CHAPTER II

METHOD OF ANALYSIS

Assumptions

Consider an inverted channel floor unit simply supported at the two end diaphragms, as shown in Fig.1, and subjected to a uniformly distributed load. The unit has uniform cross section throughout the span, consisting of vertical and horizontal rectangular plate elements.

This analysis is based on the following assumptions: The material is homogeneous and isotropic with equal moduli of elasticity in tension and compression; it is not strained beyond its elastic limit; plane sections of each individual plate remain plane after bending; the displacements are small for each plate; the supporting diaphragms are infinitely rigid in their planes but flexible normal to their planes; and the influence of the membrane forces on the bending of the plate is neglected. These assumptions lead to two fourth order differential equations, which govern the bending of the plate under the action of the normal load component and the membrane action of the plate under the in-plane load component.

The three orthogonal coordinate axes x, y and z are taken along the transverse, longitudinal and normal directions of the plate element respectively, with the origin at the mid-point of each plate as shown in Fig.2. The positive directions of the load components X, Y and Z, and the bending and membrane stress resultants are shown in Figs.3a and 3b. The displacement components u,v and w are positive in the positive directions of the coordinate axes x, y and z respectively.

Rectangular Plates in Bending

Vertical Plate

The homogeneous differential equation governing the bending of an elastic plate in small deflections (7) is

$$\nabla^4 w = 0 \tag{1}$$

The general solution of Eq.(1), which satisfies the boundary conditions at y = 0 and y = b, can be obtained in the form

$$w = \sum_{n=1}^{\infty} \zeta_n(x)\sin\beta y \tag{2}$$

where $\beta = n\pi/b$. Substituting Eq.(2) into Eq.(1) and solving the resulting differential equation yield

$$\zeta_n(x) = K(B_{1n}\sinh\beta x + B_{2n}\beta x\cosh\beta x + B_{3n}\cosh\beta x + B_{4n}\beta x\sinh\beta x) \tag{3}$$

where K is a load factor introduced to nondimensionalize the constants of integration B_{1n} to B_{4n} and will be defined for particular loading later.

The desired solution, Eq.(2), and the corresponding stress resultants take the form

$$w = \sum_{n=1}^{\infty} K[B_{1n}\sinh\beta x + B_{2n}\beta x\cosh\beta x + B_{3n}\cosh\beta x + B_{4n}\beta x \mathrm{inh}\beta x]\sin\beta y \tag{4a}$$

$$M_x = -D_w \sum_{n=1}^{\infty} \beta^2 K[B_{1n}(1-\nu)\sinh\beta x + B_{2n}\{(1-\nu)\beta x\cosh\beta x + 2\sinh\beta x\}$$
$$+ B_{3n}(1-\nu)\cosh\beta x + B_{4n}\{(1-\nu)\beta x\sinh\beta x + 2\cosh\beta x\}]\sin\beta y \tag{4b}$$

$$M_y = D_w \sum_{n=1}^{\infty} \beta^2 K[B_{1n}(1-\nu)\sinh\beta x + B_{2n}\{(1-\nu)\ \beta x\cosh\beta x-2\nu\sinh\beta x\}$$
$$+ B_{3n}(1-\nu)\cosh\beta x + B_{4n}\{(1-\nu)\beta x\sinh\beta x-2\nu\cosh\beta x\}]\sin\beta y \qquad (4c)$$

$$M_{xy} = M_{yx} = -D_w(1-\nu)\sum_{n=1}^{\infty}\beta^2 K[B_{1n}\cosh\beta x + B_{2n}\{\beta x\sinh\beta x+\cosh\beta x\}$$
$$+ B_{3n}\sinh\beta x + B_{4n}\{\beta x\cosh\beta x+\sinh\beta x\}]\cos\beta y \qquad (4d)$$

$$Q_x = -2D_w \sum_{n=1}^{\infty}\beta^3 K[B_{2n}\cosh\beta x + B_{4n}\sinh\beta x]\sin\beta y \qquad (4e)$$

$$Q_y = -2D_w \sum_{n=1}^{\infty}\beta^3 K[B_{2n}\sinh\beta x + B_{4n}\cosh\beta x]\cos\beta y \qquad (4f)$$

