CHAPTER III

A GENERALIZATION OF THE GEODESIC DIFFERENTIAL EQUATION

In chapter II, we mentioned that in differential geometry geodesics satisfy a second order ordinary differential equation of the form

$$\ddot{\psi}^i \quad = \quad G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k \quad \text{where } G^i_{jk} \text{ is analytic on open}$$

subset D of R^n for all $i,j,k = 1,2,\ldots,n$

and satisfy the functional equation given below

For each $1 \leq i \leq n$, $\psi^i(\vec{P},\alpha\vec{V},t)$ exists iff $\psi^i(\vec{P},\vec{V},\alpha t)$ exists and

$$\psi^i(\vec{P},\alpha\vec{V},t) \quad = \quad \psi^i(\vec{P},\vec{V},\alpha t) \ , \qquad i = 1,2,\ldots,n$$

Now, our problem is to determine if other types of second order ordinary differential equations satisfy a functional equation of this type. More precisely, we want to study the 2^{nd} order ordinary differential equation whose solutions satisfy a functional equation of the form $\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},f(\alpha,t))$ for some function $f(\alpha,t)$ for all $i = 1,2,\ldots, n$.

Notation 3-1

3-1.1 For all i, $1 \leq i \leq n$, $J^i_j(\vec{\psi})t^j \quad = \quad \sum_{j=1}^{\infty} J^i_j(\vec{\psi})t^j$

3-1.2 For all i, $1 \leq i \leq n$, $J^i_{j_1j_2}(\vec{\psi})\dot{\psi}^{j_1}t^{j_2} = \sum_{j_1=1}^{n}\sum_{j_2=1}^{\infty} J^i_{j_1j_2}(\vec{\psi})\dot{\psi}^{j_1}t^{j_2}$

3-1.3 For all i, k, $1 \le i \le n$ and $k = 1, 2, 3, \ldots$

$$J^i_{j_1 \ldots j_{k+1}}(\vec{\psi})\mathring{\psi}^{j_1} \ldots \mathring{\psi}^{j_k} t^{j_{k+1}}$$

$$= \sum_{j_1=1}^{n} \ldots \sum_{j_k=1}^{n} \sum_{j_{k+1}=1}^{\infty} J^i_{j_1 \ldots j_{k+1}}(\vec{\psi})\mathring{\psi}^{j_1} \ldots \mathring{\psi}^{j_k} t^{j_{k+1}}$$

<u>Introduction to theorem</u>

Let
$$\pi_1(x^1, \ldots, x^{2n+1}) = (x^1, x^2, \ldots, x^n)$$
$$\pi_2(x^1, \ldots, x^{2n+1}) = (x^{n+1}, x^{n+2}, \ldots, x^{2n})$$
$$\pi_3(x^1, \ldots, x^{2n+1}) = x^{2n+1}$$

Let Ω be an open connected subset of R^{2n+1} such that $(\vec{P}_0, \vec{0}, 0) \varepsilon\, \Omega$ for all $\vec{P}_0 \,\varepsilon\, \pi_1(\Omega)$. Let $\vec{H} : \Omega \to R^n$ be analytic on Ω. Then $\vec{H}$ determines a 2^{nd} order ordinary differential equation

(3.1.4) $$\ddot{\psi}^i = H^i(\vec{\psi}, \mathring{\vec{\psi}}, t) \quad \text{for } 1 \le i \le n .$$

By the fundamental theorem of the 2^{nd} order ordinary differential equation which we proved in chapter I, there exist neighbourhoods U of $\vec{P}_0$, V_0 of $\vec{0}$ in R^n such that for any initial point $\vec{P} = (p^1, \ldots, p^n)$ in U, initial vector $\vec{V} = (v^1, \ldots, v^n)$ in V_0, there exists an interval $I_{\vec{P},\vec{V}}$ in R such that $I_{\vec{P},\vec{V}}$ contains zero and there is a unique function $\vec{\psi}_{\vec{P},\vec{V}}$ defined on $I_{\vec{P},\vec{V}}$ into $\pi_1(\Omega)$ satisfying the differential equation (3.1.4) with $\vec{\psi}_{\vec{P},\vec{V}}(0) = \vec{P}$ and $\mathring{\vec{\psi}}_{\vec{P},\vec{V}}(0) = \vec{V}$. Write $\vec{\Phi}(\vec{P}, \vec{V}, t) = \vec{\psi}_{\vec{P},\vec{V}}(t)$.

