CHAPTER III

THE HAHN BANACH THEOREM AND ITS APPLICATION

In this chapter, we study about the Hahn Banach Theorem and its corollaries, and then, discuss its applications which will be useful in the next chapters.

Some materials are drawn from references [3], [10], [12].

3.1 The Hahn Banach Theorem

3.1.1 <u>Theorem.</u> (The Hahn Banach theorem in analytic form)

Suppose

(i) M is a subspace of a real vector space X,

(ii) $p : X \longrightarrow \mathbb{R}$ satisfies

$p(x+y) \leq p(x)+p(y)$ and $p(tx) = tp(x)$ if $x \in X$, $y \in X$, $t \geqslant 0$,

(iii) $f : M \longrightarrow \mathbb{R}$ is linear and $f(x) \leq p(x)$ on M. Then there exists a linear functional

$\Lambda : X \longrightarrow \mathbb{R}$ such that $\Lambda x = f(x)$ for $x \in M$ and $-p(x) \leq \Lambda x \leq p(x)$ for $x \in X$.

<u>Proof</u> : We assume that $M \neq X$. Let $x_0 \in X$, $x_0 \notin M$, and define

$$M_1 = \{ x + tx_0 : x \in M, t \in \mathbb{R} \} .$$

It is clear that M_1 is a vector space. Since

$$f(x)+ f(y) = f(x+y) \leq p(x+y) \leq p(x-x_0) + p(x_0+y) ,$$

we have

(1) $f(x) - p(x-x_0) \leq p(y+x_0) - f(y)$ for all x, y in M.

Let $\alpha = \sup_{x \in M} \{f(x) - p(x-x_0)\}$. Then α is a real number

and $f(x) - p(x-x_0) \leq \alpha \leq p(y+x_0) - f(y)$. Therefore

(2) $f(x) - \alpha \leq p(x-x_0)$ for all $x \in M$ and

(3) $f(y) + \alpha \leq p(y-x_0)$ for all $y \in M$.

Define f_1 on M_1 by

(4) $f_1(x + tx_0) = f(x) + t\alpha$ for all $x \in M$, $t \in \mathbb{R}$.

Then $f_1 = f$ on M, and f_1 is linear on M_1. Take $t > 0$, replace x by $t^{-1}x$ in (2), replace y by $t^{-1}y$ in (3), and multiply the resulting inequalities by t. Finally we have

(5) $f(x) - t\alpha \leq p(x-tx_0)$, $f(y) + t\alpha \leq p(x+tx_0)$

Combine (4), (5) we have $f_1 \leq p$ on M_1.

Let $\mathcal{P}$ be the collection of all ordered pairs (M', f'), where M' is a subspace of X that contains M and f' is a linear functional on M'. that extends f and satisfies $f' \leq p$ on M'. Partially ordered $\mathcal{P}$ by declaring $(M', f') \leq (M'', f'')$ to mean that $M' \subset M''$ and $f'' = f'$ on M'. Let Ω be a tollaly ordered subcollection of $\mathcal{P}$. Let M^* be the union of all those M' for which $(M', f') \in \Omega$, and let f^* be defined by $f^*(x) = f'(x)$ for all $x \in M'$ and whenever $(M', f') \in \Omega$. It is now easy to check that f^* is well-defined on M^*, that f^* is linear, and that $f^* \leq p$. Note that M^* is a vector subspace of X containing M.

Thus $(M^*, f^*) \in \mathcal{P}$ is upper bound of Ω. By Zohn's lemma, $\mathcal{P}$ has a maximal element $(\widetilde{M}, \Lambda)$. If $\widetilde{M}$ is a proper subset of X, then the first part of the proof would give a further extension of Λ, and this would contradict to the maximality of Λ. Thus $\widetilde{M} = X$.

Finally the inequality $\Lambda \leq p$ implies that

$$-p(-x) \leq -\Lambda(-x) = \Lambda(x) \quad \text{for all } x \in X.$$

This completes the proof.

3.1.2 Definition. A set M of a vector space X is called a linear variety if $M = x_0 + M_0$ where x_0 is a fixed vector and M_0 is a subspace of X.

3.1.3 Definition. A set H of a vector space X is called a homogeneous hyperplane if it is a maximal proper vector subspace of X.