$$V_x = Q_x + D_w(1-\nu)\sum_{n=1}^{\infty}\beta^3 K[B_{1n}\cosh\beta x + B_{2n}\{\beta x\sinh\beta x+\cosh\beta x\}$$
$$+ B_{3n}\sinh\beta x + B_{4n}\{\beta x\cosh\beta x+\sinh\beta x\}]\sin\beta y \qquad (4g)$$

$$V_y = Q_y - D_w(1-\nu)\sum_{n=1}^{\infty}\beta^3 K[B_{1n}\sinh\beta x + B_{2n}\{\beta x\cosh\beta x+2\sinh\beta x\}$$
$$+ B_{3n}\cosh\beta x + B_{4n}\{\beta x\sinh\beta x+2\cosh\beta x\}]\cos\beta y \qquad (4h)$$

in which $D_w = Et_w^3/\{12(1-\nu^2)\}$ is the flexural rigidity of the web plate of thickness t_w, E denotes the modulus of elasticity and ν the Poisson's ratio. M_x, M_y and M_{xy} are the transverse, longitudinal and torsional moments per unit length respectively, Q_x and Q_y the transverse and longitudinal shearing forces per unit length respectively, and V_x and V_y the transverse and longitudinal supplemented shearing forces per unit length respectively.

Horizontal Plate

The differential equation governing the bending of an elastic plate subjected to normal load component Z [7] is

$$\nabla^4 w = Z/D_d \qquad (5)$$

in which $D_d = Et_d^3/\{12(1-\nu^2)\}$ is the flexural rigidity of the deck plate of thickness t_d. The general solution of Eq.(5) is

$$w = w_p + w_c \tag{6}$$

in which w_p is a particular integral and w_c the complementary solution. A particular integral satisfying Eq.(5) and the boundry conditions at $y = 0$ and $y = b$ can be obtained by expanding w_p and Z in double series of the form

$$w_p(x,y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} W_{mn} \cos\alpha x \sin\beta y \tag{7a}$$

$$Z(x,y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} Z_{mn} \cos\alpha x \sin\beta y \tag{7b}$$

where $\alpha = m\pi/a$, $\beta = n\pi/b$, and W_{mn} and Z_{mn} are Fourier coefficients. Substituting Eqs.(7a) and (7b) into Eq.(5) gives

$$W_{mn} = \frac{Z_{mn}}{D_d(\alpha^2+\beta^2)^2} \tag{8}$$

in which

$$Z_{mn} = \frac{4}{ab}\int_0^a\int_0^b Z(x,y)\cos\alpha x \sin\beta y\,dx\,dy \qquad \text{for } m \neq 0 \tag{9a}$$

$$Z_{on} = \frac{2}{b}\int_0^b Z(x,y)\sin\beta y\,dy \qquad \text{for } m = 0 \tag{9b}$$

The complementary solution, w_c, satisfying the homogeneous part of Eq.(5) and the boundary conditions at $y = 0$ and $y = b$, can be taken in the form

$$w_c = \sum_{n=1}^{\infty} \psi_n(x)\sin\beta y \tag{10}$$

Substituting Eq.(10) into Eq.(5) and setting $Z = 0$, it can be shown that

$$\psi_n(x) = K(C_{1n}\sinh\beta x + C_{2n}\beta x\cosh\beta x + C_{3n}\cosh\beta x + C_{4n}\beta x\sinh\beta x) \tag{11}$$

where K is a load factor and C_{1n} to C_{4n} are the constants of integration.

The complete solution is then obtained by substituting Eqs.(7a), (10) and (11) into Eq.(6) which yields

$$w = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} W_{mn}\cos\alpha x\sin\beta y + \sum_{n=1}^{\infty} K[C_{1n}\sinh\beta x + C_{2n}\beta x\cosh\beta x + C_{3n}\cosh\beta x + C_{4n}\beta x\sinh\beta x]\sin\beta y \tag{12}$$

For the horizontal plate, symmetry with respect to the vertical plane at mid-width, i.e., the yz-plane, requires that all the terms in w involving odd functions of x vanish. Therefore the experession for w becomes

$$w = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} W_{mn}\cos\alpha x\sin\beta y + \sum_{n=1}^{\infty} K[A_{1n}\cosh\beta x + A_{2n}\beta x\sinh\beta x]\sin\beta y \tag{13a}$$

in which $A_{1n} = C_{3n}$ and $A_{2n} = C_{4n}$.

