CHAPTER IV

CONTINUOUS SOLUTIONS OF $f(x + y) = g(x)f(y) + g(y)f(x)$ ON THE SET OF ALL NON-NEGATIVE REAL NUMBERS

In this chapter, we apply the results in the previous chapter to obtain all the continuous functions f, g from $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$ such that

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

holds for all x, y in $\mathbb{R}^+ \cup \{0\}$.

DEFINITION 4.1 By a continuous solution of the functional equation

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$, we mean a solution (f,g) of (*) such that f,g are continuous functions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$.

We can see that any ordered pair (f,g), where f is identically zero, and g is an arbitrary continuous function on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$, is a continuous trivial solution of (*) on $\mathbb{R}^+ \cup \{0\}$. Next, we shall consider the continuous non-trivial solutions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$.

<u>THEOREM 4.2</u> Let f, g be functions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. Then (f,g) is a continuous non-zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on $\mathbb{R}^+ \cup \{0\}$ if and only if f, g are of the form

$$(4.2.1) \quad \begin{cases} f(x) = \beta q^x, \\ g(x) = \frac{1}{2} q^x \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where $\beta, q \in \mathbb{C}^*$.

<u>PROOF</u> Let (f,g) be a continuous non-zero-type solution of $(*)$ on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. By theorem 3.2, then there exists β in $\mathbb{C}^*$ and a homomorphism $\tilde{g}$ on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}^*$ such that

$$(4.2.2) \quad \begin{cases} f(x) = \beta \tilde{g}(x), \\ g(x) = \frac{1}{2} \tilde{g}(x) \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$. Since g is continuous on $\mathbb{R}^+ \cup \{0\}$ and $\tilde{g}(x) = 2g(x)$ for all x in $\mathbb{R}^+ \cup \{0\}$, hence $\tilde{g}$ is a continuous homomorphism on $\mathbb{R}^+ \cup \{0\}$. It follows from theorem 2.16 that

$$(4.2.3) \quad \tilde{g}(x) = \tilde{g}(1)^x$$

for all x in $\mathbb{R}^+ \cup \{0\}$. Let $q = \tilde{g}(1)$. From (4.2.2) and (4.2.3), it follows that f, g must be of the form (4.2.1).

Next, assume that f, g are functions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$ of the form (4.2.1). It is well-known that f, g are continuous functions on $\mathbb{R}^+ \cup \{0\}$. Observe that for any x, y in $\mathbb{R}^+ \cup \{0\}$,

$$(4.2.4) \quad g(x)f(y) + g(y)f(x) = \tfrac{1}{2}q^x \cdot \beta q^y + \tfrac{1}{2} q^y \cdot \beta q^x,$$

$$= \beta q^{x+y},$$

$$= f(x + y).$$

From (4.2.1), we see that

$$(4.2.5) \quad f(0) = \beta \neq 0 .$$

Therefore, (f,g) is a continuous non-zero-type solution of (*) on $\mathbb{R}^+ \cup \{0\}$.

Next, we shall consider the continuous non-trivial zero-type solution of (*) on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n = 2a_{n+1}$ for all n in $\mathbb{P}$. For any n in $\mathbb{P}$, Let $S_n = \langle a_n \rangle$ be the cyclic monoid generated by a_n.

We can see that S_n is a subset of S_{n+1} for all n in $\mathbb{P}$ and $S = \cup S_n$ is a dense subset of $\mathbb{R}^+ \cup \{0\}$. We shall determine a certain class of non-trivial solutions of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. First, we need the followings lemmas.

<u>LEMMA 4.3</u> Let S_n and S be as defined above. Let (f,g) be a solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. For each n in $\mathbb{P}$, let f_n, g_n be the restrictions of f, g respectively on S_n. Then

(4.3.1) $2g_n(a_n)^2 - g_n(2a_n) = \left[2g_{n+1}(a_{n+1})^2 - g_{n+1}(2a_{n+1})\right]^2$

and

(4.3.2) $g_n(a_n)^2 - g_n(2a_n) = 4g_{n+1}(a_{n+1})^2\left[g_{n+1}(a_{n+1})^2 - g_{n+1}(2a_{n+1})\right]$

for all n in $\mathbb{P}$.

<u>PROOF</u> Since g_n is the restriction of g_{n+1} and $a_n = 2a_{n+1}$, hence

$$2g_n(a_n)^2 - g_n(2a_n) = 2g_{n+1}(2a_{n+1})^2 - g_{n+1}(4a_{n+1}).$$

By lemma 3.7, it follows that

$$\begin{aligned}
2g_n(a_n)^2 - g_n(2a_n) &= 2g_{n+1}(2a_{n+1})^2 - \Big[g_{n+1}(a_{n+1})g_{n+1}(3a_{n+1}) + \\
&\quad \left[g_{n+1}(2a_{n+1}) - g_{n+1}(a_{n+1})^2\right]\Big[g_{n+1}(2a_{n+1}) + \\
&\quad 2g_{n+1}(a_{n+1})^2\Big]\Big], \\
&= 2g_{n+1}(2a_{n+1})^2 - \Big[g_{n+1}(2a_{n+1})^2 \\
&\quad + 2g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) \\
&\quad - g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) - 2g_{n+1}(a_{n+1})^4 \\
&\quad + g_{n+1}(a_{n+1})g_{n+1}(3a_{n+1})\Big], \\
&= g_{n+1}(2a_{n+1})^2 - g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) \\
&\quad + 2g_{n+1}(a_{n+1})^4 - g_{n+1}(a_{n+1})\Big[g_{n+1}(a_{n+1})g_{n+1}(2a_{n+1}) \\
&\quad + \left[g_{n+1}(2a_{n+1}) - g_{n+1}(a_{n+1})^2\right]\left[2g_{n+1}(a_{n+1})\right]\Big],
\end{aligned}$$