3.1.4 Definition. A set H of a vector space X is called a hyperplane if it is a translation of a homogeneous hyperplane.

3.1.5 Theorem. (Geometric Forms of the Hahn Banach Theorem)

(i) Let X be a real topological vector space, A an open convex set in X, M is a linear variety in X not meeting A. Then there exists a closed hyperplane H in X that contains M and that does not meet A.

(ii) Let A be an open convex set in X, L a vector subspace of X not meeting A. Then there exists a continuous linear functional Λ on X such that $\Lambda(L) = 0$, $\Lambda(A) > 0$.

Proof : (i) By translation, we may assume that M is a vector subspace of X. Let x_0 be a point in A and $C = A - x_0$ and $p = \mu_c$. Since $0 \in C$ and C is open, C is a neighborhood of 0. If $x \in C$, then by continuity of the mapping $\alpha \longmapsto \alpha x$, $\exists\, \delta > 0$ such that $\alpha x \in C$ whenever $\alpha \in (1-\delta, 1+\delta)$. Hence there exists $t > 1$ such that $tx \in C$, i.e., $x \in \frac{1}{t} C$. So $p(x) = \inf \{ t > 0 : tx \in C \} < 1$. By **2.3.4**, $p(x) = \mu_c(x) < 1$ implies $x \in C$. So we have $p(x) < 1$ if and only if $x \in C$.

Since $M \cap A = \emptyset$, $p(y - x_0) \geq 1$ for $y \in M$. Define the linear functional f on $M_1 = M + \mathbb{R} x_0$ by

$$f(y - \alpha x_0) = \alpha \qquad \text{for all } y \in M.$$

If $\alpha > 0$, we then have

$$f(y - \alpha x_0) = \alpha \leq \alpha . p\left(\frac{y}{\alpha} - x_0 \right) = p(y - \alpha x_0);$$

and if $\alpha \leq 0$

$$f(y - \alpha x_0) = \alpha \leq 0 \leq p(y - \alpha x_0).$$

Thus $f \leq p$ at all points of M_1, By 3.1.1, f can be extended into a linear functional Λ on X such that $\Lambda x \leq p(x)$ for all $x \in X$.

Take $V = C \cap (-C)$.

If $x \in V$, then $x \in C$ and $x \in -C$, hence $p(x) < 1$ and $p(-x) < 1$. Thus $\Lambda x < 1$ and $-\Lambda x = \Lambda(-x) \leq p(-x) < 1$. So we have $|\Lambda x| < 1$, i.e., Λ is bounded. By 2.2.5, Λ is continuous. Moreover $H = \mathcal{N}(\Lambda)$ is a closed hyperplane in X. Obviously $M \subset H$. Finally, $\Lambda z = 0$ for z in H and so

$$\begin{aligned} 0 = \Lambda z &= \Lambda(z - x_0) + \Lambda x_0 \\ &= \Lambda(z - x_0) + f(x_0) \\ &= \Lambda(z - x_0) - 1 \leq p(z - x_0) - 1 \end{aligned}$$

showing that $p(z - x_0) \geq 1$ for $z \in H$ which implies that $z - x_0 \notin C$, hence $z \notin A$. So we have $H \cap A = \emptyset$.

(ii) We apply (i) with L in place of M. In this case H is a homogeneous hyperplane and, being closed.

Since $H \supset L$, so $\Lambda(L) = 0$. Since $H \cap A = \emptyset$ and since A is convex, by 2.1.9 $\Lambda(A)$ must be an interval not containg 0. By changing the sign of Λ, if necessary, we can therefore arrange that $\Lambda(A) > 0$.

3.1.6 Corollary. Let X be a real topological vector space, A an open convex set in X, B a convex set in X, and suppose $A \cap B = \emptyset$. Then there exists a non-zero continuous linear functional Λ on X and a real number α such that $\Lambda(A) < \alpha$, $\Lambda(B) \geq \alpha$. (These inequalities may be reversed by changing Λ into $-\Lambda$).

Proof : The set $C = A - B$ is open and convex and does not contain 0 Hence, by 3.1.5, there exists a non-zero continuous linear functional Λ on X such that $\Lambda(c) < 0$ for all $c \in C$. So $\Lambda(a) < \Lambda(b)$ for all $a \in A$, $b \in B$.