The fundamental theorem also says that there exists an open set $V \subseteq R^{2n+1}$ such that $(\vec{P},\vec{V},0) \in V$ and the map $\vec{\Phi} : V \to \pi_1(\Omega)$ is analytic on V

From now on we shall write $\mathring{\Phi}^i(\vec{P},\vec{V},t) = \mathring{\psi}^i_{\vec{P},\vec{V}}(t)$ (i.e. we shall not use the partial derivative notation for $\phi^i(\vec{P},\vec{V},t)$)

Theorem 3-2

H Suppose there exists a neighbourhood W of (0,0) in R^2 and an analytic function $f : W \to R$ such that $\vec{\Phi}(\vec{P},\alpha\vec{V},t) = \vec{\Phi}(\vec{P},\vec{V},f(\alpha,t))$ whenever $(\vec{P},\alpha\vec{V},t) \in V$, $(\alpha,t) \in W$. Furthermore assume that $f(\alpha,0) = 0$, $f(0,t) = 0$ whenever defined.

C Then the differential equation must be either

$$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i, \quad c \neq 0 \quad \text{where}$$

$$f(\alpha,t) = \frac{1}{c}\ln(1-\alpha+\alpha e^{ct}) \text{ and } c = \frac{f_{tt}(\beta,0)}{\beta(1-\beta)} \;\forall (\beta,0) \in W,\ \beta \neq 0,1$$

or

$$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k \quad \text{where} \quad f(\alpha,t) = \alpha t$$

for all $i = 1,2,\ldots, n$.

Proof Since $\vec{H}$ is analytic on Ω, and $\vec{H}$ determine a 2nd order ordinary differential equation $\ddot{\psi}^i = H^i(\vec{\psi},\mathring{\vec{\psi}},t)$, $i = 1,\ldots, n$. For each $i = 1,2,\ldots, n$, let $H^i(\vec{\psi},\mathring{\vec{\psi}},t) = G^i(\vec{\psi}) + G^i_{j_1}(\vec{\psi})\mathring{\psi}^{j_1} + G^i_{j_1 j_2}(\vec{\psi})\mathring{\psi}^{j_1}\mathring{\psi}^{j_2} + \ldots$
$\ldots + G^i_{j_1\ldots j_k}(\vec{\psi})\mathring{\psi}^{j_1}\ldots\mathring{\psi}^{j_k} + \ldots +$

$$J^i_{j_1}(\vec{\psi})t^{j_1} + J^i_{j_1 j_2}(\vec{\psi})\mathring{\psi}^{j_1}t^{j_2} + \ldots + J^i_{j_1\ldots j_{k+1}}(\vec{\psi})\mathring{\psi}^{j_1}\ldots\mathring{\psi}^{j_k}t^{j_{k+1}} + \ldots .$$

Hence

$$(1)\quad \ddot{\psi}^i = G^i(\vec{\psi}) + G^i_{j_1}(\vec{\psi})\mathring{\psi}^{j_1} + G^i_{j_1j_2}(\vec{\psi})\mathring{\psi}^{j_1}\mathring{\psi}^{j_2} + \dots + G^i_{j_1\cdots j_k}(\vec{\psi})\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k} + \dots$$

$$+ J^i_{j_1}(\vec{\psi})t^{j_1} + J^i_{j_1j_2}(\vec{\psi})\mathring{\psi}^{j_1}t^{j_2} + \dots + J^i_{j_1\cdots j_{k+1}}(\vec{\psi})\mathring{\psi}^{j_1}\cdots\mathring{\psi}^{j_k}t^{j_{k+1}} + \dots$$

Since for each $i = 1,2,\dots, n$, Φ^i satisfies

$$(2)\quad \Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t)) \quad \text{for } (\vec{P},\alpha\vec{V},t) \in V,\ (\alpha,t) \in W.$$

Then $t = 0$, we get

$$\Phi^i(\vec{P},\alpha\vec{V},0) = \Phi^i(\vec{P},\vec{V},f(\alpha,0)) = \Phi^i(\vec{P},\vec{V},0) = p^i$$

