of H will depend on $h(F)$ and i. The following table shows how $H \in S(B)$ can be chosen so that $T \subset H$.

Table III

i / h(F)	0	1	2
0	$H = \{\infty_0, x_1, y_0\}$	$H = \{\infty_0, x_1, y_0\}$	$H = \{\infty_0, x_1, y_0\}$
1	$H = \{\infty_0, x_1, z_0\}$	$H = \{\infty_0, x_1, z_0\}$	$H = \{\infty_0, x_1, z_0\}$
2	$H = \{\infty_0, x_1, y_0\}$	$H = \{\infty_0, x_1, y_1\}$	$H = \{\infty_0, x_1, z_1\}$
3	$H = \{\infty_0, x_1, z_1\}$	$H = \{\infty_0, x_1, z_0\}$	$H = \{\infty_0, x_1, y_1\}$
4	$H = \{\infty_0, x_1, y_1\}$	$H = \{\infty_0, x_1, z_1\}$	$H = \{\infty_0, x_1, y_0\}$
5	$H = \{\infty_0, x_1, z_0\}$	$H = \{\infty_0, x_1, y_1\}$	$H = \{\infty_0, x_1, z_1\}$
6	$H = \{\infty_0, x_1, y_1\}$	$H = \{\infty_0, x_1, z_1\}$	$H = \{\infty_0, x_1, z_0\}$
7	$H = \{\infty_0, x_1, z_1\}$	$H = \{\infty_0, x_1, y_0\}$	$H = \{\infty_0, x_1, y_1\}$

case 6. $T = \{\infty_1, x_0\}$.

Similarly to case 3. there exists $F = \{x, y, z\} \in \mathcal{F}$ and $i \in \{0,1,2\}$ such that $x \in A_i$, $y \in A_{i+1}$, $z \in A_{i+2}$. The choice of H will depend on h(F) and i. The following table shows how $H \in S(B)$ can be chosen so that $T \subset H$.

Table IV

$h(F)$ \ i	0	1	2
0	$H = \{\infty_1, x_o, y_1\}$	$H = \{\infty_1, x_o, y_1\}$	$H = \{\infty_1, x_o, y_1\}$
1	$H = \{\infty_1, x_o, z_1\}$	$H = \{\infty_1, x_o, z_1\}$	$H = \{\infty_1, x_O, z_1\}$
2	$H = \{\infty_1, x_o, y_1\}$	$H = \{\infty_1, x_o, y_o\}$	$H = \{\infty_1, x_o, z_o\}$
3	$H = \{\infty_1, x_o, z_o\}$	$H = \{\infty_1, x_o, z_1\}$	$H = \{\infty_1, x_o, y_o\}$
4	$H = \{\infty_1, x_o, y_o\}$	$H = \{\infty_1, x_o, z_o\}$	$H = \{\infty_1, x_o, y_1\}$
5	$H = \{\infty_1, x_o, z_1\}$	$H = \{\infty_1, x_o, y_o\}$	$H = \{\infty_1, x_o, z_o\}$
6	$H = \{\infty_1, x_o, y_o\}$	$H = \{\infty_1, x_o, z_o\}$	$H = \{\infty_1, x_o, z_1\}$
7	$H = \{\infty_1, x_o, z_o\}$	$H = \{\infty_1, H_o, y_1\}$	$H = \{\infty_1, x_o, y_o\}$

case 7. $T = \{x_o, x_1\}$.

Let $H = \{\infty, x_o, x_1\}$. Then $H \in S(B)$ and $T \subset H$.

case 8. $T = \{x_o, y_1\}$, $x \neq y$.

Since x, y are distinct elements of A, hence there exists a unique element z in A such that $\{x, y, z\} \in S(A)$. Let $E = \{x, y, z\}$. Then either $E \in \mathcal{F}$ or $E \in S(A) - \mathcal{F}$. First we consider the case $E \in S(A) - \mathcal{F}$. By definition of g we have $g(E) = i$ for some $i \in \{0, 1\}$. Let $H = \{x_o, y_1, z_{i+1}\}$. Then $H \in S(B)$ and $T \subset H$. Next we consider the case $E \in \mathcal{F}$. By definition of $\mathcal{F}$ there exist distinct elements i, j in $\{0, 1, 2\}$ such that $x \in A_i$, $y \in A_j$. The choice of H will depend on $h(E)$ and (i, j). The following table shows how $H \in S(B)$ can be chosen so that $T \subset H$.