If $\alpha = \sup\{ \Lambda a : a \in A\}$, then α is finite, and

$$\Lambda(a) \leq \alpha \leq \Lambda(b) \text{ for all } a \in A,\ b \in B.$$

Claim that $\Lambda(a) < \alpha$ for all $a \in A$. To do this, it is enough to prove that $\Lambda(A)$ is an open interval.

Suppose $\Lambda a = 0$ for all $a \in A$. Let x_0 be a fixed element in A, $A - x_0$ is a neighborhood of 0.

By 2.1.16, for any $x \in X$, there exists a positive integer n such that $x \in n(A - x_0)$. So there exists $a \in A$ such that $x = na - nx_0$, $\Lambda x = n \Lambda a - n \Lambda x_0 = 0$ which contradicts to the fact that Λ is non-zero. Thus there exists a point $a_0 \in A$ such that $\Lambda a_0 \neq 0$. Let β be any element in $\Lambda(A)$, then there exists an $a \in A$ such that $\beta = \Lambda a$.

Consider $A - a$ which is a convex neighborhood of 0. By 2.1.13, there exists a convex balanced neighborhood U of 0 such that $U \subset A - a$.

Since Λ is non-zero, there exists $a_1 \in U$ such that $\Lambda a_1 \neq 0$. Since $a_1 \in U$ implies $-a_1 \in U$, so we may assume that $\Lambda a_1 > 0$. Since U is convex, by 2.2.3 $\Lambda(U)$ is convex. So $\Lambda(U)$ is an open interval. Since $\Lambda a_1 \in \Lambda(U)$ and $-\Lambda a_1 \in \Lambda(U)$, the open interval $(-\Lambda a_1, \Lambda a_1) \subset \Lambda(U) \subset \Lambda(A-a)$.
Put $\delta = \Lambda a_1$. So we have

$$(-\delta, \delta) \subset \Lambda(A) - \Lambda a ,$$

i.e.,

$$(\Lambda a - \delta , \Lambda a + \delta) \subset \Lambda(A).$$

Thus every point of $\Lambda(A)$ is an interior point of $\Lambda(A)$. That is $\Lambda(A)$ is an open interval. This completes the proof.

3.1.7 <u>Corollary</u>. Let X be a real locally convex topological vector space, A a compact convex set in X, B a closed convex set in X, and suppose that $A \cap B = \emptyset$. Then there exists a continuous linear functional Λ and a real number α such that $\Lambda(A) < \alpha$, $\Lambda(B) > \alpha$.

Proof : By 2.1.7. There exists a convex neighborhood U of 0 in X such that $(A + U) \cap (B + U) = \emptyset$. Applying 3.1.6 to the set A + U and B + U , we deduce the existence of a non-zero continuous linear functional on X and a real number α such that

$$\Lambda (A + U) < \alpha \quad , \quad \Lambda (B + U) \geqslant \alpha \ .$$

Since B + U and A + U are open convex sets. $\Lambda(A+U)$ and $\Lambda(B+U)$ are open intervals. Since $\Lambda(A+U) \cap \Lambda(B+U) = \emptyset$, claim that $\alpha \notin \Lambda(B+U)$. Suppose $\alpha \in \Lambda(B+U)$. There exists an open interval N such that $N \subset \Lambda(B+U)$. And since for any $\delta > 0$ $(\alpha - \delta, \alpha + \delta) \cap \Lambda(A+U) \neq \emptyset$. So we have $N \cap \Lambda(A+U) \neq \emptyset$. This implies that $\Lambda(A+U) \cap \Lambda(B+U) \neq \emptyset$. Now we have $\alpha \notin \Lambda(B+U)$. Thus $\Lambda(A+U) < \alpha$, $\Lambda(B+U) > \alpha$. Hence $\Lambda(A) < \alpha$, $\Lambda(B) > \alpha$.

3.1.8 Corollary. If X is a real topological vector space, then points of X are separtated by the continuous linear functionals on X.

Proof : Suppose x_1 and x_2 are points in X and $x_1 \neq x_2$. Take $A = \{x_1\}$, $B = \{x_2\}$. Then A is compact convex and B is closed convex. By 3.1.7, there exists $\Lambda \in X^*$ and a real number α such that $\Lambda x_1 < \alpha < \Lambda x_2$. Thus $\Lambda x_1 \neq \Lambda x_2$.