Differentiate (2) with respect to t, we get

$$(3)\quad \mathring{\Phi}^i(\vec{P},\alpha\vec{V},t) = \mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))\, f_t(\alpha,t)$$

$t = 0,$

$$\mathring{\Phi}^i(\vec{P},\alpha\vec{V},0) = \mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,0))\, f_t(\alpha,0)$$

$$\alpha v^i = v^i f_t(\alpha,0) \quad \text{for all } \vec{V},\ (\vec{P},\alpha\vec{V},t) \in V$$

Hence

$$f_t(\alpha,0) = \alpha \quad \forall\, \alpha \quad \text{where } (\alpha,0) \in W.$$

Differentiate (3) with respect to t, we obtain

$$(4)\quad \ddot{\Phi}^i(\vec{P},\alpha\vec{V},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t) + \mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t)$$

Since Φ^i satisfies equation (1) for each $i = 1,2,\dots, n$.

Hence

$$\ddot{\Phi}^i(\vec{P},\alpha\vec{V},t) = G^i(\vec{\Phi}(\vec{P},\alpha\vec{V},t)) + G^i_{j_1}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\alpha\vec{V},t)$$

$$+ G^i_{j_1j_2}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\alpha\vec{V},t)+\ldots+ G^i_{j_1\ldots j_k}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\alpha\vec{V},t)+\ldots$$

$$+ J^i_{j_1}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))t^{j_1} + J^i_{j_1j_2}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\alpha\vec{V},t)t^{j_2} + \ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\alpha\vec{V},t)t^{j_{k+1}} + \ldots$$

$$\ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t)) = G^i(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t))+G^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))$$

$$+ G^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\vec{V},f(\alpha,t))+ \ldots$$

$$+ G^i_{j_1\ldots j_k}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\vec{V},f(\alpha,t)) + \ldots$$

$$+ J^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))t^{j_1} + J^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))t^{j_2}+ \ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\vec{V},f(\alpha,t))t^{j_{k+1}} + \ldots$$

Substitute these two equation into equation (4) :

$$(5)\quad G^i(\vec{\Phi}(\vec{P},\alpha\vec{V},t))+G^i_{j_1}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\alpha\vec{V},t)+G^i_{j_1j_2}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\alpha\vec{V},t)$$

$$+\ldots+ G^i_{j_1\ldots j_k}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\alpha\vec{V},t)+\ldots+ J^i_{j_1}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))t^{j_1}$$

$$+ J^i_{j_1j_2}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\alpha\vec{V},t)t^{j_2}+\ldots+ J^i_{j_1\ldots j_{k+1}}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\alpha\vec{V},t)t^{j_{k+1}}$$

$$+ \ldots.$$

$$= G^i(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))f_t^2(\alpha,t)+G^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)+$$

$$G^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t) + \ldots$$

$$+ G^i_{j_1\ldots j_k}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t) + \ldots$$

$$+ J^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))t^{j_1}f_t^2(\alpha,t)+J^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))t^{j_2}f_t^2(\alpha,t)+\ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\vec{V},f(\alpha,t))t^{j_{k+1}}f_t^2(\alpha,t)+\ldots$$

$$+ \mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t).$$

When $t = 0$, equation (5) becomes :

$$G^i(\vec{P})+G^i_{j_1}(\vec{P})\alpha v^{j_1}+G^i_{j_1j_2}(\vec{P})\alpha^2v^{j_1}v^{j_2}+\ldots+G^i_{j_1\ldots j_k}(\vec{P})\alpha^kv^{j_1}\ldots v^{j_k}+\ldots$$

$$= G^i(\vec{P})\alpha^2+G^i_{j_1}(\vec{P})v^{j_1}\alpha^2+G^i_{j_1j_2}(\vec{P})\alpha^2v^{j_1}v^{j_2}+\ldots+G^i_{j_1\ldots j_k}(\vec{P})\alpha^2v^{j_1}\ldots v^{j_k}+\ldots$$

$$+ v^if_{tt}(\alpha,0)$$

Hence $v^if_{tt}(\alpha,0) = (1-\alpha^2)G^i(\vec{P})+(\alpha-\alpha^2)G^i_{j_1}(\vec{P})v^{j_1}+(\alpha^3-\alpha^2)G^i_{j_1j_2j_3}(\vec{P})v^{j_1}v^{j_2}v^{j_3}$

