CHAPTER 5

VILLE THEOREM ON CONTINUOUS GAME

5.1 Introduction

In this chapter, Ville theorem is extended to continuous two-person zero sum game

Let B_1 and B_2 be Banach spaces and let a set P in B_1 and a set Q in B_2 be sets of distribution functions on $[0,1]$.

Suppose that the payoff function for a continuous game is M which is a continuous function on $[0,1] \times [0,1]$ and suppose that player I chooses x from $[0,1]$ by means of the distribution function $p \in P$ and that player II chooses y from $[0,1]$ by means of the distribution function $q \in Q$

Let A be the mapping from P into B_2^* defined by

$$Ap = \psi$$

where $$\psi(y) = \int_0^1 M(x,y)dp(x) \qquad (y \in [0,1],\ p \in P).$$

Then the total expectation of player I will be the bilinear form

$$(Ap, q) = \int_0^1 \int_0^1 M(x,y)dp(x)dq(y) \qquad \text{for } q \in Q.$$

It is assumed that to each $q \in Q$ there exists a $p \in P$ such that $(Ap,q) \geq 0$. 3.3.7 asserts the existence of a $p_0 \in P$ for which $(Ap_0,q) \geq 0$ for all $q \in Q$ if

(i) The image of P under A is weak*-compact and reqularly convex.

(ii) The cone $\bigcup_{\lambda \geqslant 0} \lambda Q$ is convex and closed .

5.2.2 Remark. Let B be a topological vector space. If E is weak*-open in $B_2^* \cap C[0,1]$, then E is open in $C[0,1]$

Proof : For any $f_0 \in E$, there exists $r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n$ such that

$$f_0 \in \left\{ f \in C[0,1] : |f(x_i) - f_0(x_i)| < r_i, \quad i = 1,2,\ldots, n \right\} \subset E$$

Let $r = \min \left\{ r_1, r_2, \ldots, r_n \right\}$.

Then $f_0 \in \left\{ f \in C[0,1] : |f(x) - f_0(x)| < r, \ \forall x \in [0,1] \right\} \subset E$.

So E is open in $C[0,1]$.

5.2.3 Remark. If $K \subset B^* \cap C[0,1]$ and K is compact in $C[0,1]$, then K is weak*-compact in B^*

Proof : Let $\{V_\lambda\}$ be an open covering of K where V_λ is weak*-open in B^*, i.e., $K \subset \bigcup_\lambda V_\lambda$. Then $K \subset \bigcup_\lambda (V_\lambda \cap C[0,1])$ where $V_\lambda \cap C[0,1]$ is weak*-open in $B^* \cap C[0,1]$. By 5.2.2, $V_\lambda \cap C[0,1]$ is open in $C[0,1]$. Since K is compact in $C[0,1]$, $K \subset \bigcup_{i=1}^{n} (V_{\lambda_i} \cap C[0,1])$. Hence $K \subset \bigcup_{i=1}^{n} V_{\lambda_i}$, that is K is weak*-compact in B^*

5.2.4 Theorem. Let B_1 and B_2 be the spacesof bounded Borel measures on $[0,1]$. And let

$$P = \left\{ \mu \in B_1 : \mu(E) \geq 0 \text{ for all } E \in \text{Borel algebra } M \text{ and } \mu([0,1]) = 1 \right\},$$

$$Q = \left\{ \sigma \in B_2 : \sigma(E) \geq 0 \text{ for all } E \in \text{Borel algebra } M \text{ and } \sigma([0,1]) = 1 \right\}.$$

Let $K(x,y)$ be a continuous function on $[0,1] \times [0,1]$ and A be a linear mapping from P into B_2^* defined

by $\quad A\mu = \psi$ where

$$\psi(y) = \int_0^1 K(x,y)\,d\mu(x) \qquad (y \in [0,1], \mu \in P).$$

If to each $\sigma \in Q$ there exists corresponding $\mu \in P$ such that

$\int_0^1 \int_0^1 K(x,y)\,d\mu\, d\sigma \geqslant 0$, then there exists a fixed $\mu_0 \in P$ such that

$$\int_0^1 \int_0^1 K(x,y)\,d\mu_0\, d\sigma \geqslant 0 \text{ for all } \sigma \in Q.$$

Proof : First, we have to show that $A(P)$ is weak*-compact regularly convex . Since K is continuous on a compact set $[0,1] \times [0,1]$, K is uniformly continuous and there exists a real number m such that

$$|K(x,y)| \leq m \qquad \text{for } x \in [0,1], \; y \in [0,1].$$

Hence $\left|A\mu(y)\right| = \left|\int_0^1 K(x,y)\,d\mu(x)\right| \leq \int_0^1 |K(x,y)|\,d\mu(x) \leq m$

for $y \in [0,1]$ and $\mu \in P$. Thus $A(P)$ is uniformly bounded.