The corresponding stress resultants become

$$M_x = D_d \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\alpha^2+\nu\beta^2)W_{mn}\cos\alpha x\sin\beta y - D_d\sum_{n=1}^{\infty}\beta^2K[A_{1n}(1-\nu)\cosh\beta x + A_{2n}\{(1-\nu)\beta x\sinh\beta x+2\cosh\beta x\}]\sin\beta y \tag{13b}$$

$$M_y = D_d \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\beta^2+\nu\alpha^2)W_{mn}\cos\alpha x\sin\beta y + D_d\sum_{n=1}^{\infty}\beta^2K[A_{1n}(1-\nu)\cosh\beta x + A_{2n}\{(1-\nu)\beta x\sinh\beta x-2\nu\cosh\beta x\}]\sin\beta y \tag{13c}$$

$$M_{xy} = M_{yx} = D_d(1-\nu)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha\beta W_{mn}\sin\alpha x\cos\beta y - D_d(1-\nu)\sum_{n=1}^{\infty}\beta^2K[A_{1n}\sinh\beta x + A_{2n}\{\beta x\cosh\beta x+\sinh\beta x\}]\cos\beta y \tag{13d}$$

$$Q_x = -D_d\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha(\alpha^2+\beta^2)W_{mn}\sin\alpha x\sin\beta y - 2D_d\sum_{n=1}^{\infty}\beta^3K[A_{2n}\sinh\beta x]\sin\beta y \tag{13e}$$

$$Q_y = D_d\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta(\alpha^2+\beta^2)W_{mn}\cos\alpha x\cos\beta y - 2D_d\sum_{n=1}^{\infty}\beta^3K[A_{2n}\cosh\beta x]\cos\beta y \tag{13f}$$

$$V_x = Q_x - D_d(1-\nu)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha\beta^2W_{mn}\sin\alpha x\sin\beta y$$
$$+ D_d(1-\nu)\sum_{n=1}^{\infty}\beta^3K[A_{1n}\sinh\beta x + A_{2n}\{\beta x\cosh\beta x+\sinh\beta x\}]\sin\beta y \qquad (13g)$$

$$V_y = Q_y + D_d(1-\nu)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha^2\beta W_{mn}\cos\alpha x\cos\beta y$$
$$- D_d(1-\nu)\sum_{n=1}^{\infty}\beta^3K[A_{1n}\cosh\beta x + A_{2n}\{\beta x\sinh\beta x+2\cosh\beta x\}]\cos\beta y \qquad (13h)$$

There are altogether 6 constants of intergration involved in the bending analysis of the vertical and horizontal plates.

Rectangular Plates in Plane Stress

Vertical Plate

The homogeneous differential equation governing the membrane action of the elastic plate [8] is

$$\nabla^4\phi = 0 \qquad (14)$$

where ϕ is an Airy's stress function related to the stress resultants by

$$N_x = \frac{\partial^2\phi}{\partial y^2} \qquad (15a)$$

$$N_y = \frac{\partial^2\phi}{\partial x^2} \qquad (15b)$$

$$N_{xy} = N_{yx} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad (15c)$$

in which N_x, N_y and N_{xy} are the transverse normal force, the longitudinal normal force and the membrane shearing force, per unit length, respectively.

The general solution of Eq.(14) which satisfies the boundary conditions at the transverse edges y = 0 and y = b can be taken in the form

$$\phi = \sum_{n=1}^{\infty}\xi_n(x)\sin\beta y \qquad (16)$$

Substituting Eq.(16) into Eq.(14) and solving the resulting differential equation lead to

$$\xi_n(x) = L(B_{5n}\sinh\beta x + B_{6n}\beta x\cosh\beta x + B_{7n}\cosh\beta x + B_{8n}\beta x\sinh\beta x) \quad (17)$$

where L is another load factor which nondimensionalizes the constants of integration. B_{5n} to B_{8n}.