$+\ldots+ (\alpha^k-\alpha^2)G^i_{j_1\ldots j_k}(\vec{P})v^{j_1}\ldots v^{j_k}+\ldots$ for all $(\vec{P},\alpha\vec{V},t) \in V$

and $(\alpha,t) \in W$

By corollary (1-? 5), we conclude that for all $i = 1,2,\ldots,n$ and for all sufficiently small α

$$(1-\alpha^2)\ G^i(\vec{P}) = 0$$

$$\alpha^2(1-\alpha)(1+\alpha+\alpha^2+\ldots+\alpha^{k-3})G^i_{j_1\ldots j_k}(\vec{P}) = 0 \quad \text{for all } k = 3,4,\ldots$$

$$\alpha(1-\alpha)G^i_{j_1}(\vec{P})v^{j_1} = v^i\ f_{tt}(\alpha,0)$$

Choose $\alpha > 0$, $\alpha \neq 1$ such that $(\alpha,t) \in W$.

Hence for $1 \leq i \leq n$, we get

$$(5.1) \qquad \begin{aligned} G^i(\vec{P}) &\equiv 0 \\ G^i_{j_1\ldots j_k}(\vec{P}) &\equiv 0 \quad , \quad k = 3,4,\ldots \end{aligned}$$

Then again we choose $\alpha_0 \neq 0,1$ such that $(\alpha_0,0) \in W$.

Hence

$$\alpha_0(1-\alpha_0)G^i_{j_1}(\vec{P})v^{j_1} = f_{tt}(\alpha_0,0)\delta^i_{j_1}v^{j_1} \quad \text{where } \delta^i_j = \begin{cases} 1, & j=i \\ 0, & j\neq i \end{cases}$$

Thus

$$G^i_{j_1}(\vec{P}) = \frac{f_{tt}(\alpha_0,0)}{\alpha_0(1-\alpha_0)}\ \delta^i_{j_1}\ .$$

Let $c = \dfrac{f_{tt}(\alpha_0,0)}{(1-\alpha_0)\alpha_0}$. Then c is independent of α_0 by theorem 1-3.7.

Hence $$G^i_{j_1}(\vec{P}) = \begin{cases} c\delta^i_{j_1} & , \ c \neq 0 \\ 0 & , \ c = 0\ . \end{cases}$$

<u>Case 1 $c \neq 0$</u>

Substitute (5.1) into (1) we obtain

$$\ddot{\psi}^i = c\dot{\psi}^i + G^i_{j_1j_2}(\vec{\psi})\dot{\psi}^{j_1}\dot{\psi}^{j_2} + J^i_{j_1}(\vec{\psi})t^{j_1} + J^i_{j_1j_2}(\vec{\psi})\dot{\psi}^{j_1}t^{j_2} + \ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_k}t^{j_{k+1}} + \ldots\ .$$

Differentiate equation (2) with respect to α :

$$(6) \qquad v^j \frac{\partial \Phi^i}{\partial v^j}(\vec{P},\alpha\vec{V},t) = \mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_\alpha(\alpha,t)$$

Differentiate equation (6) with respect to α :

$$(7) \quad v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\alpha\vec{V},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_\alpha^2(\alpha,t)+\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{\alpha\alpha}(\alpha,t)$$

Substitute $\alpha = 0$ into equation (2), we get

$$\Phi^i(\vec{P},\vec{0},t) = \Phi^i(\vec{P},\vec{V},0) = p^i \qquad (\text{since } f(0,t) = 0)$$

Hence

$$(8) \qquad \Phi^i(\vec{P},\vec{0},t) = p^i$$

$\alpha = 0$, equation (6) becomes

$$v^j \frac{\partial \Phi^i}{\partial v^j}(\vec{P},\vec{0},t) = \mathring{\Phi}^i(\vec{P},\vec{V},f(0,t))f_\alpha(0,t)$$

$$= \mathring{\Phi}^i(\vec{P},\vec{V},0)\, f_\alpha(0,t)$$

$$= v^j \delta^i_j\, f_\alpha(0,t)$$

Hence

$$(9) \qquad \frac{\partial \Phi^i}{\partial v^j}(\vec{P},\vec{0},t) = f_\alpha(0,t)\delta^i_j$$