3.1.9 Theorem. Suppose M is a subspace of a locally convex topological vector space X and $x_0 \in X$. If x_0 is not in $\bar{M}$ (clousure of M), then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ but $\Lambda x = 0$ for every $x \in M$.

Proof : Take $A = \{x_0\}$ and $B = \bar{M}$. By 3.1.8, there exists $\Lambda \in X^*$ such that Λx_0 and $\Lambda(M)$ are disjoint. Thus $\Lambda(M)$ is a proper subspace of scalar field. This forces $\Lambda(M) = \{0\}$ and $\Lambda x_0 \neq 0$. The desired functional is obtained by deviding Λ by Λx_0.

3.1.10 Theorem. If f is a continuous linear functional on a subspace M of a locally convex topological vector space X, there exists $\Lambda \in X^*$ such that $\Lambda = f$ on M.

Proof : Assume, without loss of generality, that f is not identically 0 on M. Put $M_0 = \{x \in M : f(x) = 0\}$ and pick $x_0 \in M$ such that $f(x_0) = 1$. Since f is continuous, x_0 is not in the M-closure of M_0, there exists a neighborhood V of x_0 in X such that $(V \cap M) \cap M_0 = \emptyset$, hence $V \cap M_0 = \emptyset$. Thus x_0 is not in $\bar{M}_0$. By 3.1.9, there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ and $\Lambda = 0$ on M_0.

If $x \in M$, then $x - f(x)x_0 \in M_0$, since $f(x_0) = 1$. Hence $\Lambda x - f(x) = \Lambda x - f(x) \Lambda x_0 = \Lambda(x - f(x)x_0) = 0$. Thus $\Lambda = f$ on M.

3.2 Weak Topology

3.2.1 Definition. Let τ_1 and τ_2 be two topologies on a set X, and assume $\tau_1 \subset \tau_2$ that is, every τ_1- open set is also τ_2- open. Then we say that τ_1 is weaker than τ_2 , or that τ_2 is stronger than τ_1

3.2.2 Definition. Let X be a set and $\mathcal{F}$ a nonempty family of mappings $f : X \longrightarrow Y_f$ where each Y_f is a topological space.

Let τ be the collection of all unions of finite intersection of set $f^{-1}(V)$, with $f \in \mathcal{F}$ and V be open in Y_f, i.e., $G \in \tau$ if and only if for any $x \in G$ there are $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and $V_1, V_2, \ldots, V_n$ which are open in $Y_{f_1}, Y_{f_2}, \ldots, Y_{f_n}$ such that $x \in f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \ldots f_n^{-1}(V_n) \subset G$. Then τ is a topology on X, and it is in fact the weakest topology on X that makes every $f \in \mathcal{F}$ continuous : If τ' is any other topology with that property, then $\tau \subset \tau'$. This τ is called <u>the weak topology on X induced by $\mathcal{F}$</u>, or, more suceinctly, the <u>$\mathcal{F}$-topology</u> of X.

3.2.3 <u>Definition</u>. Let X be a vector space and X' a vector space of linear functionals on X. X' separates points of X if for any distinct points x_1, x_2 in X, there exists $f \in X'$ such that $f(x_1) \neq f(x_2)$.

3.2.4 <u>Lemma</u>. Suppose $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and Λ are linear functionals on a vector space X. Let

$$N = \{ x : \Lambda_1 x = \Lambda_2 x = \ldots = \Lambda_n x = 0 \}.$$

The following three properties are then equivalent :

(i) There are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\Lambda = \alpha_1 \Lambda_1 + \alpha_2 \Lambda_2 + \ldots + \alpha_n \Lambda_n .$$

(ii) There exists $\gamma < \infty$ such that

$$|\Lambda x| \leq \gamma \max_{1 \leq i \leq n} |\Lambda_i x| \qquad \text{for all } x \in X$$

(iii) $\Lambda x = 0$ for every $x \in N$

Proof : It is clear that (i) implies (ii) and (ii) implies (iii). Assume (iii) holds. Let $\mathbb{K}$ be the scalar field. Define

$\pi : X \longrightarrow \mathbb{K}^n$ by

$$\pi(x) = (\Lambda_1 x, \Lambda_2 x, \ldots, \Lambda_n x)$$

If $\pi(x) = \pi(x')$ then (iii) implies $\Lambda x = \Lambda x'$. Since π is a linear mapping, $\pi(X)$ is a vector subspace of $\mathbb{K}^n$. Define F on $\pi(X)$ by $F(\pi(x)) = \Lambda x$. We can see that F is well-defined and linear on $\pi(X)$. Since $\pi(X)$ has finite dimension, F is continuous linear functional on $\pi(X)$. By 3.1.10, F can be extended to $\mathbb{K}^n$.