$$= g_{n+1}(2a_{n+1})^2 - g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) + 2g_{n+1}(a_{n+1})^4$$

$$- g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) - 2g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1})$$

$$+ 2g_{n+1}(a_{n+1})^4 ,$$

$$= g_{n+1}(2a_{n+1})^2 - 4g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) + 4g_{n+1}(a_{n+1})^4,$$

$$= \left[g_{n+1}(2a_{n+1}) - 2g_{n+1}(a_{n+1})^2 \right]^2 .$$

Therefore, (4.3.1) holds for all n in $\mathbb{P}$.

To prove (4.3.2). Since g_n is the restriction of g_{n+1} and $a_n = 2a_{n+1}$, hence

(4.3.3) $\quad g_n(a_n)^2 = g_{n+1}(2a_{n+1})^2$

and

$$g_n(2a_n) = g_{n+1}(4a_{n+1}).$$

By lemma 3.7, we have

(4.3.4) $\quad g_n(2a_n) = g_{n+1}(a_{n+1})g_{n+1}(3a_{n+1}) + \left[g_{n+1}(2a_{n+1}) - g_{n+1}(a_{n+1})^2 \right]\left[g_{n+1}(2a_{n+1}) + 2g_{n+1}(a_{n+1})^2 \right],$

$$= g_{n+1}(a_{n+1}) \left\{ g_{n+1}(a_{n+1})g_{n+1}(2a_{n+1}) + \left[g_{n+1}(2a_{n+1}) - g_{n+1}(a_{n+1})^2 \right]\left[2g_{n+1}(a_{n+1}) \right] \right\} + \left[g_{n+1}(2a_{n+1}) - g_{n+1}(a_{n+1})^2 \right]\left[g_{n+1}(2a_{n+1}) + 2g_{n+1}(a_{n+1})^2 \right],$$

$$= g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}) + 2g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1})$$

$$- 2g_{n+1}(a_{n+1})^4 + g_{n+1}(2a_{n+1})^2$$

$$+ 2g_{n+1}(a_{n+1})^2 g_{n+1}(2g_{n+1}) - g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1})$$

$$- 2g_{n+1}(a_{n+1})^4,$$

$$= g_{n+1}(2a_{n+1})^2 + 4g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1})$$

$$- 4g_{n+1}(a_{n+1})^4.$$

It follows from (4.3.3) and (4.3.4) that

$$g_n(a_n)^2 - g_n(2a_n) = 4g_{n+1}(a_{n+1})^4 - 4g_{n+1}(a_{n+1})^2 g_{n+1}(2a_{n+1}),$$

$$= 4g_{n+1}(a_{n+1})^2 \left[g_{n+1}(a_{n+1})^2 - g_{n+1}(2a_{n+1}) \right].$$

Therefore, (4.3.2) holds.

<u>LEMMA 4.4</u> Let S_n, S be as in lemma 4.3. Let (f,g) be a non-trivial zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. For each n in $\mathbb{P}$, let f_n, g_n be the restrictions of f, g respectively on S_n.

If (f_n, g_n) is also a non-trivial zero-type solution of $(*)$ on S_n for all n in $\mathbb{P}$ and there exists an n_0 in $\mathbb{P}$ such that (f_{n_0}, g_{n_0}) is a type I solution of $(*)$ then (f_n, g_n) is a type I solution of (*) for all n in $\mathbb{P}$ and f, g are given by

$$(4.4.1)\quad \begin{cases} f(ta_n) = \beta\left(q_{11}^{\frac{t}{2^{n-1}}} - q_{21}^{\frac{t}{2^{n-1}}}\right), \\ g(ta_n) = \frac{1}{2}\left(q_{11}^{\frac{t}{2^{n-1}}} - q_{21}^{\frac{t}{2^{n-1}}}\right) \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where β, q_{11}, $q_{21} \in \mathbb{C}^{\#}$, and $q_{11} \neq q_{21}$, where $q_{ij}^{\frac{1}{2^{n-1}}} = q_{in}$; i, j = 1, 2.

<u>PROOF</u> Let (f,g) be a non-trivial zero-type solution of (*) on S into $\mathbb{C}$ such that (f_n, g_n) is also a non-trivial zero-type solution of (*) on S_n for all n in $\mathbb{P}$.

Let (f_{n_0}, g_{n_0}) be a type I solution of (*) on S_{n_0}, Hence

$$(4.4.2)\quad 2g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) \neq 0$$

and

$$(4.4.3)\quad g_{n_0}(a_{n_0})^2 \neq g_{n_0}(2a_{n_0}).$$

It follows from (4.3.1) in lemma 4.3 that

$$(4.4.4)\quad 2g_n(a_n)^2 - g_n(2a_n) \neq 0 \text{ iff } 2g_{n+1}(a_{n+1})^2 - g_{n+1}(2a_{n+1}) \neq 0$$

for all n in $\mathbb{P}$. By using (4.4.2) and (4.4.4) we can verify, by mathematical induction, that

$$(4.4.5)\quad 2g_n(a_n)^2 - g_n(2a_n) \neq 0$$

for all n in $\mathbb{P}$ such that $n \geq n_0$.