The complete solution is obtained by substituting Eq.(17) into Eq.(16) which yields

$$\phi = \sum_{n=1}^{\infty} L(B_{5n}\sinh\beta x + B_{6n}\beta x\cosh\beta x + B_{7n}\cosh\beta x + B_{8n}\beta x\sinh\beta x)\sin\beta y \quad (18)$$

The membrane stress resultants can be determined by substituting Eq.(18) into Eqs.(15a) to (15c), giving

$$N_x = -\sum_{n=1}^{\infty}\beta^2 L[B_{5n}\sinh\beta x + B_{6n}\beta x\cosh\beta x + B_{7n}\cosh\beta x + B_{8n}\beta x\sinh\beta x]\sin\beta y \quad (19a)$$

$$N_y = \sum_{n=1}^{\infty}\beta^2 L[B_{5n}\sinh\beta x + B_{6n}\{\beta x\cosh\beta x+2\sinh\beta x\} + B_{7n}\cosh\beta x + B_{8n}\{\beta x\sinh\beta x+2\cosh\beta x\}]\sin\beta y \quad (19b)$$

$$N_{xy} = N_{yx} = -\sum_{n=1}^{\infty}\beta^2 L[B_{5n}\cosh\beta x + B_{6n}\{\beta x\sinh\beta x+\cosh\beta x\} + B_{7n}\sinh\beta x + B_{8n}\{\beta x\cosh\beta x+\sinh\beta x\}]\cos\beta y \quad (19c)$$

The relationship between the transverse and longitudinal displacements of the middle plane and the membrane stress resultants are

$$\frac{\partial u}{\partial x} = \frac{1}{Et_w}\left[N_x - \nu N_y\right] \quad (20a)$$

$$\frac{\partial v}{\partial y} = \frac{1}{Et_w}\left[N_y - \nu N_x\right] \quad (20b)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+\nu)}{Et_w} N_{xy} \tag{20c}$$

Substituting Eqs.(19a) and (19b) into Eqs.(20a) and (20b) and solving for u and v yield

$$u = -\frac{1}{Et_w}\sum_{n=1}^{\infty} \beta L[B_{5n}(1+\nu)\cosh\beta x + B_{6n}\{(1+\nu)\beta x\sinh\beta x - (1-\nu)\cosh\beta x\}$$
$$+ B_{7n}(1+\nu)\sinh\beta x + B_{8n}\{(1+\nu)\beta x\cosh\beta x - (1-\nu)\sinh\beta x\}]\sin\beta y$$
$$+ \frac{1}{Et_w} R(y) \tag{21a}$$

$$v = -\frac{1}{Et_w}\sum_{n=1}^{\infty} \beta L[B_{5n}(1+\nu)\sinh\beta x + B_{6n}\{(1+\nu)\beta x\cosh\beta x + 2\sinh\beta x\}$$
$$+ B_{7n}(1+\nu)\cosh\beta x + B_{8n}\{(1+\nu)\beta x\sinh\beta x + 2\cosh\beta x\}]\cos\beta y$$
$$+ \frac{1}{Et_w} S(x) \tag{21b}$$

where $\frac{1}{Et_w} R(y)$ and $\frac{1}{Et_w} S(x)$ are constants of integration. These can expand into single series of the form

$$R(y) = \sum_{n=1}^{\infty} R_n \sin\beta y \tag{22a}$$

$$S(x) = \frac{S_0}{2} + \sum_{m=1}^{\infty} S_m \cos\gamma x \tag{22b}$$

where $\gamma = m\pi/c$ and

$$R_n = \frac{2}{b}\int_0^b R(y)\sin\beta y\,dy \tag{23a}$$

$$S_m = \frac{2}{c}\int_0^c S(x)\cos\gamma x\,dx \tag{23b}$$

Substituting Eqs.(19c), (21a) and (21b) into Eq.(20c), multiplying both sides of the resulting equation by cosβy and integrating with respect to y from zero to b, taking the advantage of the orthogonality condition, lead to

$$R_n = 0 \qquad \text{for all values of } n \tag{24a}$$

Repeating the same procedure, but multiplying both sides by $\sin\gamma x$ and integrating with respect to x from zero to c yields

$$S_m = 0 \qquad \text{for all values of } m \tag{24b}$$

consequently Eqs.(21a) and (21b) become

$$u = -\frac{1}{Et_w}\sum_{n=1}^{\infty}\beta L[B_{5n}(1+\nu)\cosh\beta x + B_{6n}\{(1+\nu)\beta x\sinh\beta x-(1-\nu)\cosh\beta x\}$$
$$+ B_{7n}(1+\nu)\sinh\beta x + B_{8n}\{(1+\nu)\beta x\cosh\beta x-(1-\nu)\sinh\beta x\}]\sin\beta y \tag{25a}$$

$$v = -\frac{1}{Et_w}\sum_{n=1}^{\infty}\beta L[B_{5n}(1+\nu)\sinh\beta x + B_{6n}\{(1+\nu)\beta x\cosh\beta x+2\sinh\beta x\}$$
$$+ B_{7n}(1+\nu)\cosh\beta x + B_{8n}\{(1+\nu)\beta x\sinh\beta x+2\cosh\beta x\}]\cos\beta y \tag{25b}$$

in which u an v are the displacement components in the x- and y-directions respectively.