Since

$$\ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t)) = c\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))+G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\vec{V},f(\alpha,t))$$

$$+ J^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))t^{j_1}+ J^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\,\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))t^{j_2}+ \ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\mathring{\Phi}^{j_1}\ldots\mathring{\Phi}^{j_k})(\vec{P},\vec{V},f(\alpha,t))t^{j_{k+1}}+ \ldots$$

Substitute this equation into equation (7), we get

$$(10) \qquad v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\alpha\vec{V},t) = f_\alpha^2(\alpha,t)c\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))$$

$$+ f_\alpha^2(\alpha,t)G^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\dot{\Phi}^{j_1}\dot{\Phi}^{j_2})(\vec{P},\vec{V},f(\alpha,t)) + f_\alpha^2(\alpha,t)J^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))t^{j_1}$$

$$+ f_\alpha^2(\alpha,t)J^i_{j_1j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\dot{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))t^{j_2} + \ldots$$

$$+ f_\alpha^2(\alpha,t)J^i_{j_1\cdots j_k}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\dot{\Phi}^{j_1}\ldots\dot{\Phi}^{j_{k-1}})(\vec{P},\vec{V},f(\alpha,t))t^{j_k} + \ldots$$

$$+ \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{\alpha\alpha}(\alpha,t)$$

$\alpha = 0$, equation (10) becomes

$$v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = f_\alpha^2(0,t)cv^j\delta^i_j + f_\alpha^2(0,t)\, G^i_{j_1j_2}(\vec{P})v^{j_1}v^{j_2}$$

$$+ f_\alpha^2(0,t)J^i_{j_1}(\vec{P})t^{j_1} + f_\alpha^2(0,t)J^i_{j_1j_2}(\vec{P})v^{j_1}t^{j_2} + f_\alpha^2(0,t)J^i_{j_1j_2j_3}(\vec{P})v^{j_1}v^{j_2}t^{j_3} + \ldots$$

$$+ f_\alpha^2(0,t)J^i_{j_1\ldots j_k}(\vec{P})v^{j_1}\ldots v^{j_{k-1}}t^{j_k} + \ldots + v^j\delta^i_j f_{\alpha\alpha}(0,t)$$

Rearrange terms, we obtain

$$(10.1) \qquad v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = f_\alpha^2(0,t)J^i_{j_1}(\vec{P})t^{j_1}$$

$$+ [(cf_\alpha^2(0,t)+f_{\alpha\alpha}(0,t))\delta^i_{j_1} + f_\alpha^2(0,t)t^{j_2}J^i_{j_1j_2}(\vec{P})]v^{j_1}$$

$$+ [f_\alpha^2(0,t)(G^i_{j_1j_2}(\vec{P})+J^i_{j_1j_2j_3}(\vec{P})t^{j_3})]\, v^{j_1}v^{j_2}$$

$$+ [f_\alpha^2(0,t)J^i_{j_1\cdots j_4}(\vec{P})t^{j_4}]v^{j_1}v^{j_2}v^{j_3}$$

$$\vdots$$

$$+ \left[f_\alpha^2(0,t) J^i_{j_1 \cdot\cdot j_{k+1}}(\vec{P})\, t^{j_{k+1}}\right] v^{j_1} \ldots v^{j_k}$$

$$\vdots$$

Next, we shall prove that

(a) $$G^i_{j_1 j_2}(\vec{P}) f_\alpha^2(0,t) = \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t)$$

(b) $$c f_\alpha^2(0,t) + f_{\alpha\alpha}(0,t) = 0$$

(c) $$\exists t_0 \neq 0 \,,\, f_\alpha(0,t_0) \neq 0.$$

Since $\ddot{\Phi}^i(\vec{P},\vec{V},t) = c\mathring{\Phi}^i(\vec{P},\vec{V},t) + G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},t))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\vec{V},t)$

$+ J^i_{j_1}(\vec{\Phi}(\vec{P},\vec{V},t))t^{j_1} + J^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},t)t^{j_2} + \ldots$

$t = 0$, we get

$$\ddot{\Phi}^i(\vec{P},\vec{V},0) = c\mathring{\Phi}^i(\vec{P},\vec{V},0) + G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},0))(\mathring{\Phi}^{j_1}\mathring{\Phi}^{j_2})(\vec{P},\vec{V},0)$$