Given $\varepsilon > 0$, by uniform continuity of K, there exists $\delta > 0$ such that

$$|K(x,y_1) - K(x,y_2)| < \varepsilon \qquad \text{whenever } |y_1 - y_2| < \delta \text{, and}$$

for $x \in [0,1]$, $y_1, y_2 \in [0,1]$.

So $$\left| A\mu(y_1) - A\mu(y_2) \right| = \left| \int_0^1 (K(x,y_1)d\mu(x) - \int_0^1 K(x,y_2)d\mu(x) \right|$$

$$\leq \int_0^1 \left| K(x,y_1) - K(x,y_2) \right| d\mu(x)$$

$$< \varepsilon \qquad \text{whenever } |y_1 - y_2| < \delta ,$$

and for all $\mu \in P$.

Thus A(P) is equicontinuous. Since by 2.5.19 $A(P) \subset C[0,1]$, by Ascoli theorem, A(P) is compact. Hence, by 5.2.3, A(P) is weak*-compact in B_2^*.

Let μ_1, μ_2 be any elements in P. For $0 \leq t \leq 1$,

$[t\mu_1 + (1-t)\mu_2](E) = t\mu_1(E) + (1-t)\mu_2(E)$ which is non-negative, since $\mu_1(E)$ and $\mu_2(E)$ are non-negative.

And $[t\mu_1+(1-t)\mu_2]([0,1]) = t\mu_1([0,1])+(1-t)\mu_2([0,1]) = 1.$

Thus P is convex

Consider $A\mu_1 \in A(P)$ and $A\mu_2 \in A(P)$, for $0 \leq t \leq 1$

$$tA\mu_1(y)+(1-t)A\mu_2(y) = t\int_0^1 K(x,y)d\mu_1(x)+(1-t)\int_0^1 K(x,y)d\mu_2(x) \quad \text{for } y \in [0,1].$$

$$= \int_0^1 K(x,y)d(t\mu_1+(1-t)\mu_2)(x) \text{ for } y \in [0,1]$$

By convexity of P, A(P) is convex. So, by 3.3.2 A(P) is weak*-compact regularly convex.

Let $X = \bigcup_{\lambda \geq 0} \lambda Q$. Since Q is convex, X is a convex cone. And since Q is closed and $0 \notin Q$, by 5.1.1, X is closed in B_2.

From the hypothesis to each $\sigma \in Q$, there exists $\mu \in P$ such that $\int_0^1\int_0^1 K(x,y)d\mu(x)d\sigma(y) \geqslant 0$. Therefore to each $\lambda\sigma \in X$, there exists $A\mu \in A(P)$ such that $\lambda\int_0^1\int_0^1 K(x,y)d\mu(x)d\sigma(y) \geqslant 0$ and hence $\int_0^1\int_0^1 K(x,y)d\mu(x)d(\lambda\sigma)(y) \geqslant 0$.

By 3.3.7, there exists $A\mu_0 \in A(P)$ such that

$$\int_0^1\int_0^1 K(x,y)d\mu_0(x)d(\lambda\sigma)(y) \geqslant 0 \quad \text{for all} \quad \sigma \in Q,$$

$$\lambda\int_0^1\int_0^1 K(x,y)d\mu_0(x)d\sigma(y) \geqslant 0 \quad \text{for all} \quad \sigma \in Q,$$

i.e., there exists $\mu_0 \in P$ such that

$$\int_0^1\int_0^1 K(x,y)d\mu_0(x)d\sigma(y) \geqslant 0 \quad \text{for all} \quad \sigma \in Q.$$

This completes the proof.