Suppose that

$$(4.4.6) \quad 2g_{k_0}(a_{k_0})^2 - g_{k_0}(2a_{k_0}) = 0$$

for some k_0 in $\mathbb{P}$ such that $k_0 < n_0$. By using (4.4.4) and (4.4.6), we can verify, by mathematical induction, that

$$(4.4.7) \quad 2g_k(a_k)^2 - g_k(2a_k) = 0$$

for all $k \geq k_0$. But this is contrary to (4.4.2).

Therefore

$$(4.4.8) \quad 2g_n(a_n)^2 - g_n(2a_n) \neq 0$$

for all n in $\mathbb{P}$ such that $n < n_0$. From (4.4.5) and (4.4.8), we have

$$(4.4.9) \quad 2g_n(a_n)^2 - g_n(2a_n) \neq 0$$

for all n in $\mathbb{P}$.

It follows from (4.4.3) that

$$(4.4.10) \quad g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) \neq 0.$$

From (4.3.2) in lemma 4.3, we have

$$(4.4.11) \quad g_n(a_n)^2 - g_n(2a_n) = 4g_{n+1}(a_{n+1})^2 \left[g_{n+1}(a_{n+1})^2 - g_{n+1}(2a_{n+1}) \right]$$

for all n in $\mathbb{P}$. From (4.4.10) and (4.4.11), we can see that if

$$g_k(a_k)^2 - g_k(2a_k) \neq 0$$

for any k in $\mathbb{P}$ such that $k \geq n_0$, then

$$g_{k+1}(a_{k+1})^2 - g_{k+1}(2a_{k+1}) \neq 0.$$

Hence, by mathematical induction, we can conclude that

(4.4.12) $$g_n(a_n)^2 - g_n(2a_n) \neq 0$$

for all n in $\mathbb{P}$ such that $n \geq n_0$.

Now, suppose that there exists k_0 in $\mathbb{P}$ such that $k_0 < n_0$ and

(4.4.43) $$g_{k_0}(a_{k_0})^2 - g_{k_0}(2a_{k_0}) = 0.$$

Assume that

(4.4.14) $$g_k(a_k)^2 - g_k(2a_k) = 0$$

for any k in $\mathbb{P}$ such that $k \geq k_0$. From (4.4.11) and (4.4.14), it follows that

(4.4.15) $$g_{k+1}(a_{k+1}) = 0 \text{ or } g_{k+1}(a_{k+1})^2 - g_{k+1}(2a_{k+1}) = 0$$

If

(4.4.16) $$g_{k+1}(a_{k+1}) = 0,$$

we have

(4.4.17)
$$\begin{aligned} f_k(a_k) &= f(a_k), \\ &= f(2a_{k+1}), \\ &= 2g(a_{k+1})f(a_{k+1}), \\ &= 2g_{k+1}(a_{k+1})\, f_{k+1}(a_{k+1}), \\ &= 0, \end{aligned}$$

which is a contradiction. Hence we must have

$$g_{k+1}(a_{k+1})^2 - g_{k+1}(2a_{k+1}) = 0.$$

Hence, by mathematical induction, we can conclude that

(4.4.18) $g_n(a_n)^2 - g_n(2a_n) = 0$

for all n in $\mathbb{P}$ such that $n \geq k_0$. This is contrary to (4.4.10). Therefore,

(4.4.19) $g_n(a_n)^2 - g_n(2a_n) \neq 0$

for all n in $\mathbb{P}$ such that $n < n_0$. From (4.4.12) and (4.4.19), we have

(4.4.20) $g_n(a_n)^2 - g_n(2a_n) \neq 0$

for all n in $\mathbb{P}$.

From (4.4.9) and (4.4.20), we see that for each n in $\mathbb{P}$, (f_n, g_n) is a type I solution of (*). Hence, by lemma 3.9, f_n, g_n must be of the form

$$(4.4.21) \quad \begin{cases} f_n(ta_n) = \beta_n (q_{1n}^t - q_{2n}^t) , \\ g_n(ta_n) = \frac{1}{2} (q_{1n}^t + q_{2n}^t) \end{cases}$$

for all t in $\mathbb{N}$, where β_n, q_{1n}, $q_{2n} \in \mathbb{C}^*$, $q_{1n} \neq q_{2n}$.

Since (f_n, g_n), (f_{n+1}, g_{n+1}) are the restrictions of (f, g) and $a_n = 2a_{n+1}$, hence we have

(4.4.22) $g_n(a_n) = g_{n+1}(2a_{n+1})$,

(4.4.23) $g_n(2a_n) = g_{n+1}(4a_{n+1})$,

(4.4.24) $f_n(a_n) = f_{n+1}(2a_{n+1})$.

By applying (4.4.21) to the above identities, we have, respectively,

(4.4.25) $\frac{1}{2}(q_{1n} + q_{2n}) = \frac{1}{2}(q^2_{1(n+1)} + q^2_{2(n+1)})$,

(4.4.26) $\frac{1}{2}(q^2_{1n} + q^2_{2n}) = \frac{1}{2}(q^4_{1(n+1)} + q^4_{2(n+1)})$,

(4.4.27) $\beta_n(q_{1n} - q_{2n}) = \beta_{n+1}(q^2_{1(n+1)} - q^2_{2(n+1)})$.