Table V

$h(E)$ \ (i,j)	(0,1)	(0,2)	(1,0)	(1,2)	(2,0)	(2,1)
0	$H = \{\infty_1, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$
1	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{\infty_1, x_0, y_1\}$
2	$H = \{\infty_1, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$
3	$H = \{\infty_0, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$
4	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$
5	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$
6	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_0, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_1, x_0, y_1\}$
7	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_1, x_0, y_1\}$	$H = \{x_0, y_1, z_0\}$	$H = \{\infty_0, x_0, y_1\}$

Hence $(B, S(B))$ is m - STS.

3.4 Construction of 2n + 7 - STS from n - STS

3.4.1 Definition. Let (A,S(A)) be a STS. By an automorphism of (A,S(A)) we mean any permutation f on A such that $f(H) \in S(A)$ if $H \in S(A)$.

3.4.2 Theorem. Let (A,S(A)) be a STS. Let F be the set of all automorphisms of (A,S(A)). Then F is a group under composition.

Proof : It is known that the set P of all permutations on A is a group under composition. Hence to show that F is a group under composition, it suffices to show that F is a subgroup of P. Let f, g be any elements of F. To show that $fg \in F$, let B be any triple in S(A). By definition of g we have $g(B) \in S(A)$ so that $fg(B) = f(g(B)) \in S(A)$. Thus $fg \in F$. Let h be any element of F. Since $h \in P$, hence h^{-1} exists. We shall show that $h^{-1} \in F$. Let $B = \{r,s,t\}$ be any triple in S(A). Suppose that $h^{-1}(B) \notin S(A)$. Since $r \neq s$ and h^{-1} is one to one, hence $h^{-1}(r) \neq h^{-1}(s)$. Thus $h^{-1}(r)$ and $h^{-1}(s)$ are distinct elements of A so that there exists a unique element w in A such that $\{h^{-1}(r), h^{-1}(s), w\} \in S(A)$. But $h^{-1}(B) = \{h^{-1}(r), h^{-1}(s), h^{-1}(t)\} \notin S(A)$. Therefore $w \neq h^{-1}(t)$. Since $h^{-1}(A) = A$ and $w \in A$, hence there exists u in A such that $h^{-1}(u) = w$. Let $C = \{h^{-1}(r), h^{-1}(s), h^{-1}(u)\}$. Then $C \in S(A)$ and $h(C) = \{hh^{-1}(r), hh^{-1}(s), hh^{-1}(u)\} = \{r,s,u\}$. Thus u = t so that $w = h^{-1}(u) = h^{-1}(t)$. This contradicts the preceding remark that $w \neq h^{-1}(t)$. Hence $h^{-1}(B) \in S(A)$. Therefore $h^{-1} \in F$. Thus F is a subgroup of P.

In what follows the set of all automorphisms of a STS (A,S(A)) will be called the automorphism group of (A,S(A)).

3.4.3 Definition. A permutation group S on a finite set A is said to be transitive on a subset B of A if for any x,y in B there exists $f \in S$ such that $f(x) = y$.

3.4.4 Definition. Let (A,S(A)) be n - STS. Let G be a subgroup of the automorphism group of (A,S(A)) such that G is transitive on A and $|G| = n$. If $A_o \in S(A)$, $\mathcal{F} \subset S(A)$, $p \in A$, have the following properties :

(i) $|\mathcal{F}| = n$

(ii) $\{ g(A_o) \ / \ g \in G \} = \mathcal{F}$

(iii) for each x in A there exists $E \in S(A) - \mathcal{F}$ such that $x \in E$,

(iv) p is contained in exactly three triples in $\mathcal{F}$.

We say that G, A_o, $\mathcal{F}$ and p satisfy Property II with respect to (A,S(A)).

A STS (A,S(A)) is said to be a STS with Property II[1] if there exist G, A_o, $\mathcal{F}$ and p such that G, A_o, $\mathcal{F}$ and p satisfy Property II with respect to (A,S(A)).

3.4.4 Theorem. Let (A,S(A)) be n - STS with G, A_o, $\mathcal{F}$ and p

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[1] A STS of order $n \equiv 3 \pmod 6$ with Property II exists. This will be proved in Section 5.4 of Chapter V.

sastisfying Property II with respect to (A,S(A)). Suppose that H_1, H_2, H_3 are distinct triples in $\mathcal{F}$ that contain p. Let P = $H_1 \cup H_2 \cup H_3 - \{p\}$ and $B = (A \times \{0,1\}) \cup C$, where $C = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ and $C \cap (A \times \{0,1\}) = \phi$. Construct S(C) as S(A) of Example (iii) in Chapter II so that (C,S(C)) is 7 - STS. Let f be a mapping on S(A) - $\mathcal{F}$ into $\{0,1\}$ and h be a bijection on $C - \{\infty_0\}$ onto P. If S(B) is the family of the following 3-subsets of B:

(i) $\{\infty_0, x_0, x_1\}$, where $x \in A$,

(ii) the triples in S(C),

(iii) $\{x_i, y_i, z_i\}, \{x_i, y_{i+1}, z_{i+1}\}, \{x_{i+1}, y_i, z_{i+1}\}, \{x_{i+1}, y_{i+1}, z_i\}$, where $\{x,y,z\} \in S(A) - \mathcal{F}$ and $f(\{x,y,z\}) = i$, $i \in \{0,1\}$,

(iv) $\{x_0, y_0, z_0\}, \{x_1, y_1, z_1\}$, where $\{x,y,z\} \in \mathcal{F}$

(v) $\{\infty_r, (g(h(\infty_r)))_0, (g(p))_1\}$, where $g \in G$, $r \in \{1,2,\ldots,6\}$.

Then (B,S(B)) is 2n + 7 - STS.

<u>Proof</u> : It can be seen from the construction of B that $|B| = 2n + 7$. Let m = 2n + 7. We shall show that the total number of 3-subsets in S(B) is at most $\frac{1}{6}$ m(m - 1). The total number of 3-subsets in S(B) of the form (i) - (ii) is n + 7. Since for each of $(\frac{1}{6}n(n-1) - n)$ triples in S(A) - $\mathcal{F}$ we can form exactly 4 3-subsets in S(B) of the form (iii), hence the total number of 3-subsets in S(B) of the form (iii) is at most $(4)(\frac{1}{6}n(n-1) - n)$. Similarly the total number of 3-subsets in S(B) of the form (iv) is at most 2n. Moreover the total number of 3-subsets in S(B) of the form (v) is at most 6n. Hence the total number of 3-subsets in S(B) is at most

$n + 7 + (4)(\frac{1}{6}\, n(n - 1) - n) + 2n + 6n = \frac{4n^2 + 26n + 42}{6} =$

$\frac{1}{6}\,(2n + 7)(2n + 6) = \frac{1}{6}\, m(m - 1)$. Thus to prove that $(B, S(B))$ is m - STS, it suffices to show that for any 2-subset T of B there exists a 3-subset H in $S(B)$ such that $T \subset H$. To show this let T be any 2-subset of B. We shall show by cases that there exists a 3-subset H in $S(B)$ such that $T \subset H$.

case 1. $T \subset C$

Since $(C, S(C))$ is a STS, hence there exists a 3-subset H in $S(C)$ such that $T \subset H$. But $S(C) \subset S(B)$. Therefore $H \in S(B)$.

case 2. $T = \{\infty_0, x_i\}$, $i \in \{0,1\}$.

Let $H = \{\infty_0, x_i, x_{i+1}\}$. Then $H \in S(B)$ and $T \subset H$.

case 3. $T = \{\infty_r, x_0\}$, $1 \leq r \leq 6$.

Let $a = h(\infty_r)$. Then $a \in A$. By definition of G there exists $g \in G$ such that $g(a) = x$. Let $H = \{\infty_r, x_0, (g(p))_1\}$. Then $H \in S(B)$ and $T \subset H$.

case 4. $T = \{\infty_r, x_1\}$, $1 \leq r \leq 6$.

It follows from definition of G that there exists $g \in G$ such that $g(x) = p$. Since G is a group, hence g^{-1} exists and $g^{-1}(p) = x$. Let $H = \{\infty_r, (g^{-1}(h(\infty_r)))_0, x_1\}$. Then $H \in S(B)$ and $T \subset H$.

case 5. $T = \{x_i, x_j\}$, $0 \leq i,j \leq 1$, $i \neq j$.

Let $H = \{\infty_0, x_i, x_j\}$. Then $H \in S(B)$ and $T \subset H$.

case 6. $T = \{x_i, y_i\}$, $x \neq y$, $0 \leq i \leq 1$.

Since x, y are distinct elements of A, hence there exists a unique element z in A such that $\{x, y, z\} \in S(A)$. Let $E = \{x, y, z\}$.

Then either $E \in \mathcal{F}$ or $E \in S(A) - \mathcal{F}$. If $E \in \mathcal{F}$, let $H = \{x_i, y_i, z_i\}$. Then $H \in S(B)$ and $T \subset H$. In case $E \in S(A) - \mathcal{F}$ we have $f(E) = j$ for some $j \in \{0,1\}$. Let $H = \{x_i, y_i, z_j\}$. Then $H \in S(B)$ and $T \subset H$.

case 7. $T = \{x_i, y_j\}$, $x \neq y$, $0 \leq i,j \leq 1, i \neq j$.