Let
$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots, 1)$$

Then $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{K}^n$.

Put $F(e_j) = \alpha_j$

For any $(u_1, u_2, \ldots, u_n) \in \mathbb{K}^n$ we have

$$F((u_1, u_2, \ldots, u_n)) = F(u_1 e_1 + u_2 e_2 + \ldots, + u_n e_n)$$
$$= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n .$$

Thus $\Lambda x = F(\pi(x)) = F((\Lambda_1 x, \Lambda_2 x, \ldots, \Lambda_n x)) = \sum_{i=1}^{n} \alpha_i \Lambda_i x$,

which is (i).

3.2.5 Theorem. Suppose X is a vector space and X' is a separating vector space of linear functionals on X. Then the X'-topology τ' makes X into a locally convex space whose dual space is X'.

Proof : For any $\Lambda \in X'$, if we define $p(x) = |\Lambda x|$ then we can see that those p form a separating family of seminorms on X. Further we can see that the topology generated by those p is of course the X'-topology. By 2.3.7, X is locally convex with respect to this topology.

Obviously, by the definition of X'-topology $X' \subset X^*$. It remains to show that $X^* \subset X'$. If $\Lambda \in X^*$, there exists a neighborhood V of 0 in X such that $|\Lambda x| < 1$ for all x in V. Since the sets of type

$$V = \{x : |\Lambda_i x| < r_i \ , \ r_i > 0 \quad i = 1,2,\ldots, n\}$$ where

$\Lambda_i \in X'$ form a local base of X with respect to X'-topology. So $|\Lambda_i x| < r_i$, $i = 1,2,\ldots, n$ implies $|\Lambda x| < 1$.

Let $N = \{ x : \Lambda_1 x = \Lambda_2 x = \ldots = \Lambda_n x = 0\}$.

Fix $x \in N$, given $\varepsilon > 0$, we have $\left|\Lambda_i \left(\frac{x}{\varepsilon}\right)\right| = 0 < r_i$, $i = 1,2,\ldots,n$; hence $\left|\Lambda\left(\frac{x}{\varepsilon}\right)\right| < 1$, i.e., $|\Lambda x| < \varepsilon$. Since ε is arbitrary, $|\Lambda x| = 0$, that is $\Lambda x = 0$ for all x in N. By 3.2.4, there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\Lambda = \alpha_1 \Lambda_1 + \ldots + \alpha_n \Lambda_n$. Since $\Lambda_i \in X'$ and X' is a vector space, $\Lambda \in X'$. This completes the proof.

3.2.6 Definition. Suppose X is a topological vector space (with topology τ) whose dual X^* separates points on X. The X^*-topology of X is called the weak topology of X.

We shall let X_w denote X topologized by this weak topology τ_w. 3.2.5 implies that X_w is a locally convex space whose dual is also X^*.

Since every $\Lambda \in X^*$ is τ-continuous and since τ_w is the weakest topology on X with that property, we have $\tau_w \subset \tau$. In this context, the given topology τ will often be called the original topology of X.

3.2.7 Definition. A sequence $\{x_n\}$ in X is said to converge to 0 originally if for any original neighborhood V of 0, there exists N such that for all $n \geq N$, $x_n \in V$.

A sequence $\{x_n\}$ in X is said to converge to 0 weakly if for any weakly neighborhood V of 0, there exists N such that for all $n \geq N$, $x_n \in V$.

3.2.8 Remark. A sequence $\{x_n\}$ weakly converges to 0 if and only if Λx_n converges to 0 for every $\Lambda \in X^*$.

Proof : Since every weak neighborhood of 0 contains a neighborhood of the form.

$$V = \{ x : |\Lambda x| < \varepsilon \}.$$

Since $\{x_n\}$ weakly converges to 0, given $\varepsilon > 0$ there exists N such that $x_n \in V$ for all $n \geq N$, i.e., $|\Lambda x_n| < \varepsilon$ for all $n \geq N$. That is $\{\Lambda x_n\}$ converges to 0. Conversely, if Λx_n converges to 0 for every $\Lambda \in X^*$. Consider $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and $r_1, r_2, \ldots, r_n$. Since $\Lambda_i x_n$ converge to 0 for $i = 1, 2, \ldots, m$, there exists N_i such that $|\Lambda_i x_n| < r_i$ for all $n \geqslant N_i$

Take $N = \max \{ N_1, N_2, \ldots, N_m \}$.