Horizontal Plate

For the horizontal plate, due to symmetry, the stress function, the membrane stress resultants and the displacement components simplify into the forms

$$\phi = \sum_{n=1}^{\infty} L(A_{3n}\cosh\beta x + A_{4n}\beta x\sinh\beta x)\sin\beta y \tag{26a}$$

$$N_x = -\sum_{n=1}^{\infty}\beta^2 L[A_{3n}\cosh\beta x + A_{4n}\beta x\sinh\beta x]\sin\beta y \tag{26b}$$

$$N_y = \sum_{n=1}^{\infty} \beta^2 L[A_{3n} \cosh\beta x + A_{4n}\{\beta x \sinh\beta x + 2\cosh\beta x\}] \sin\beta y \tag{26c}$$

$$N_{xy} = N_{yx} = -\sum_{n=1}^{\infty} \beta^2 L[A_{3n} \sinh\beta x + A_{4n}\{\beta x \cosh\beta x + \sinh\beta x\}] \cos\beta y \tag{26d}$$

$$u = -\frac{1}{Et_d} \sum_{n=1}^{\infty} \beta L[A_{3n}(1+\nu)\sinh\beta x + A_{4n}\{(1+\nu)\beta x \cosh\beta x - (1-\nu)\sinh\beta x\}] \sin\beta y \tag{26e}$$

$$v = -\frac{1}{Et_d} \sum_{n=1}^{\infty} \beta L[A_{3n}(1+\nu)\cosh\beta x + A_{4n}\{(1+\nu)\beta x \sinh\beta x + 2\cosh\beta x\}] \cos\beta y \tag{26f}$$

in which L is a load factor and A_{3n} and A_{4n} are the constants of integration.

As in the bending analysis, six constants of integration are involved in the membrane analysis of the vertical and horizontal plates.

Applied Load and Load Factors

In case of a uniformly distributed load, q, applied on the horizontal plate, the Fourier coefficient Z_{mn} defined by Eqs.(9a) and (9b) are given respectively by

$$Z_{mn} = 0 \qquad \text{for } m \neq 0 \text{ or even } n \tag{27a}$$

$$Z_{on} = \frac{4q}{n\pi} \qquad \text{for } m = 0 \text{ and odd } n \tag{27b}$$

The corresponding load factors, K and L, are

$$K = \frac{b^4 q}{D_d} \tag{28a}$$

$$L = b^3 q \tag{28b}$$

in which K and L are the load factors in bending and membrane analysis respectively.

Boundary Conditions

The twelve arbritary constants, A_{1n} to A_{4n} for the horizontal plate and B_{1n} to B_{8n} for the vertical plate, involved in the general solutions, Eqs.(4a), (13a), (18) and (26a), are determined from the following 12 boundary conditions. The positive directions of the stress resultants and displacements are shown in Fig.4 to facilitate the derivation of these equations.

At the joint $x_d = a$ and $x_w = -c$,

$$(N_{xd})_{x_d=a} + (V_{xw})_{x_w=-c} = 0 \tag{29a}$$

$$(M_{xd})_{x_d=a} - (M_{xw})_{x_w=-c} = 0 \tag{29b}$$

$$(V_{xd})_{x_d=a} - (N_{xw})_{x_w=-c} = 0 \tag{29c}$$

$$(N_{xyd})_{x_d=a} - (N_{xyw})_{x_w=-c} = 0 \tag{29d}$$

$$(w_d)_{x_d=a} = (u_w)_{x_w=-c} \tag{29e}$$

$$\left[\frac{\partial w_d}{\partial x_d}\right]_{x_d=a} = \left[\frac{\partial w_w}{\partial x_w}\right]_{x_w=-c} \tag{29f}$$

$$(u_d)_{x_d=a} = -(w_w)_{x_w=-c} \tag{29g}$$

$$(v_d)_{x_d=a} = (v_w)_{x_w=-c} \tag{29h}$$

At the free edge $x_w = c$,

$$(N_{xw})_{x_w=c} = 0 \tag{30a}$$

$$(N_{xyw})_{x_w=c} = 0 \tag{30b}$$

$$(M_{xw})_{x_w=c} = 0 \tag{30c}$$

$$(V_{xw})_{x_w=c} = 0 \tag{30d}$$

The subscripts d and w are used for the horizontal and vertical plates respectively.