Without loss of generality we may assume that $i = 0$ and $j = 1$. Thus $T = \{x_0, y_1\}$. Since x,y are distinct elements of A, hence there exists a unique element z in A such that $\{x,y,z\} \in S(A)$. Then either $\{x,y,z\} \in S(A) - \mathcal{F}$ or $\{x,y,z\} \in \mathcal{F}$. First we consider the case $\{x,y,z\} \in S(A) - \mathcal{F}$. By definition of f we have $f(\{x,y,z\}) = r$ for some $r \in \{0,1\}$. Let $H = \{x_0, y_1, z_{r+1}\}$. Then $H \in S(B)$ and $T \subset H$. Next we assume that $\{x,y,z\} \in \mathcal{F}$. Since G is transitive on A and $p,y \in A$, hence there exists $g \in G$ such that $g(p) = y$. Since $g(A) = A$, it follows that there exist w, t in A such that $g(w) = x$ and $g(t) = z$. Let $H_1' = \{x,y,z\}$ and $H_2' = \{p,w,t\}$. Thus $g(H_2') = H_1'$ so that $H_2' = g^{-1}(H_1')$. It follows from definition of $\mathcal{F}$ that there exists $g_1 \in G$ such that $H_1' = g_1(A_0)$. Therefore $H_2' = g^{-1}g_1(A_0)$. But $g^{-1}g_1 \in G$. Hence $H_2' \in \mathcal{F}$. That is $w \in P$. Since $h(C - \{\infty_0\}) = P$, hence there exists $r \in \{1,2,\ldots,6\}$ such that $h(\infty_r) = w$. Let $H = \{\infty_r, x_0, y_1\}$. Then $H \in S(B)$ and $T \subset H$.

Therefore $(B,S(B))$ is $2n + 7$ - STS.

3.5 Construction of $n_1 n_2$ - STS from n_1- STS and n_2- STS

By an extension of a STS we shall mean a STS which contains a subsystem isomorphic to the given STS. In this section and the next section two methods of constructing extensions of STS will be discussed. For convenience we shall introduce a concept of "triple system", which is more general than that of STS.

3.5.1 Definition . A triple system, abbreviated as TS, is an ordered pair $(A,S(A))$, where A is a finite set and $S(A)$ is a family of 3-subsets of A.

We say that $(A,S(A))$ is a TS of order n if A contains n elements.

3.5.2 Definition. Let $(A,S(A))$, $(B,S(B))$ be any TS of order n_1, n_2 respectively. By the cartesian product extension of $(A,S(A))$ and $(B,S(B))$ we mean the TS $(A \times B, S(A \times B))$, where $S(A \times B)$ consists of all 3-subsets of $A \times B$ of the following forms :

(i) $\{(a_i,b_{j_1}),(a_i,b_{j_2}),(a_i,b_{j_3})\}$, where $\{b_{j_1},b_{j_2},b_{j_3}\} \in S(B)$ and $a_i \in A$,

(ii) $\{(a_{i_1},b_j),(a_{i_2},b_j),(a_{i_3},b_j)\}$, where $\{a_{i_1},a_{i_2},a_{i_3}\} \in S(A)$ and $b_j \in B$,

(iii) $\{(a_{i_1},b_{j_1}),(a_{i_2},b_{j_2}),(a_{i_3},b_{j_3})\}$, $\{(a_{i_1},b_{j_1}),(a_{i_2},b_{j_3}),(a_{i_3},b_{j_2})\}$,

$\{(a_{i_2},b_{j_1}),(a_{i_1},b_{j_2}),(a_{i_3},b_{j_3})\}$, $\{(a_{i_2},b_{j_1}),(a_{i_1},b_{j_3}),(a_{i_3},b_{j_2})\}$,

$\{(a_{i_3},b_{j_1}),(a_{i_2},b_{j_2}),(a_{i_1},b_{j_3})\}$, $\{(a_{i_3},b_{j_1}),(a_{i_2},b_{j_3}),(a_{i_1},b_{j_2})\}$,

where $\{a_{i_1},a_{i_2},a_{i_3}\} \in S(A)$ and $\{b_{j_1},b_{j_2},b_{j_3}\} \in S(B)$.

3.5.3 Proposition. If (A,S(A)) and (B,S(B)) are STS of orders n_1 and n_2 respectively. Then the cartesian product extension of (A,S(A)) amd (B,S(B)) is n_1n_2- STS which is an extension of both (A,S(A)) and (B,S(B)).