By multiplying both sides of (4.4.25) and (4.4.26) by 2, we have

(4.4.28) $q_{1n} + q_{2n} = q^2_{1(n+1)} + q^2_{2(n+1)}$

and

(4.4.29) $q^2_{1n} + q^2_{2n} = q^4_{1(n+1)} + q^4_{2(n+1)}$

respectively. By squaring both sides of (4.4.27) and (4.4.28), we have

(4.4.30) $\beta^2_n (q_{1n} - q_{2n})^2 = \beta^2_{n+1}(q^2_{1(n+1)} - q^2_{2(n+1)})^2$

and

(4.4.31) $q^2_{1n} + 2q_{1n}q_{2n} + q^2_{2n} = q^4_{1(n+1)} + 2q^2_{1(n+1)}q^2_{2(n+1)} + q^4_{2(n+1)}$

respectively.

From (4.4.29) and (4.4.31), by substraction, it follows that

(4.4.32) $2q_{1n}\, q_{2n} = 2q^2_{1(n+1)}\, q^2_{2(n+1)}$.

From (4.4.29) and (4.4.32), it follows that

$$q^2_{1n} - 2q_{1n}q_{2n} + q^2_{2n} = q^4_{1(n+1)} - 2q^2_{1(n+1)}q^2_{2(n+1)} + q^4_{2(n+1)}.$$

Hence

$$(4.4.33) \qquad (q_{1n} - q_{2n})^2 = (q^2_{1(n+1)} - q^2_{2(n+1)})^2 .$$

From (4.4.30), (4.4.33), and the fact that $q_{1n} \neq q_{2n}$, we have

$$(4.4.34) \qquad \beta^2_n = \beta^2_{n+1} .$$

Therefore,

$$(4.4.35) \qquad \beta_n = \beta_{n+1} \text{ or } \beta = -\beta_{n+1} .$$

If $\beta_n = \beta_{n+1}$ then, by (4.4.27) and (4.4.28), it follows that

$$(4.4.36) \qquad q_{1n} = q^2_{1(n+1)} \text{ and } q_{2n} = q^2_{2(n+1)} .$$

Hence, f_{n+1}, g_{n+1} will be of the form

$$(4.4.37) \qquad \begin{cases} f_{n+1}(ta_{n+1}) = \beta_n \left(q_{1n}^{\frac{t}{2}} - q_{2n}^{\frac{t}{2}}\right), \\ \\ g_{n+1}(ta_{n+1}) = \frac{1}{2} \left(q_{1n}^{\frac{t}{2}} + q_{2n}^{\frac{t}{2}}\right) \end{cases}$$

for all t in $\mathbb{N}$, where $q_{in}^{\frac{1}{2}}$ is defined to be $q_{i(n+1)}$, $i = 1,2,$.

If $\beta_n = -\beta_{n+1}$ then, by (4.4.27) and (4.4.28), it follows that

(4.4.38) $\quad q_{1n} = q^2_{2(n+1)}$ and $q_{2n} = q^2_{1(n+1)}$.

Hence, f_{n+1}, g_{n+1} can be written in the form

$$(4.4.39) \quad \begin{cases} f_{n+1}(t\, a_{n+1}) = \beta_n (q_{1n}^{\frac{t}{2}} - q_{2n}^{\frac{t}{2}}), \\ g_{n+1}(t\, a_{n+1}) = \frac{1}{2}(q_{1n}^{\frac{t}{2}} + q_{2n}^{\frac{t}{2}}) \end{cases}$$

where $q_{in}^{\frac{1}{2}}$ is defined to be $q_{j(n+1)}$; i, j = 1,2 and $i \neq j$.

We can see that in the case $\beta_n = -\beta_{n+1}$, by changing notations $q_{1(n+1)}$ to be $q_{2(n+1)}$ and $q_{2(n+1)}$ to be $q_{1(n+1)}$, we may write f_{n+1}, g_{n+1} in the form (4.4.37). Hence there is no loss of generality in assuming that $\beta_n = \beta_{n+1}$.

Therefore, f_{n+1}, g_{n+1} are of the form

$$f_{n+1}(t\, a_{n+1}) = \beta_n(q_{1n}^{\frac{t}{2}} - q_{2n}^{\frac{t}{2}}),$$
$$g_{n+1}(t\, a_{n+1}) = \frac{1}{2}(q_{1n}^{\frac{t}{2}} + q_{2n}^{\frac{t}{2}})$$

for all t in $\mathbb{N}$, where $q_{in}^{\frac{1}{2}}$ is defined to be $q_{i(n+1)}$; i = 1, 2. Then f, g are of the form (4.4.1).

<u>LEMMA 4.5</u> Let S_n, S be as in lemma 4.3. Let (f,g) be a non-trivial zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. For each n in $\mathbb{P}$, let f_n, g_n be the restrictions of f, g respectively on S_n.