It is evident that the boundary conditions at $y = 0$ and $y = b$,

$$(N_{yd})_{y=o,b} = (M_{yd})_{y=o,b} = (u_d)_{y=o,b} = (w_d)_{y=o,b} = 0 \tag{31a}$$

$$(N_{yw})_{y=o,b} = (M_{yw})_{y=o,b} = (u_w)_{y=o,b} = (w_w)_{y=o,b} = 0 \tag{31b}$$

and the boundary conditions on the plane of symmetry at the mid-width of the horizontal plate,

$$(N_{xyd})_{x_d=o} = (V_{xd})_{x_d=o} = (u_d)_{x_d=o} = \left[\frac{\partial w_d}{\partial x_d}\right]_{x_d=o} = 0 \tag{32}$$

are satisfied by the general solutions.

Solution

Substituting the expressions for the stress resultants and displacements, Eqs.(4), (13), (19), (25) and (26), from the bending and membrane analysis into the 12 boundary conditions, Eqs.(29) and (30), and simplifying lead to a set of 12 simultaneous equations for each harmonic n,

$$\begin{aligned}
&L[A_{3n}\cosh\beta a + A_{4n}\beta a\sinh\beta a] \\
&-D_w\beta K[B_{1n}(1-\nu)\cosh\beta c + B_{2n}\{(1-\nu)\beta c\sinh\beta c-(1+\nu)\cosh\beta c\} \\
&- B_{3n}(1-\nu)\sinh\beta c - B_{4n}\{(1-\nu)\beta c\cosh\beta c-(1+\nu)\sinh\beta c\}] = 0
\end{aligned} \tag{33a}$$

$$D_d[A_{1n}(1-\nu)\cosh\beta a + A_{2n}\{(1-\nu)\beta a\sinh\beta a+2\cosh\beta a\}]$$
$$+ D_w[B_{1n}(1-\nu)\sinh\beta c + B_{2n}\{(1-\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$- B_{3n}(1-\nu)\cosh\beta c - B_{4n}\{(1-\nu)\beta c\sinh\beta c+2\cosh\beta c\}]$$
$$= \frac{D_d}{K\beta^2}\sum_{m=0}^{\infty}(\alpha^2+\nu\beta^2)W_{mn}\cos m\pi \tag{33b}$$

$$D_d\beta K[A_{1n}(1-\nu)\sinh\beta a + A_{2n}\{(1-\nu)\beta a\cosh\beta a-(1+\nu)\sinh\beta a\}]$$
$$- L[B_{5n}\sinh\beta c + B_{6n}\beta c\cosh\beta c - B_{7n}\cosh\beta c - B_{8n}\beta c\sinh\beta c]$$
$$= \frac{D_d}{\beta^2}\sum_{m=0}^{\infty}\alpha\{\alpha^2+(2-\nu)\beta^2\}W_{mn}\sin m\pi \tag{33c}$$

$$A_{3n}\sinh\beta a + A_{4n}\{\beta a\cosh\beta a+\sinh\beta a\} - B_{5n}\cosh\beta c - B_{6n}\{\beta c\sinh\beta c+\cosh\beta c\}$$
$$+ B_{7n}\sinh\beta c + B_{8n}\{\beta c\cosh\beta c+\sinh\beta c\} = 0 \tag{33d}$$

$$K[A_{1n}\cosh\beta a + A_{2n}\beta a\sinh\beta a]$$
$$+ \frac{\beta L}{Et_w}[B_{5n}(1+\nu)\cosh\beta c + B_{6n}\{(1+\nu)\beta c\sinh\beta c-(1-\nu)\cosh\beta c\}$$
$$- B_{7n}(1+\nu)\sinh\beta c - B_{8n}\{(1+\nu)\beta c\cosh\beta c-(1-\nu)\sinh\beta c\}]$$
$$= -\sum_{m=0}^{\infty}W_{mn}\cos m\pi \tag{33e}$$