Proof: Let (A,S(A)), (B,S(B)) be n_1- STS, n_2- STS respectively, where $A = \{a_1, \dots, a_{n_1}\}$, $B = \{b_1, \dots, b_{n_2}\}$. Let (C,S(C)) be the cartesian product extension of (A,S(A)) and (B,S(B)). For $i = 1, \dots, n_1$ and $j = 1, \dots, n_2$ denote the ordered pair (a_i, b_j) of $A \times B$ by c_{ij}. It follows from definition that C has n_1n_2 elements. We first show that (C,S(C)) is n_1n_2 - STS. By counting the number of 3-subsets in S(C), we see that there are at most $(n_1)(\frac{1}{6}n_2(n_2-1))$ and $(n_2)(\frac{1}{6}n_1(n_1-1))$ 3-subsets of the form (i) and (ii) respectively. Moreover, there are at most $(6)(\frac{1}{6}n_1(n_1-1))(\frac{1}{6}n_2(n_2-1))$ 3-subsets of the form (iii). Hence the total number of 3-subsets in S(C) is at most $(n_1)(\frac{1}{6}n_2(n_2-1)) + (n_2)(\frac{1}{6}n_1(n_1-1)) + (6)(\frac{1}{6}n_1(n_1-1)(\frac{1}{6}n_2(n_2-1))= (\frac{1}{6}n_1n_2)(n_1n_2-1)$. Thus to prove that (C,S(C)) is n_1n_2 - STS, it suffices to show that for any 2-subset T of C there exists a 3-subset H in S(C) such that $T \subset H$. Let $T = \{c_{i_1j_1}, c_{i_2j_2}\}$ be any 2-subset of C.

case 1. $i_1 = i_2 = i$.

Thus $j_1 \neq j_2$ and hence b_{j_1}, b_{j_2} are distinct elements of B. Since (B,S(B)) is a STS, it follows that there exists a unique element b_{j_3} in B such that $\{b_{j_1}, b_{j_2}, b_{j_3}\} \in S(B)$. Let $H = \{c_{ij_1}, c_{ij_2}, c_{ij_3}\}$. Then $H \in S(C)$ and $T \subset H$.

case 2. $j_1 = j_2 = j$.

Thus $i_1 \neq i_2$ and hence a_{i_1}, a_{i_2} are distinct elements of A. Since (A,S(A)) is a STS, it follows that there exists a unique element a_{i_3} in A such that $\{a_{i_1}, a_{i_2}, a_{i_3}\} \in S(A)$. Let H = $\{c_{i_1 j}, c_{i_2 j}, c_{i_3 j}\}$. Then $H \in S(C)$ and $T \subset H$.

case 3. $i_1 \neq i_2$, $j_1 \neq j_2$.

Thus a_{i_1}, a_{i_2} are distinct elements of A and hence there exist a unique element a_{i_3} in A such that $\{a_{i_1}, a_{i_2}, a_{i_3}\} \in S(A)$. Also b_{j_1} and b_{j_2} are distinct elements of B so that there exists a unique element b_{j_3} in B such that $\{b_{j_1}, b_{j_2}, b_{j_3}\} \in S(B)$. Let $H = \{c_{i_1 j_2}, c_{i_2 j_2}, c_{i_3 j_3}\}$. Then $H \in S(C)$ and $T \subset H$.

Hence (C,S(C)) is $n_1 n_2$- STS.

We must show that (C,S(C) has subsystems of order n_1, n_2 which are isomorphic to (A,S(A)), (B,S(B)) respectively. For $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, let

$$B_i = \{ c_{ij} \ / \ j = 1, \ldots, n_2 \},$$

$$S(B_i) = \left\{ \{c_{ij_1}, c_{ij_2}, c_{ij_3}\} / \{b_{j_1}, b_{j_2}, b_{i_3}\} \in S(B) \right\},$$

$$A_j = \{ c_{ij} \ / \ i = 1, \ldots, n_1 \},$$

$$S(A_j) = \{\{c_{i_1 j}, c_{i_2 j}, c_{i_3 j}\} / \{a_{i_1}, a_{i_2}, a_{i_3}\} \in S(A)\}.$$

We abserve that $B_i = \{a_i\} \times B$,

$$S(B_i) = \{\{a_i\} \times \{b_{j_1}, b_{j_2}, b_{j_3}\} / \{b_{j_1}, b_{j_2}, b_{j_3}\} \in S(B)\},$$

$$A_j = A \times \{b_j\},$$

$$S(A_j) = \{\{a_{i_1}, a_{i_2}, a_{i_3}\} \times \{b_j\} / \{a_{i_1}, a_{i_2}, a_{i_3}\} \in S(A)\}.$$

Hence by Corollary 3.1.3, $(A_j, S(A_j))$, $(B_i, S(B_i))$ are STS of orders n_1, n_2 which are isomorphic to $(A,S(A))$, $(B,S(B))$ respectively. Since members of $S(A_j)$, $S(B_i)$ are 3-subsets in $S(C)$, hence $(A_j, S(A_j))$, $(B_i, S(B_i))$ are subsystems of $(C,S(C))$ of orders n_1, n_2 which are isomorphic to $(A,S(A))$, $(B,S(B))$ respectively. Therefore $(C,S(C))$ is an extension of both $(A,S(A))$ and $(B,S(B))$.