If (f_n, g_n) is also a non-trivial zero-type solution of (*) on S_n for all n in $\mathbb{P}$ and there exists an n_0 in $\mathbb{P}$ such that (f_{n_0}, g_{n_0}) is a type II solution of (*) on S_{n_0}, then (f_n, g_n) is also a type II solution of (*) on S_n for all n in $\mathbb{P}$ and f, g are given by

$$(4.5.1) \qquad \begin{cases} f(t\,a_n) = \dfrac{t}{2^{n-1}}\,\beta_1\, q_1^{\frac{t}{2^{n-1}}}, \\ g(t\,a_n) = q_1^{\frac{t}{2^{n-1}}} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where $\beta_1, q_1 \in \mathbb{C}^*$ and $q_1^{\frac{1}{2^{n-1}}}$ is defined to be q_n

PROOF Let (f,g) be a non-trivial zero-type solution of (*) on S into C such that (f_n, g_n) is also a non-trivial zero-type solution of (*) on S_n for all n in $\mathbb{P}$. Let (f_{n_0}, g_{n_0}) be a type II solution of (*) on S_{n_0}. Hence

$$(4.5.2) \qquad 2g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) \neq 0$$

and

$$(4.5.3) \qquad g_{n_0}(a_{n_0})^2 = g_{n_0}(2a_{n_0}).$$

By using (4.5.2), (4.5.3), lemma 4.3, and the same argument as in the proof of lemma 4.4, we can verify that

$$(4.5.4) \qquad 2g_n(a_n)^2 - g_n(2a_n) \neq 0$$

for all n in $\mathbb{P}$, and

(4.5.5) $$g_n(a_n)^2 = g_n(2a_n)$$

for all n in $\mathbb{P}$.

From (4.5.4) and (4.5.5), we see that for each n in $\mathbb{P}$, (f_n, g_n) is a type II solution of (✱). Hence, by lemma 3.10, f_n, g_n must be of the form

(4.5.6) $$\begin{cases} f_n(t\, a_n) = t\, \beta_n\, q_n^t , \\ g_n(t\, a_n) = q_n^t \end{cases}$$

for all t in $\mathbb{N}$, where $\beta_n, q_n \in \mathbb{C}^*$.

Since (f_n, g_n), (f_{n+1}, g_{n+1}) are the restrictions of (f, g) and $a_n = 2a_{n+1}$, hence we have

(4.5.7) $$g_n(a_n) = g_{n+1}(2a_{n+1})$$

and

(4.5.8) $$f_n(a_n) = f_{n+1}(2a_{n+1}) .$$

By applying (4.5.6) to the above identities, we have

(4.5.9) $$q_n = q_{n+1}^2$$

and

(4.5.10) $$\beta_n q_n = 2\beta_{n+1}\, q_{n+1}^2 .$$

From (4.5.9) and (4.5.10), it follows that

(4.5.11) $$\beta_n = 2\beta_{n+1} ,$$

Therefore, from (4.5.6), (4.5.9), and (4.5.11), f_{n+1}, g_{n+1} are of the form

$$f_{n+1}(t\, a_{n+1}) = t\,\frac{\beta_n}{2}\, q_n^{\frac{t}{2}} ,$$

$$g_{n+1}(t\, a_{n+1}) = q_n^{\frac{t}{2}}$$

for all t in $\mathbb{N}$ where $q_n^{\frac{1}{2}}$ is defined to be q_{n+1}. Hence f, g are of the form (4.5.1).

<u>LEMMA 4.6</u> Let S_n, S be as in lemma 4.3. Let (f,g) be a non-trivial zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. For any n in $\mathbb{P}$, let f_n, g_n be the restrictions of f, g respectively on S_n.

If there exists an n_0 in $\mathbb{P}$ such that (f_{n_0}, g_{n_0}) is a type III solution of (*) on S_{n_0}, then (f_n, g_n) is also a type III solution of (*) on S_n for all n in $\mathbb{P}$, and f, g are given by

$$(4.6.1) \quad \begin{cases} f(t\, a_n) = \begin{cases} 0 & \text{if } t = 0 , \\ \beta_1 q_1^{\frac{t}{2^{n-1}}} & \text{otherwise} , \end{cases} \\ g(t\, a_n) = \begin{cases} 1 & \text{if } t = 0 , \\ \frac{1}{2}\, q_1^{\frac{t}{2^{n-1}}} & \text{otherwise} \end{cases} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where β_1, $q_1 \in \mathbb{C}^*$ and $q_1^{\frac{1}{2^{n-1}}}$ is defined to be q_n.

PROOF Let (f,g) be a nontrivial zero-type solution of (*) on S into $\mathbb{C}$. Let (f_{n_0}, g_{n_0}) be a type III solution of (*) on S_{n_0}.

Hence

$$(4.6.2) \qquad 2g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) = 0$$

and

$$(4.6.3) \qquad g_{n_0}(a_{n_0}) \neq 0.$$

By using (4.6.2), lemma 4.3, and the same argument as in the proof of lemma 4.4, we can verify that

$$(4.6.4) \qquad 2g_n(a_n)^2 - g_n(2a_n) = 0$$

for all n in $\mathbb{P}$.

From (4.6.4), it follows that

$$(4.6.5) \qquad g_n(2a_n) = 2g_n(a_n)^2$$

for all n in $\mathbb{P}$. By using (4.6.3) and (4.6.5), we can verify, by mathematical induction, that

$$(4.6.6) \qquad g_n(a_n) \neq 0$$

for all n in $\mathbb{P}$ such that $n \geq n_0$.

Suppose that

$$(4.6.7) \qquad g_{k_0}(a_{k_0}) = 0$$

for some k_0 in $\mathbb{P}$ such that $k_0 < n_0$. By using (4.6.5) and (4.6.7), we can verify, by mathematical induction, that

$$(4.6.8) \qquad g_k(a_k) = 0$$

for all $k \geq k_0$. But this is contrary to (4.6.3).