5.2.5 Theorem. Let B_1 and B_2 be the spaces of functions of bounded variation on $[0,1]$. Let P and Q, in B_1 and B_2 respectively, be the sets of distribution functions, i.e., the sets of non-decreasing functions of total variations 1. Let K(x,y) be a continuous function on $[0,1]\times[0,1]$ and A a linear mapping from P into B_2^* defined by

$$Ap = \psi$$

where $$\psi(y) = \int_0^1 K(x,y)dp(x) \qquad (y \in [0,1], \quad p \in P)$$

If to each $q \in Q$ there exists $p \in P$ such that

5.2 The Ville theorem

In this section we shall state and prove the Ville theorem on continuous two-person zero sum game which B_1 and B_2 mentioned in section 5.1 are as follows :

(i) spaces of bounded Borel measures on $[0,1]$,

(ii) spaces of bounded variation functions on $[0,1]$,

(iii) space of bounded variation functions on $[0,1]$, and space of bounded Borel measures on $[0,1]$, respectively.

The proof needs the following remarks.

5.2.1 Remark. If Q is a closed subset of B^+ (set of positive measure) and $0 \notin Q$, then $A = \bigcup_{c \geq 0} cQ$ is closed subset in B

Proof : Let $\{\alpha_i\}$ be sequence of point in A such that $\alpha_i \longrightarrow \alpha$ in B. We want to show that $\alpha \in A$.

Choose $c_i \geq 0$ and $\sigma_i \in Q$ such that $\alpha_i = c_i \sigma_i$.

We claim that $\overline{\lim}\ ci < +\infty$.

Suppose $\overline{\lim}\ c_i = +\infty$. There exists subsequence c_i such that $\lim c_i = +\infty$. Then

$$0 \leq \sigma_i = c_i^{-1}(c_i \sigma_i) = c_i^{-1} \alpha_i ,$$

which would entail that $\lim_i \sigma_i = 0$ and so, since Q is closed, $0 \in Q$. This would contradict our hypothesis. So $\overline{\lim}\ c_i < +\infty$. In that case there exists subsequence c_i which converges to c, say, and at the same time subsequence σ_i converging to σ. Then $c_i \sigma_i$ is convergent, and its limit, $c\sigma$, belongs to cQ. Hence $c\sigma \in A$, i.e., α_i converges to $\alpha = c\sigma$ and $\alpha \in A$.

$$\int_0^1 \int_0^1 K(x,y)dp(x)dq(y) \geqslant 0,$$ then there exists $p_0 \in P$

such that $$\int_0^1 \int_0^1 K(x,y)dp_0(x)dq(y) \geqslant 0$$ for all $q \in Q$.

Proof : Since K is continuous function on a compact set $[0,1] \times [0,1]$, K is uniformly continuous and there exists a real number m such that

$$|K(x,y)| \leq m \quad \text{for} \quad x \in [0,1], \quad y \in [0,1].$$ Hence

$$|Ap(y)| = \left| \int_0^1 K(x,y)dp(x) \right| \leq \int_0^1 |K(x,y)| \, dp(x) \leq m \, V_0^1(P) = m$$

for $p \in P$. Thus $A(P)$ is uniformly bounded.

Given $\varepsilon > 0$, by uniform continuity of K, there exists $\delta > 0$ such that for all $x \in [0,1]$

$$|K(x,y_1) - K(x,y_2)| < \varepsilon \qquad \text{whenever} \quad |y_1 - y_2| < \delta .$$

So $$|Ap(y_1) - Ap(y_2)| = \left| \int_0^1 K(x,y_1)dp(x) - \int_0^1 K(x,y_2)dp(x) \right|$$

$$\leq \int_0^1 |K(x,y_1) - K(x,y_2)| \, dp(x)$$

$$< \varepsilon \qquad \text{whenever} \quad |y_1 - y_2| < \delta .$$

Thus $A(P)$ is equicontinuous. By Ascoli theorem $A(P)$ is compact in $C[0,1]$ and hence, by 5.2.3, $A(P)$ is weak*-compact in B_2^*. Since $A(P)$ is convex, by 3.3.2, $A(P)$ is regularly convex.

