CHAPTER V

TORSION-FREE LOCALLY CYCLIC DECOMPOSABLE GROUPS

The materials of this chapter are drawn from reference [3].

In this chapter, the problem as to which torsion-free groups are locally cyclic decomposable is solved. Namely, we prove :

5.1 Theorem. Let G be a group. Then the following are equivalent :

1. G is strongly torsion-free.
2. G is torsion-free locally cyclic decomposable.
3. G is the union of a family $\{G_k / k \in K\}$ of abelian torsion-free subgroups such that

$$G_j \cap G_k = \{1\}$$

if $G_j \neq G_k$.

Proof : 1 implies 2 :

Let G be a strongly torsion-free group. Then G is torsion-free by Remark 2.6. For $g \in G$. Let

$$\langle g \rangle = \{x \in G / x^m \in [g] \text{ for some non-zero integer } m\}$$

Then $\langle g \rangle$ is a commutative subgroup of G which can be easily shown as follows :

If $g = 1$, then, by Remark 2.6 (a), $\langle g \rangle = \{1\}$ and the claim is true. Thus we may and shall assume that $g \neq 1$.

<u>$\langle g \rangle$ is commutative</u> : Let $x, y \in \langle g \rangle$. If one of the x and y is 1, then x and y obviously commute. Assume $x \neq 1 \neq y$, then by Remark 2.6 (c) $x^m = y^n$ for some non-zero integers m,n.

$$\text{Hence } (xyx^{-1})^n = \underbrace{(xyx^{-1})(xyx^{-1}) \ldots (xyx^{-1})}_{n \text{ times.}}$$

$$= xy^n x^{-1}$$

$$= xx^m x^{-1}$$

$$= x^m$$

$$= y^n$$

i.e., $(xyx^{-1})^n = y^n$.

Since G is strongly trosion-free, we have $xyx^{-1} = y$, i.e., $xy = yx$.

Hence $\langle g \rangle$ is commutative.

<u>$\langle g \rangle$ is a subgroup of G:</u> We obviously have $1 \in \langle g \rangle$. If $x, y \in \langle g \rangle$, then x^m, $y^n \in [g]$ for some non-zero integers m and n. Since x and y commute, $(xy)^{mn} = x^{mn}y^{mn} \in [g]$, and since $mn \neq 0$, $xy \in \langle g \rangle$. Finally, if $x \in \langle g \rangle \setminus \{1\}$, then $x^m \in [g]$, for some non-zero integer m so that

$$(x^{-1})^m = (x^m)^{-1} \in [g] ;$$

i.e. $x^{-1} \in \langle g \rangle$.

Hence $\langle g \rangle$ is a commutative subgroup of G.

Moreover, we shall show that $\langle g \rangle$ is isomorphic to a subgroup of the additive group $\mathbb{Q}$ of rational numbers, and, therefore, by Theorem 3.8, $\langle g \rangle$ is locally cyclic.

The case when $g = 1$ is obvious; assume $g \neq 1$. For any $x \in \langle g \rangle \setminus \{1\}$, $x^m = g^n$ for some non-zero integers m and n, by Remark 2.6 (c). Define $\varphi(x) = n/_m$ and $\varphi(1) = 0$.

<u>φ is a well-defined function from $\langle g \rangle$ into $\mathbb{Q}$</u>: Let $x \in \langle g \rangle \setminus \{1\}$. Suppose $x^m = g^n$ and $x^s = g^t$ for some non-zero integers m, n, s and t. Then

$$x^{ms} = g^{ns} \quad \text{and} \quad x^{sm} = g^{tm}$$

so that $g^{ns} = g^{tm}$. Since $g \neq 1$ and G is torsion-free, $ns = tm$; i.e. $\frac{n}{m} = \frac{t}{s}$.

Hence φ is well-defined.

<u>φ is 1 - 1</u> : Let $x, y \in \langle g \rangle$ such that $\varphi(x) = \varphi(y) = n/_m$. We may and shall assume that $x \neq 1$. Then $x^m = g^n$ and $y^m = g^n$ and $m \neq 0 \neq n$ so that $x^m = y^m$. Since G is strongly torsion-free $x = y$. Hence φ is 1 - 1.

<u>φ is a homomorphism</u> : Let $x, y \in \langle g \rangle$. We may and shall assume that $x \neq 1 \neq y$, and that $\varphi(x) = \frac{n}{m}$ and $\varphi(y) = \frac{t}{s}$. Then n, m, t, s are all different from 0 and $x^m = g^n$ and $y^s = g^t$. Since $\langle g \rangle$ is commutative, we have

$$\begin{aligned}(xy)^{ms} &= x^{ms}\, y^{ms}\\ &= g^{ns}\, g^{mt}\\ &= g^{ns+mt}\end{aligned}$$

so that

$$\begin{aligned}\varphi(xy) &= \frac{ns + mt}{ms}\\ &= \frac{n}{m} + \frac{t}{s}\\ &= \varphi(x) + \varphi(y)\end{aligned}$$

Thus φ is a homomorphism.

Hence $\langle g \rangle$ is isomorphic to a subgroup of $\mathbb{Q}$.

Finally, we shall show that G is locally cyclic decomposable. But we already have $G = \bigsqcup_{g \in G} \langle g \rangle$, since each $\langle g \rangle$ is locally cyclic subgroup of G, we only need to show that $\langle g \rangle \cap \langle h \rangle = \{1\}$ if $\langle g \rangle \neq \langle h \rangle$. We shall prove the contrapositive.

Suppose there is a $1 \neq x \in \langle g \rangle \cap \langle h \rangle$; then $g \neq 1 \neq h$, and therefore there are non-zero integers m,n,s and t such that $x^m = g^n$, $x^s = h^t$. Then $x^{ms} = g^{ns}$ and $x^{sm} = h^{tm}$ so that $g^{ns} = h^{tm}$. Since $ns \neq 0 \neq tm$, $g \in \langle h \rangle$ and $h \in \langle g \rangle$ so that $\langle g \rangle \subset \langle h \rangle$ and $\langle h \rangle \subset \langle g \rangle$; i.e., $\langle g \rangle = \langle h \rangle$.

Thus (2) holds.

<u>2 implies 3.</u>

Since subgroups of torsion-free group are torsion-free and since Locally cyclic subgroups are commutative, by Lemma 3.4, 2 implies 3.

<u>3 implies 1.</u>

Suppose $G = \bigcup_{k \in K} G_k$, each G_k is an abelian torsion-free subgroup of G and $G_j \cap G_k = \{1\}$ if $G_j \neq G_k$.

Let $x, y \in G \setminus \{1\}$, and n be a non-zero integer such that $x^n = y^n$. Then $x \in G_j$ and $y \in G_k$ for some $j, k \in K$. Since G_j and G_k are torsion-free, then $1 \neq x^n \in G_j$ and $1 \neq y^n \in G_k$. Hence $1 \neq x^n = y^n \in G_j \cap G_k \neq \{1\}$ so that $G_j = G_k$. Thus $x^n = y^n$ is in the abelian torsion-free subgroup $G_j = G_k$, so that $x = y$ by Remark 2.6 (b). Hence G is strongly torsion-free.