Therefore,

$$(4.6.9) \qquad g_n(a_n) \neq 0$$

for all n in $\mathbb{P}$ such that $n < n_0$. From (4.6.6) and (4.6.9), we have

$$(4.6.10) \qquad g_n(a_n) \neq 0$$

for all n in $\mathbb{P}$.

From (4.6.4) and (4.6.10), it follows that for each n in $\mathbb{P}$, (f_n, g_n) is a type III solution of (*). By lemma 3.11, f_n, g_n must be of the form

$$(4.6.11) \quad \begin{cases} f_n(t\, a_n) = \begin{cases} 0 & \text{if } t = 0, \\ \beta_n\, q_n^t & \text{otherwise}, \end{cases} \\ \\ g_n(t\, a_n) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{1}{2}\, q_n^t & \text{otherwise} \end{cases} \end{cases}$$

for all t in $\mathbb{N}$, where $\beta_n, q_n \in \mathbb{C}^*$.

Since (f_n, g_n) and (f_{n+1}, g_{n+1}) are the restrictions of (f, g) and $a_n = 2a_{n+1}$, hence we have

$$(4.6.12) \qquad g_n(a_n) = g_{n+1}(2a_{n+1}),$$

(4.6.13) $f_n(a_n) = f_{n+1}(2a_{n+1})$.

By applying (4.6.11) to the above identities, we have, respectively,

(4.6.14) $\frac{1}{2} q_n = \frac{1}{2} q_{n+1}^2$,

(4.6.15) $\beta_n q_n = \beta_{n+1} q_{n+1}^2$.

It follows that

(4.6.15) $q_n = q_{n+1}^2$,

(4.6.16) $\beta_n = \beta_{n+1}$.

From (4.6.11), (4.6.15), and (4.6.16), we can see that f_{n+1}, g_{n+1} are of the form

$$f_{n+1}(t\, a_{n+1}) = \begin{cases} 0 & \text{if } t = 0 \ , \\ \beta_n\, q_n^{\frac{t}{2}} & \text{otherwise} \ , \end{cases}$$

$$g_{n+1}(t\, a_{n+1}) = \begin{cases} 1 & \text{if } t = 0 \ , \\ \frac{1}{2}\, q_n^{\frac{t}{2}} & \text{otherwise} \end{cases}$$

for all t in $\mathbb{N}$, where $q_n^{\frac{1}{2}}$ is defined to be q_{n+1}. Therefore f, g are of the form (4.6.1).

<u>LEMMA 4.7</u> Let S_n, S be as in lemma 4.3. Let (f,g) be a non-trivial zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$. For each n in P, let f_n, g_n be the restrictions of f, g respectively on S_n. Then f_n, g_n cannot be a type IV solution of (*) on S_n for any n in $\mathbb{P}$.

PROOF Let (f,g) be a non-trivial zero-type solution of (*) on S into $\mathbb{C}$. Suppose there exists n_0 in $\mathbb{P}$ such that (f_{n_0}, g_{n_0}) is a type IV solution of (*) on S_{n_0}. Hence

(4.7.1) $$2g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) = 0$$

and

(4.7.2) $$g_{n_0}(a_{n_0}) = 0.$$

By lemma 3.12, we have

(4.7.3) $$\begin{cases} f_{n_0}(t\, a_{n_0}) = \begin{cases} \beta_{n_0} & \text{if } t = 1, \\ 0 & \text{otherwise}, \end{cases} \\ g_{n_0}(t\, a_{n_0}) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

for all t in $\mathbb{N}$, where $\beta_{n_0} \in \mathbb{C}^*$. Inparticular, we have

(4.7.4) $$f_{n_0}(a_{n_0}) = \beta_{n_0} \neq 0.$$

Since g_{n_0}, g_{n_0+1} are the restrictions of g and $a_{n_0} = 2a_{n_0+1}$, hence

(4.7.5) $$\begin{aligned} g_{n_0+1}(2a_{n_0+1}) &= g(2a_{n_0+1}), \\ &= g(a_{n_0}), \\ &= g_{n_0}(a_{n_0}), \\ &= 0. \end{aligned}$$

By (4.3.1) in lemma 4.3, we have

$$(4.7.6) \qquad 2g_{n_0}(a_{n_0})^2 - g_{n_0}(2a_{n_0}) = \left[2g_{n_0+1}(a_{n_0+1})^2 - g_{n_0+1}(2a_{n_0+1})\right]^2.$$

From (4.7.1) and (4.7.6), it follows that

$$(4.7.7) \qquad 2g_{n_0+1}(a_{n_0+1})^2 - g_{n_0+1}(2a_{n_0+1}) = 0 .$$

From (4.7.5) and (4.7.7), it follows that

$$(4.7.8) \qquad g_{n_0+1}(a_{n_0+1}) = 0.$$

Therefore (f_{n_0+1}, g_{n_0+1}) is also a type IV solution of (*) on S_{n_0+1}. Hence, by lemma 3.12, we have

$$(4.7.9) \qquad f_{n_0+1}(t\, a_{n_0+1}) = \begin{cases} \beta_{n_0+1} & \text{if } t = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all t in $\mathbb{N}$, where $\beta_n \in \mathbb{C}^*$. In particular, we have

$$\begin{aligned} f_{n_0}(a_{n_0}) &= f(a_{n_0}), \\ &= f(2a_{n_0+1}) , \\ &= f_{n_0+1}(2a_{n_0+1}) , \\ &= 0, \end{aligned}$$

which is contrary to (4.7.4). Therefore f_n, g_n cannot be a type IV solution of (*) on S_n for any n.