$$A_{1n}\sinh\beta a + A_{2n}\{\beta a\cosh\beta a+\sinh\beta a\} - B_{1n}\cosh\beta c - B_{2n}\{\beta c\sinh\beta c+\cosh\beta c\}$$
$$+ B_{3n}\sinh\beta c + B_{4n}\{\beta c\cosh\beta c+\sinh\beta c\} = \frac{1}{\beta K}\sum_{m=0}^{\infty}\alpha W_{mn}\sin m\pi \tag{33f}$$

$$\frac{\beta L}{Et_d}[A_{3n}(1+\nu)\sinh\beta a + A_{4n}\{(1+\nu)\beta a\cosh\beta a-(1-\nu)\sinh\beta a\}]$$
$$+ K[B_{1n}\sinh\beta c + B_{2n}\beta c\cosh\beta c - B_{3n}\cosh\beta c - B_{4n}\beta c\sinh\beta c] = 0 \tag{33g}$$

$$\frac{1}{Et_d}[A_{3n}(1+\nu)\cosh\beta a + A_{4n}\{(1+\nu)\beta a\sinh\beta a+2\cosh\beta a\}]$$
$$+ \frac{1}{Et_w}[B_{5n}(1+\nu)\sinh\beta c + B_{6n}\{(1+\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$- B_{7n}(1+\nu)\cosh\beta c - B_{8n}\{(1+\nu)\beta c\sinh\beta c+2\cosh\beta c\}] = 0 \tag{33h}$$

$$B_{5n}\sinh\beta c + B_{6n}\beta c\cosh\beta c + B_{7n}\cosh\beta c + B_{8n}\beta c\sinh\beta c = 0 \tag{33i}$$

$$B_{5n}\cosh\beta c+B_{6n}\{\beta c\sinh\beta c+\cosh\beta c\}+B_{7n}\sinh\beta c+B_{8n}\{\beta c\cosh\beta c+\sinh\beta c\} = 0 \quad (33j)$$

$$B_{1n}(1-\nu)\sinh\beta c + B_{2n}\{(1-\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$+ B_{3n}(1-\nu)\cosh\beta c + B_{4n}\{(1-\nu)\beta c\sinh\beta c+2\cosh\beta c\} = 0 \quad (33k)$$

$$B_{1n}(1-\nu)\cosh\beta c + B_{2n}\{(1-\nu)\beta c\sinh\beta c-(1+\nu)\cosh\beta c\}$$
$$+ B_{3n}(1-\nu)\sinh\beta c + B_{4n}\{(1-\nu)\beta c\cosh\beta c-(1+\nu)\sinh\beta c\} = 0 \quad (33l)$$

in which $n = 1,2,3,...$

Finally, substituting the Fourier coefficients Z_{mn} and Z_{on} from Eqs.(27) and the load factors K and L from Eqs.(28) into these simultaneous equations and converting into dimensionless forms,

$$A_{3n}\cosh\beta a + A_{4n}\beta a\sinh\beta a - B_{1n}\frac{1}{R_2^3}n\pi(1-\nu)\cosh\beta c$$
$$- B_{2n}\frac{1}{R_2^3}n\pi\{(1-\nu)\beta c\sinh\beta c-(1+\nu)\cosh\beta c\} + B_{3n}\frac{1}{R_2^3}n\pi(1-\nu)\sinh\beta c$$
$$+ B_{4n}\frac{1}{R_2^3}n\pi\{(1-\nu)\beta c\cosh\beta c-(1+\nu)\sinh\beta c\} = 0 \quad (34a)$$

$$A_{1n}(1-\nu)\cosh\beta a + A_{2n}\{(1-\nu)\beta a\sinh\beta a+2\cosh\beta a\}$$
$$+ B_{1n}\frac{1}{R_2^3}(1-\nu)\sinh\beta c + B_{2n}\frac{1}{R_2^3}\{(1-\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$- B_{3n}\frac{1}{R_2^3}(1-\nu)\cosh\beta c - B_{4n}\frac{1}{R_2^3}\{(1-\nu)\beta c\sinh\beta c+2\cosh\beta c\} = \frac{4\nu}{n^5\pi^5} \quad (34b)$$

$$A_{1n}\,n\pi(1-\nu)\sinh\beta a + A_{2n}\,n\pi\{(1-\nu)\beta a\cosh\beta a-(1+\nu)\sinh\beta a\}$$
$$- B_{5n}\sinh\beta c - B_{6n}\beta c\cosh\beta c + B_{7n}\cosh\beta c + B_{8n}\beta c\cosh\beta c = 0 \quad (34c)$$