So for all $n \geqslant N$, $x_n \in V = \{ x : |\Lambda_i x| < r_i \}$. Thus $\{x_n\}$ weakly converges to 0.

3.2.9 Theorem. Suppose E is a convex subset of a locally convex space X. Then the weak closure $\bar{E}_w$ of E is equal to its original closure $\bar{E}$.

Proof : $\bar{E}_w$ is weakly closed, hence originally closed, so that $\bar{E} \subset \bar{E}_w$. To obtain the opposite inclusion, choose $x_0 \in X$, $x_0 \notin \bar{E}$. 3.1.7 shows that there exists $\Lambda \in X^*$ and a real number α such that

$$\Lambda x_0 < \alpha \quad \text{and} \quad \Lambda x > \alpha \quad \text{for all } x \in \bar{E}.$$

The set $\{ x : \Lambda x < \alpha \}$ is therefore a weak neighborhood of x_0 that does not intersect E. Thus x_0 is not in $\bar{E}_w$. This proves $\bar{E}_w \subset \bar{E}$.

3.2.10 Definition. Let X be a topological vector space whose dual is X^*. For a fixed x in X define f_x by

$$f_x(\Lambda) = \Lambda x \quad \text{for all } \Lambda \in X^*.$$

we can see that f_x is a linear functional on X . Let $\hat{X}$ be the collection of all linear functionals on X. If Λ_1 and Λ_2 belong to X^* and $\Lambda_1 \neq \Lambda_2$, there exists $x \in X$ such that $\Lambda_1 x \neq \Lambda_2 x$, i.e., there exists $f_x \in \hat{X}$ such that $f_x(\Lambda_1) \neq f_x(\Lambda_2)$. Thus $\hat{X}$ separates points of X^*. By 3.2.5, $\hat{X}$ can define a locally convex vector topology for X^*. This topology is called <u>the weak* topology</u> of X^*.

3.3 Application on the Hahn Banach Theorem

Let B be a Banach space and B^* its dual. The elements of B are denoted by x,y,.... and those of B^* by f,g,.... The value of the functional f at x is written in the form (f,x). The weak*-topology in B^* is the weak topology induced by the elements of B as functionals.

3.3.1 <u>Definition</u>. A set F of B^* is called <u>regularly convex</u> if to each $f_0 \notin F$ there exists an element x_0 such that

$$(f_0, x_0) > \sup_{f \in F} (f, x_0)$$

3.3.2 <u>Remark</u>. (i) If F is a regularly convex set in B^*, F is convex and weak*-closed.

(ii) If F is convex and weak*-compact set in B^*, F is regularly convex.

<u>Proof</u> : (i) For each $f \notin F$. By regularity of F, there exists $x_0 \in B$ such that

$$(f, x_0) > \sup_{g \in F} (g, x_0).$$

Thus, for any g, h in F and a real number t such that $0 \leq t \leq 1$,

$(tg+(1-t)h,x_0) = t(g,x_0)+(1-t)(h,x_0) < t(f,x_0)+(1-t)(f,x_0)=(f,x_0)$,

i.e., $f \neq tg + (1-t)h$ for any g, h in F.

So $f \notin tF +(1-t)F$. That is, F is convex.

Next, we will show that F is w^*-closed. For any $f_0 \in F^c$, by regularity of F, there exists x_0 such that

$$(f_0, x_0) > \sup_{f \in F} (f, x_0) = a .$$

Let $V = \{ f \in B^* : |(f,x_0)| < \varepsilon ,\ \varepsilon = (f_0, x_0) - a \}$. Then V is a weak*-neighborhood of 0 ; hence $V + f_0$ is a weak*-neighborhood of f_0.