<u>THEOREM 4.8</u> Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n = 2a_{n+1}$. For any n in $\mathbb{P}$, let $S_n = \langle a_n \rangle$ be the cyclic monoid generated by a_n. Let $S = \cup S_n$. For any functions f, g on S into $\mathbb{C}$, let f_n, g_n be the restrictions of f, g respectively on S_n. If (f,g) is a non-trivial zero-type solution of

$$(*) \qquad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into $\mathbb{C}$ such that (f_n, g_n) is also a non-trivial zero-type solution of (*) on S_n for each n in $\mathbb{P}$ then f, g are of the form

$$(4.8.1) \qquad \begin{cases} f(ta_n) = \beta_1\left(q_{11}^{\frac{t}{2^{n-1}}} - q_{21}^{\frac{t}{2^{n-1}}}\right), \\ g(ta_n) = \frac{1}{2}\left(q_{11}^{\frac{t}{2^{n-1}}} + q_{21}^{\frac{t}{2^{n-1}}}\right) \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where $\beta_1, q_{11}, q_{21} \in \mathbb{C}^*$ such that $q_{11} \neq q_{21}$, or

$$(4.8.2) \qquad \begin{cases} f(ta_n) = \frac{t}{2^{n-1}}\,\beta_1 q_1^{\frac{t}{2^{n-1}}}, \\ g(ta_n) = q_1^{\frac{t}{2^{n-1}}} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where $\beta_1, q_1 \in \mathbb{C}^*$ or

$$(4.8.3) \qquad \begin{cases} f(ta_n) = \begin{cases} 0 & \text{if } t = 0, \\ \beta_1 q_1^{\frac{t}{2^{n-1}}} & \text{otherwise,} \end{cases} \\ g(ta_n) = \begin{cases} 1, & \text{if } t = 0, \\ \frac{1}{2} q_1^{\frac{t}{2^{n-1}}} & \text{otherwise,} \end{cases} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where β_1, $q_1 \in \mathbb{C}^*$.

PROOF Let (f,g) be a non-trivial zero-type solution of (*) on S into $\mathbb{C}$. Let f_n, g_n be the restrictions of f, g respectively on S_n. By our assumption, (f_n, g_n) is also a non-trivial zero-type solution of (*) on S_n for all n in $\mathbb{P}$. By lemma 4.7, we know that (f_n, g_n) cannot be a type IV solution of (*). By lemma 4.4 - 4.6, then f, g are of the form (4.8.1) - (4.8.3) respectively.

THEOREM 4.9 Let f, g be functions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. Then (f, g) is a continuous non-trivial zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on $\mathbb{R}^+ \cup \{0\}$ if and only if f, g are of the form

$$(4.9.1) \quad \begin{cases} f(x) = \beta(q_1^x - q_2^x), \\ g(x) = \frac{1}{2}(q_1^x + q_2^x) \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where β, q_1, $q_2 \in \mathbb{C}^*$ such that $q_1 \neq q_2$, or

$$(4.9.2) \quad \begin{cases} f(x) = x\beta q^x, \\ g(x) = q^x \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where β, $q \in \mathbb{C}^*$.

PROOF Let (f,g) be a continuous non-trivial zero-type solution of (*) on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. Then there exists $a \in \mathbb{R}^+$ such that $f(\frac{a}{2}) \neq 0$.

For any n in $\mathbb{P}$, define

$$S_n = \{ \frac{ta}{2^n} / t \in \mathbb{N} \} .$$

Let $S = \bigcup S_n$. Let f_n, g_n be the restrictions of f, g respectively on S_n. Since $\frac{a}{2} \in S_n$ for all n in $\mathbb{P}$ and $f(\frac{a}{2}) \neq 0$, hence (f_n, g_n) is a non-trivial zero-type solution of (*) on S_n for all n in $\mathbb{P}$.

Let f_S, g_S be the restrictions of f, g respectively on S. By theorem 4.8, f_S, g_S must be of the form

$$(4.9.3) \quad \begin{cases} f_S(\frac{ta}{2^n}) = \beta_1 (q_{11}^{\frac{t}{2^{n-1}}} - q_{21}^{\frac{t}{2^{n-1}}}), \\ g_S(\frac{ta}{2^n}) = \frac{1}{2} (q_{11}^{\frac{t}{2^{n-1}}} + q_{21}^{\frac{t}{2^{n-1}}}) \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where $\beta_1, q_{11}, q_{21} \in \mathbb{C}^*$ such that $q_{11} \neq q_{21}$, or

$$(4.9.4) \quad \begin{cases} f_S(\frac{ta}{2^n}) = \frac{t}{2^{n-1}} \beta_1 q_1^{\frac{t}{2^{n-1}}}, \\ g_S(\frac{ta}{2^n}) = q_1^{\frac{t}{2^{n-1}}} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where $\beta_1, q_1 \in \mathbb{C}^*$, or

$$(4.9.5) \quad \begin{cases} f_S(\frac{ta}{2^n}) = \begin{cases} 0 & \text{if } t = 0, \\ \beta_1 q_1^{\frac{t}{2^{n-1}}} & \text{otherwise,} \end{cases} \\ g_S(\frac{ta}{2^n}) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{1}{2} q_1^{\frac{t}{2^{n-1}}} & \text{otherwise} \end{cases} \end{cases}$$

for all t in $\mathbb{N}$, for all n in $\mathbb{P}$, where β_1, $q_1 \in \mathbb{C}^*$.