Let $X = \bigcup_{\lambda \geqslant 0} \lambda Q$. Then X is closed convex cone. From the hypothesis, to each $q \in Q$ there exists $p \in P$ such that

$$\int_0^1 \int_0^1 K(x,y)dp(x)dq(y) \geqslant 0.$$ Therefore to each $\lambda q \in X$ there exists $p \in P$ such that

$$\int_0^1\int_0^1 K(x,y)dp(x)d\lambda q(y) = \lambda\int_0^1\int_0^1 K(x,y)dp(x)dq(y) \geqslant 0.$$

By 3.3.7, there exists $p_0 \in P$ such that

$$\int_0^1\int_0^1 K(x,y)dp_0(x)d(\lambda q)(y) \geqslant 0 \qquad \text{for all} \quad q \in Q,$$

$$\lambda\int_0^1\int_0^1 K(x,y)dp_0(x)dq(y) \geqslant 0 \qquad \text{for all} \quad q \in Q,$$

i.e., there exists $p_0 \in P$ such that

$$\int_0^1\int_0^1 K(x,y)dp_0(x)dq(y) \geqslant 0 \qquad \text{for all} \quad q \in Q.$$

The proof is complete.

5.2.6 <u>Theorem.</u> Let B_1 be a space of functions of bounded variation on $[0,1]$ and B_2 a space of bounded Borel measures. Let P in B_1 be the set of non-decreasing function of total variation 1 and Q in B_2 a set of positive measure such that $\mu([0,1]) = 1$. And let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$, and A be a linear mapping from P into B_2^* defined by $Ap = \psi$

where $\psi(y) = \int_0^1 K(x,y)dp(x) \qquad (y \in [0,1],\ p \in P)$

If to each $\mu \in Q$ there exists $p \in P$ such that $\int_0^1\int_0^1 K(x,y)dp(x)d\mu(y) \geqslant 0$, then there exists $p_0 \in P$ such

that $\int_0^1\int_0^1 K(x,y)dp_0(x)d\mu(y) \qquad$ for all $\mu \in Q$.

<u>Proof</u> : Since K is continuous function on a compact set $[0,1]\times[0,1]$, K is uniformly continuous and there exists a real number m such that

$$|K(x,y)| \leq m \quad \text{for } x \in [0,1],\ y \in [0,1] .$$ Hence

$$|Ap(y)| = \left|\int_0^1 K(x,y)dp(x)\right| \leq \int_0^1 |K(x,y)|\, dp(x) \leq m \overset{1}{\underset{0}{V}}(P) = m \text{ for } p \in P.$$

Thus A(P) is uniformly bounded.

Given $\varepsilon > 0$, by uniform continuity of K, there exists $\delta > 0$ such that for all $x \in [0,1]$, $y_1, y_2 \in [0,1]$

$$|K(x,y_1) - K(x,y_2)| < \varepsilon \qquad \text{whenever } |y_1 - y_2| < \delta$$

$$\begin{aligned} |Ap(y_1)-Ap(y_2)| &= \left|\int_0^1 K(x,y_1)dp(x) - \int_0^1 K(x,y_2)dp(x)\right| \\ &\leq \int_0^1 |K(x,y_1) - K(x,y_2)|\, dp(x) \\ &< \varepsilon \end{aligned}$$

whenever $|y_1 - y_2| < \delta$ and for $p \in P$. Thus A(P) is equicontinuous. By Ascoli theorem A(P) is compact in $C[0,1]$ and hence by 5.2.3, A(P) is weak*-compact in B_2^*. Since A(P) is convex, by 3.3.2, A(P) is regularly convex. Let $X = \bigcup_{\lambda \geq 0} \lambda Q$. Then X is closed convex cone. From the hypothesis, to each $\mu \in Q$, there exists, $p \in P$ such that $\int_0^1 \int_0^1 K(x,y)dp(x)d\mu(y) \geq 0$. Therefore to each $\lambda\mu \in X$ there exists $Ap \in A(P)$ such that

$$\lambda \int_0^1 \int_0^1 K(x,y)dp(x)d\mu(y) \geq 0 \text{ and hence}$$

$$\int_0^1 \int_0^1 K(x,y)dp(x)d(\lambda\mu)(y) \geq 0.$$ By 3.3.7, there exists

$Ap_0 \in A(P)$ such that

$$\int_0^1 \int_0^1 K(x,y)dp_0(x)d(\lambda\mu)(y) \geq 0 \quad \text{for all} \quad \mu \in Q ,$$

$$\lambda \int_0^1 \int_0^1 K(x,y)dp_0(x)d\mu(y) \geq 0 \quad \text{for all} \quad \mu \in Q ,$$

i.e., there exists $p_0 \in P$ such that

$$\int_0^1 \int_0^1 K(x,y)dp_0(x)d\mu(y) \geq 0 \quad \text{for all} \quad \mu \in Q .$$

The proof is complete.