In the sequel we shall refer to the method of constructing the cartesian product extension $(C,S(C))$ of $(A,S(A))$ and $(B,S(B))$ as described in Definition 3.5.2 as <u>Method I</u>. Moreover $(C,S(C))$ is said to be <u>constructed by Method I from $(A,S(A))$ and $(B,S(B))$</u>.

3.5.4 <u>Remark</u>. It follows from Corollary 2.2.7 that the cartesian product extension of STS $(A,S(A))$ and $(B,S(B))$ will contain a subsystem of order k if at least one of $(A,S(A))$ and $(B,S(B)$ does.

As a consequence of Proposition 3.5.3 and Remark 3.5.4 we have the following

3.5.5 Theorem. If any positive integer n can be written in the form $n = n_1 n_2$, where n_1- STS and n_2- STS exist, then n - STS exists. Furthermore, if (n_1,k) - STS or (n_2,k) - STS exists then (n,k)-STS exists.

3.6 Construction of $n_3 + n_1(n_2 - n_3)$ - STS from n_1- STS and (n_2,n_3)-STS.

3.6.1 Definition. Let $(C,S(C)),(B,S(B)),(A,S(A))$ be TS of orders n_1,n_2,n_3 respectively such that $A \subset B$ and $S(A) \subset S(B)$, where $A = \{a_1,\ldots,a_{n_3}\}$, $B = \{a_1,\ldots,a_{n_3},b_1,\ldots,b_s\}$, $s = n_2 - n_3$, $C = \{1,\ldots,n_1\}$. For $i = 1,\ldots,n_1$, let $N_i = \{i\} \underset{n_1}{\times} B - A$ and denote the element (i,b_p) of N_i by b_{ip}. Let $N = A \cup (\bigcup_{i=1} N_i)$. By Moore decomposition extension of $(C,S(C))$, $(B,S(B))$ and $(A,S(A))$ we mean the TS $(N,S(N))$, where $S(N)$ consists of all 3-subsets of N of the following forms :

(i) $\{a_i,a_j,a_k\}$, where $\{a_i,a_j,a_k\} \in S(A)$,

(ii) $\{a_m,b_{ip},b_{iq}\}$, where $\{a_m,b_p,b_q\} \in S(B)$, $i = 1,\ldots,n_1$,

(iii) $\{b_{ip},b_{iq},b_{ir}\}$, where $\{b_p,b_q,b_r\} \in S(B)$, $i = 1,2,\ldots,n_1$,

(iv) $\{b_{ip},b_{jq},b_{kr}\}$, where $\{i,j,k\} \in S(C)$ and $p+q+r \equiv 0 \pmod{s}$.

3.6.2 Proposition. If $(C,S(C))$, $(B,S(B))$, $(A,S(A))$ are STS of orders n_1,n_2,n_3 respectively, then Moore decomposition extension $(N,S(N))$ of $(C,S(C))$, $(B,S(B))$ and $(A,S(A))$ is $n_3 + n_1(n_2 - n_3)$ - STS which is an extension of $(C,S(C))$, $(B,S(B))$ and $(A,S(A))$.

Proof : Let $(C,S(C))$, $(B,S(B))$, $(A,S(A))$ be STS of order n_1,n_2,n_3, where $A = \{a_1,\ldots,a_{n_3}\}$, $B = \{a_1,\ldots,a_{n_3},b_1,\ldots,b_s\}$, $s = n_2 - n_3$, $C = \{1,\ldots,n_1\}$. Let $(N,S(N))$ be Moore decomposition extension of $(C,S(C))$, $(B,S(B))$ and $(A,S(A))$. It follows from definition that N has $n_3 + n_1(n_2 - n_3)$ elements. Let $n = n_3 + n_1(n_2 - n_3)$. We first show that $(N,S(N))$ is n - STS. To determine the number of 3-subsets in $S(N)$ of the form (i) - (iv) let

$A(a_m)$ = the total number of triples in $S(A)$ that contain a_m

$B(a_m)$ = the total number of triples in $S(B)$ that contain a_m

$B_2(a_m)$ = the total number of triples in $S(B)$ of the form $\{a_m, b_i, b_j\}$