<u>CASE 1</u> If f_S, g_S are of the form (4.9.3). Let x be any point in $\mathbb{R}^+ \cup \{0\}$. Let $\{\frac{t_n a}{2^n}\}$ be a sequence in S which tends to x. Since f, g are continuous on $\mathbb{R}^+ \cup \{0\}$, hence

$$f(x) = \lim_{n \to \infty} f_S \left(\frac{t_n a}{2^n}\right),$$

$$= \lim_{n \to \infty} \beta_1 \left(q_{11}^{\frac{t_n}{2^{n-1}}} - q_{21}^{\frac{t_n}{2^{n-1}}}\right),$$

$$= \beta_1 \lim_{n \to \infty} \left(q_{11}^{\frac{t_n}{2^n} \cdot \frac{2}{a}} - q_{21}^{\frac{t_n}{2^n} \cdot \frac{2}{a}}\right),$$

$$= \beta_1 \left(q_{11}^{\frac{2x}{a}} - q_{21}^{\frac{2x}{a}}\right).$$

Similarly it can be shown that

$$g(x) = \frac{1}{2} \left(q_{11}^{\frac{2x}{a}} + q_{21}^{\frac{2x}{a}}\right).$$

Let $\beta = \beta_1$, $q_1 = q_{11}^{\frac{2}{a}}$, $q_2 = q_{21}^{\frac{2}{a}}$. Then f, g are of the form

$$(4.9.6) \quad \begin{cases} f(x) = \beta(q_1^x - q_2^x), \\ g(x) = \frac{1}{2}(q_1^x + q_2^x) \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$. Since β_1, q_{11}, $q_{21} \in \mathbb{C}^*$, hence

$$(4.9.7) \quad \beta_1, q_1, q_2 \in \mathbb{C}^*.$$

Because of (f, g) is a non-trivial solution of (*), hence $f \not\equiv 0$. Therefore we must have

(4.9.8) $q_1 \neq q_2$.

Hence, f, g are of the form (4.9.1).

<u>CASE 2</u>. If f_S, g_S are of the form (4.9.4). Let x be any point in $\mathbb{R}^+ \cup \{0\}$. Let $\{\frac{t_n a}{2^n}\}$ be a sequence in S which tends to x.

Since f, g are continuous, hence by an argument similar to that in case 1 we have

$$f(x) = \frac{2\beta_1}{a} \, x \, q_1^{\frac{2x}{a}} ,$$

$$g(x) = q_1^{\frac{2x}{a}} .$$

Let $\beta = \frac{2\beta_1}{a}$, $q = q_1^{\frac{2}{a}}$. Then f, g are of the form

$$(4.9.9) \quad \begin{cases} f(x) = x\beta \, q^x, \\ g(x) = q^x . \end{cases}$$

Since β_1, $q_1 \in \mathbb{C}^*$, $a \in \mathbb{R}^+$, hence

(4.9.10) $\beta, q \in \mathbb{C}^*$.

Therefore f, g are of the form (4.9.2) .

CASE 3 If f_S, g_S are of the form (4.9.5). Observe that $\{\frac{a}{2^n}\}$ tends to 0. Hence we have

$$\lim_{n\to\infty} f_S\left(\frac{a}{2^n}\right) = \beta_1 \lim_{n\to\infty} q_1^{\frac{1}{2^{n-1}}} ,$$

$$= \beta_1 ,$$

$$\neq 0.$$

That is $\lim_{n\to\infty} f_S\left(\frac{a}{2^n}\right) \neq f(0)$. Therefore f is not continuous at 0, which is contrary to the assumption that (f,g) is a continuous solution of (*) on $\mathbb{R}^+ \cup \{0\}$. Therefore f_S, g_S cannot be of the form (4.9.5).

We can verify directly that any function f, g of the form (4.9.1) or (4.9.2) are continuous and (f,g) is a non-trivial zero-type solution of (*) on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$.

We may now summarize the results obtained in theorem 4.2 and theorem 4.9 in the following theorem.

THEOREM 4.10 Let f, g be functions on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$. Then (f,g) is a continuous non-trivial solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on $\mathbb{R}^+ \cup \{0\}$ if and only if f, g are of the form

$$(4.10.1)\quad \begin{cases} f(x) = \beta q^x, \\ g(x) = \frac{1}{2} q^x \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where $\beta, q \in \mathbb{C}^*$, or

$$(4.10.2)\quad \begin{cases} f(x) = \beta(q_1^x - q_2^x), \\ g(x) = \frac{1}{2}(q_1^x + q_2^x) \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where $\beta, q_1, q_2 \in \mathbb{C}^*$ such that $q_1 \neq q_2$, or

$$(4.10.3)\quad \begin{cases} f(x) = x\beta q^x, \\ g(x) = q^x \end{cases}$$

for all x in $\mathbb{R}^+ \cup \{0\}$, where $\beta, q \in \mathbb{C}^*$.