$$A_{3n}\sinh\beta a + A_{4n}\{\beta a\cosh\beta a+\sinh\beta a\} - B_{5n}\cosh\beta c - B_{6n}\{\beta c\sinh\beta c+\cosh\beta c\}$$
$$+ B_{7n}\sinh\beta c + B_{8n}\{\beta c\cosh\beta c+\sinh\beta c\} = 0 \quad (34d)$$

$$A_{1n}\cosh\beta a + A_{2n}\beta a\sinh\beta a + B_{5n}R_1^2R_2R_4\cosh\beta c$$
$$+ B_{6n}R_1^2R_2\{R_4\beta c\sinh\beta c-R_3\cosh\beta c\} - B_{7n}R_1^2R_2R_4\sinh\beta c$$
$$- B_{8n}R_1^2R_2\{R_4\beta c\cosh\beta c-R_3\sinh\beta c\} = -\frac{4}{n^5\pi^5} \tag{32e}$$

$$A_{1n}\sinh\beta a + A_{2n}\{\beta a\cosh\beta a+\sinh\beta a\} - B_{1n}\cosh\beta c - B_{2n}\{\beta c\sinh\beta c+\cosh\beta c\}$$
$$+ B_{3n}\sinh\beta c + B_{4n}\{\beta c\cosh\beta c+\sinh\beta c\} = 0 \tag{32f}$$

$$A_{3n}R_1^2R_4\sinh\beta a + A_{4n}R_1^2\{R_4\beta a\cosh\beta a-R_3\sinh\beta a\}$$
$$+ B_{1n}\sinh\beta c + B_{2n}\beta c\cosh\beta c - B_{3n}\cosh\beta c - B_{4n}\beta c\sinh\beta c = 0 \tag{32g}$$

$$A_{3n}(1+\nu)\cosh\beta a + A_{4n}\{(1+\nu)\beta a\sinh\beta a+2\cosh\beta a\}$$
$$+ B_{5n}R_2(1+\nu)\sinh\beta c + B_{6n}R_2\{(1+\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$- B_{7n}R_2(1+\nu)\cosh\beta c - B_{8n}R_2\{(1+\nu)\beta c\sinh\beta c+2\cosh\beta c\} = 0 \tag{32h}$$

$$B_{5n}\sinh\beta c + B_{6n}\beta c\cosh\beta c + B_{7n}\cosh\beta c + B_{8n}\beta c\sinh\beta c = 0 \tag{32i}$$

$$B_{5n}\cosh\beta c+B_{6n}\{\beta c\sinh\beta c+\cosh\beta c\}+B_{7n}\sinh\beta c+B_{8n}\{\beta c\cosh\beta c+\sinh\beta c\} = 0 \tag{32j}$$

$$B_{1n}(1-\nu)\sinh\beta c + B_{2n}\{(1-\nu)\beta c\cosh\beta c+2\sinh\beta c\}$$
$$+ B_{3n}(1+\nu)\cosh\beta c + B_{4n}\{(1-\nu)\beta c\sinh\beta c+2\cosh\beta c\} = 0 \tag{32k}$$

$$B_{1n}(1-\nu)\cosh\beta c + B_{2n}\{(1-\nu)\beta c\sinh\beta c-(1+\nu)\cosh\beta c\}$$
$$+ B_{3n}(1-\nu)\sinh\beta c + B_{4n}\{(1-\nu)\beta c\cosh\beta c-(1+\nu)\sinh\beta c\} = 0 \tag{32l}$$

in which $\beta a = n\pi\frac{a}{b}$, $\beta c = n\pi\frac{c}{b}$, $R_1 = \frac{t_d}{b}$, $R_2 = \frac{t_d}{t_w}$, $R_3 = \frac{n\pi}{12(1+\nu)}$, $R_4 = \frac{n\pi}{12(1-\nu)}$ and $n = 1,3,5,\ldots$

The dimensionless parameters involve in these 12 simultaneous equations to be solved for the constants of intergration are ν, $\frac{a}{b}$, $\frac{c}{b}$, $\frac{t_d}{b}$ and $\frac{t_d}{t_w}$.

Once the constants of integration are determined, the corresponding stress resultants and displacements at any point (x,y) can then be obtained by means of the appropriate equations.