B_2 = the total number of triples in $S(B)$ of the form $\{a_i, b_j, b_k\}$

B_3 = the total number of triples in $S(B)$ of the form $\{b_i, b_j, b_k\}$

With these notations the total number of 3-subsets in $S(N)$ of the form (i), (ii), (iii) are $|S(A)|$, n_1B_2, n_1B_3 respectively. Let $a_m \in A$. Consider any triple H in $S(B)$ such that $a_m \in H$. Suppose that $H = \{a_m, x, y\}$. Since $(A,S(A))$ is a subsystem of $(B,S(B))$, hence either x,y are both in A or are both in $B - A$. Therefore H must be of the form $\{a_m, a_i, a_j\}$ or $\{a_m, b_i, b_j\}$. Hence

$$\begin{aligned} B_2(a_m) &= B(a_m) - A(a_m) \\ &= \frac{1}{2}(n_2 - 1) - \frac{1}{2}(n_3 - 1) \\ &= \frac{1}{2}(n_2 - n_3) \end{aligned}$$

Therefore

$$B_2 = B_2(a_1) + \ldots + B_2(a_{n_3})$$
$$= n_3 \frac{(n_2 - n_3)}{2}$$

Observe that $|S(B)| = |S(A)| + B_2 + B_3$. Hence

$$B_3 = |S(B)| - |S(A)| - B_2$$
$$= \frac{n_2(n_2 - 1)}{6} - \frac{n_3(n_3 - 1)}{6} - \frac{n_3(n_2 - n_3)}{2}$$
$$= \frac{1}{6}\left[n_2(n_2 - 1) - n_3(n_3 - 1) - 3n_3(n_2 - n_3)\right]$$

Therefore the total number of 3-subsets in S(N) of the forms (i) - (iii) is

$$|S(A)| + n_1B_2 + n_1B_3 = \frac{1}{6} n_3(n_3 - 1) + \frac{1}{2}n_1n_3(n_2 - n_3) + \frac{1}{6}n_1\left[n_2(n_2 - 1) - n_3(n_3 - 1) - 3n_3(n_2 - n_3)\right]$$
$$= \frac{1}{6} n_3(n_3 - 1) + \frac{1}{6}n_1\left[n_2(n_2 - 1) - n_3(n_3 - 1)\right]$$

To determine the total number of 3-subsets in S(N) of the form (iv) we observe that the total number of ordered triples (p,q,r) such that $p + q + r \equiv 0 \pmod{s}$ and $1 \leq p,q \leq s$ is s^2. Thus the total number of 3-subsets in S(N) of the form (iv) is $\frac{1}{6} n_1(n_1 - 1)(s^2)$.
Therefore the total number of 3-subsets in S(N) of the form (i)-(iv) is at most $\frac{1}{6} n_3(n_3 - 1) + \frac{1}{6} n_1 \left[n_2(n_2 - 1) - n_3(n_3 - 1)\right] + \frac{1}{6}n_1(n_1 - 1)(s^2) = \frac{1}{6} n(n - 1)$. Thus to show that (N,S(N)) is n - STS, it suffices to show that for any 2-subset T of N there exists a 3-subset H in S(N) such that $T \subseteq H$. Let T be any 2-subset of N. We shall show by cases that there exists a 3-subset H in S(N) such that $T \subseteq H$.

case 1 $T = \{a_i, a_j\}$, $1 \leq i, j \leq n_3$, $i \neq j$.

Since a_i and a_j are distinct elements of A, hence there exists a unique element a_k in A such that $\{a_i, a_j, a_k\} \in S(A)$. Let $H = \{a_i, a_j, a_k\}$. Then $H \in S(N)$ and $T \subset H$.

case 2. $T = \{a_m, b_{ij}\}$, $1 \leq m \leq n_3$, $1 \leq i \leq n_1$, $1 \leq j \leq s$.

Since $(B, S(B))$ is a STS and a_m, b_j are distinct elements of B, hence there exists a unique element x in B such that $\{a_m, b_j, x\} \in S(B)$. By Proposition 2.2.3 we have $x \in B - A$ so that $x = b_k$, where $b_k \neq b_j$. Let $H = \{a_m, b_{ij}, b_{ik}\}$. Then $H \in S(N)$ and $T \subset H$.

case 3. $T = \{b_{ip}, b_{iq}\}$, $1 \leq i \leq n_1$, $1 \leq p, q \leq s, p \neq q$.

Since $p \neq q$, hence b_p and b_q are distinct elements of B so that there exists a unique element x in B such that $\{b_p, b_q, x\} \in S(B)$. Then x is in either A or B - A. If $x \in A$, we see that $x = a_m$ for some a_m in A. Let $H = \{a_m, b_{ip}, b_{iq}\}$. Then $H \in S(N)$ and $T \subset H$. In case $x \in B - A$, we have $x = b_r$, where $1 \leq r \leq s$. Let $H = \{b_{ip}, b_{iq}, b_{ir}\}$. Then $H \in S(N)$ and $T \subset H$.

case 4. $T = \{b_{ip}, b_{jp}\}$, $1 \leq p \leq s$, $1 \leq i, j \leq n_1$, $i \neq j$.