For each $f \in V$, $f+f_0 \in V+f_0$; $|(f, x_0)| < \varepsilon$

$$-\varepsilon < (f, x_0) < \varepsilon ,$$

$$a-(f_0,x_0) < (f,x_0) < (f_0,x_0)-a ,$$

$$(f,x_0) + (f_0,x_0) > a ,$$

i.e., $(f+f_0, x_0) > a$.

Thus $f + f_0 \notin F$.

So we have a weak*-neighborhood $V+f_0$ of f_0 such that $V+f_0 \subset F^c$. That is F^c is weak*-open.

(ii) Since B^* is a Banach space, B^* is locally convex. For any $f_0 \notin F$, $\{f_0\}$ is closed and convex in B^*. By 3.2.9 , $\{f_0\}$ is weak*-closed. Since F is convex and weak*-compact by 3.1.7., there exists $x_0 \in B$ and a real number α such that

$$(f_0, x_0) > \alpha$$

$$(f, x_0) < \alpha \qquad \text{for all } f \in F.$$

That is $(f_0, x_0) > \sup_{f \in F} (f, x_0)$.

Thus F is regularly convex.

3.3.3 <u>Theorm.</u> If F and G are regularly convex disjoint sets of B^* and if one of them is weak*-compact then there exists an element x_0 and a real number α such that

$$(f, x_0) > \alpha \qquad \text{for all } f \text{ in } F ;$$

$$(g, x_0) < \alpha \qquad \text{for all } g \text{ in } G .$$

<u>Proof</u> : Since F and G are regularly convex, by 3.3.2 F and G are weak*-closed convex sets. Assume G to be weak*-compact. By 3.2.5, B^* is a locally convex topological vector space. By 3.1.7, there exists $x_0 \in B$, a real number α such that

$$(f, x_0) > \alpha \qquad \text{for all } f \text{ in } F ;$$

$$(g, x_0) < \alpha \qquad \text{for all } g \text{ in } G .$$

3.3.4 <u>Definition.</u> A set X in B is called a <u>cone</u> if $x \in X$ implies $\lambda x \in X$ for every real, non-negative number λ .

3.3.5 <u>Remark.</u> (i) A convex closed cone X determines a cone F in B^* by :

(1) $$F = \left\{ f : (f, x) \geq 0 \quad \text{for all } x \text{ in } X \right\}.$$

(ii) A cone F is regularly convex.

3.3.6 Remark. Let F be a cone determined by a closed convex cone X, i.e., $F = \{ f : (f,x) \geqslant 0 \text{ for all } x \text{ in } X \}$.
$x \in X$ if and only if $(f,x) \geqslant 0$ for all f in F.

Proof : If $x \in X$, it is obvious that $(f,x) \geqslant 0$ for all f in F. Conversely, we prove by contrapositive. For each x_0 which is not in X, $\{x_0\}$ is convex and compact. And X is closed convex set. By 3.1.7, there exists $f_0 \in B^*$ and a real number α such that

$$(f_0, x_0) < \alpha \quad ;$$

$$(f_0, x) > \alpha \quad \text{for all } x \text{ in } X.$$

Since $0 \in X$, α must be negative. Furthermore $(f_0,x) \geqslant 0$ for every $x \in X$, since otherwise $(f_0, \lambda x)$ would be less than α for λ sufficiently large. Hence f_0 must belong to F, and for this f_0, the value (f_0,x_0) is negative.

3.3.7 Theorem. Let G be a weak*-compact regularly convex set in B^* and X a closed convex cone in B with the following properties : to each $x \in X$ there exists a $g \in G$ for which $(g,x) \geqslant 0$, then there exists a $g_0 \in G$ such that for all $x \in X$, $(g_0,x) \geqslant 0$.

Proof : Let F be the cone in B^* determined by X according to (1). This theorem states simply that $F \cap G \neq \emptyset$. Assume by contradiction that F and G have no point in common. Then by 3.3.3, there exists x_0 and a real number α such that $(f,x_0) > \alpha$

for all f in F, $(g,x_0) < \alpha$ for all g in G. Since F contains 0, the number α must be negative. Furthermore $(f,x_0) \geqslant 0$ for all f in F, since otherwise for some large λ, $(\lambda f,x_0)$ would be less than α. By 3.3.6, x_0 must belong to X, and since $(g,x_0) < \alpha < 0$ for all g in G, the assumption of this theorem is contradicted. Thus F and G must intersect. This completes the proof.