Since $i \neq j$, hence i and j are distinct elements of C so that there exists a unique element k in C such that $\{i, j, k\} \in S(C)$. For a given $p \in \{1, 2, \ldots, s\}$, there exists $q \in \{1, 2, \ldots, s\}$ such that $p + p + q \equiv 0 \pmod{s}$. Let $H = \{b_{ip}, b_{jp}, b_{kq}\}$. Then $H \in S(N)$ and $T \subset H$.

case 5. $T = \{b_{ip}, b_{jq}\}$, $1 \leq p, q \leq s$, $p \neq q$, $1 \leq i, j \leq n_1$, $i \neq j$.

By a similar argument as in case 4 we can see that there exists a 3-subset H in S(N) such that $T \subset H$.

Hence (N,S(N)) is $n_3 + n_1(n_2 - n_3)$ - STS.

Since $S(A) \subset S(N)$ and (A,S(A)) is a STS, hence (A,S(A)) is a subsystem of (N,S(N)) of order n_3.

To show that (N,S(N)) has subsystems of orders n_1, n_2 which are isomorphic to (C,S(C)) and (B,S(B)) respectively, let

$$M = \{ b_{ks} / k = 1,\dots,n_1 \}$$

$$S(M) = \{ \{ b_{is}, b_{js}, b_{ks} \} / \{i,j,k\} \in S(C) \},$$

$$B_i = A \cup N_1, \ i = 1,2,\dots,n_1,$$

and let $S(B_i)$ be the family of the following 3-subsets of B_i

(1) $\{a_i, a_j, a_k\}$, where $\{a_i, a_j, a_k\} \in S(A)$

(2) $\{a_m, b_{ij}, b_{ik}\}$, where $\{a_m, b_j, b_k\} \in S(B)$,

(3) $\{b_{ij}, b_{ik}, b_{ir}\}$, where $\{b_j, b_k, b_r\} \in S(B)$.

We observe that $M = C \times \{b_s\}$ and $S(M) = \{ \{i,j,k\} \times \{b_s\} / \{i,j,k\} \in S(C) \}$. Thus it follows from Corollary 3.1.3 that (M,S(M)) is n_1- STS which is isomorphic to (C,S(C)). Since $S(M) \subseteq S(N)$, hence (M,S(M)) is a subsystem of (N,S(N)) of order n_1 which is isomorphic to (C,S(C)). To see that $(B_i, S(B_i))$ is a subsystme of (N,S(N)) of order n_2, which is isomorphic to (B,S(B)), let f_i be a mapping on B into B_i defined by $f_i(a_m) = a_m$, $1 \le m \le n_3$, $f_i(b_j) = b_{ij}$, $1 \le j \le s$. We can see that f_i is a one to one mapping on B onto B_i and $S(B_i) = \{ f_i(H) / H \in S(B) \}$. Thus it follows from Lemma 3.1.2

that $(B_i, S(B_i))$ is n_2- STS which is isomorphic to $(B, S(B))$. Since $S(B_i) \subset S(N)$, hence $(B_i, S(B_i))$ is a subsystem of $(N, S(N))$ of order n_2.

Therefore $(N, S(N))$ is n - STS which is an extension of $(C, S(C))$, $(B, S(B))$ and $(A, S(A))$.

In the sequel we shall refer to the method of constructing Moore decomposition extension $(N, S(N))$ of STS $(C, S(C))$, $(B, S(B))$ and $(A, S(A))$ as described in Definition 3.6.1 as Method II. Moreover $(N, S(N))$ is said to be constructed by Method II from $(C, S(C))$ $(B, S(B))$ and $(A, S(A))$.

3.6.3 Remark. It follows from Corollary 2.2.7 that Moore decompo--sition extension of STS $(C, S(C))$, $(B, S(B))$ and $(A, S(A))$ will contain a subsystem of order k if at least one of $(C, S(C))$, $(B, S(B))$ and $(A, S(A))$ does.

As a consequence of Proposition 3.6.2 and Remark 3.6.3 we have the following

3.6.4 Theorem. If any positive integer n can be written in the form $n = n_3 + n_1(n_2 - n_3)$, where n_1- STS and (n_2, n_3) - STS exists, then n - STS exists. Furthermore if (n_1, k) - STS or (n_2, k) - STS exists, then (n, k) - STS exists.