(11) $$\ddot{\Phi}^i(\vec{P},\vec{V},0) = cv^i + G^i_{j_1 j_2}(\vec{P}) v^{j_1} v^{j_2}$$

Substitute $\alpha = 0$ into (7), we get

$$v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(0,t)) f_\alpha^2(0,t) + \mathring{\Phi}^i(\vec{P},\vec{V},f(0,t)) f_{\alpha\alpha}(0,t)$$

$$= \ddot{\Phi}^i(\vec{P},\vec{V},0) f_\alpha^2(0,t) + \mathring{\Phi}^i(\vec{P},\vec{V},0) f_{\alpha\alpha}(0,t).$$

Substitute (11) into the above equation we obtain

$$v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = [cv^i + G^i_{j_1 j_2}(\vec{P}) v^{j_1} v^{j_2}] f^2_\alpha(0,t) + v^i f_{\alpha\alpha}(0,t)$$

Hence

$$(12) \quad v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = [cf^2_\alpha(0,t) + f_{\alpha\alpha}(0,t)] v^i + G^i_{j_1 j_2}(\vec{P}) f^2_\alpha(0,t) v^{j_1} v^{j_2}$$

Thus, by corollary 1-3.5 we conclude that

$$G^i_{j_1 j_2}(\vec{P}) f^2_\alpha(0,t) = \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) \qquad \forall\, t$$

and $cf^2_\alpha(0,t) + f_{\alpha\alpha}(0,t) = 0 \qquad \forall\, t$

Then (a) and (b) are proved.

In order to prove (c), let us first prove the following lemmas.

Lemma 3-2.1 For each $i = 1,2,\ldots,n$; Φ^i satisfies the functional equation (2)

$$\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t))$$

then Φ^i satisfies equation

$$(13) \quad v^{j_1} \ldots v^{j_k} \frac{\partial^k \Phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1} \ldots \partial v^{j_k}} = f_\alpha(\alpha,t) \frac{\partial^{k-1} \dot{\Phi}^i}{\partial \alpha^{k-1}}(\vec{P},\vec{V},f(\alpha,t))$$

$$+ \binom{k-1}{1} \frac{\partial^{k-2} \dot{\Phi}^i}{\partial \alpha^{k-2}}(\vec{P},\vec{V},f(\alpha,t)) \frac{\partial f_\alpha(\alpha,t)}{\partial \alpha} + \binom{k-1}{2} \frac{\partial^{k-3} \dot{\Phi}^i}{\partial \alpha^{k-3}}(\vec{P},\vec{V},f(\alpha,t)) \frac{\partial^2 f_\alpha(\alpha,t)}{\partial \alpha^2}$$

$$+\binom{k-1}{3}\frac{\partial^{k-4}\mathring{\Phi}^i}{\partial\alpha^{k-4}}(\vec{P},\vec{V},f(\alpha,t))\frac{\partial^3 f_\alpha(\alpha,t)}{\partial\alpha^3}+\ldots+\binom{k-1}{k-2}\frac{\partial\mathring{\Phi}^i}{\partial\alpha}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^{k-2}f_\alpha(\alpha,t)}{\partial\alpha^{k-2}}$$

$$+\binom{k-1}{k-1}\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^{k-1}f_\alpha(\alpha,t)}{\partial\alpha^{k-1}}\quad \text{for all}\quad k = 2,3,\ldots$$

Proof. We will prove this lemma by induction on k. From equation (7), we get

$$v^{j_1}v^{j_2}\frac{\partial^2\Phi^i}{\partial v^j\partial v^k}(\vec{P},\alpha\vec{V},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_\alpha^2(\alpha,t)+\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{\alpha\alpha}(\alpha,t)$$

$$= f_\alpha(\alpha,t)\ \frac{\partial\mathring{\Phi}^i}{\partial\alpha}(\vec{P},\vec{V},f(\alpha,t))+\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{\alpha\alpha}(\alpha,t)$$

Hence equation (13) is true for k = 2.

Assume eq (13) is true for all k = 2,3,..., n.

Hence equation (13) is true for k = n, replacing k in eq (13) with n and call the new equation (13*).

Since

$$v^{j_1}v^{j_2}\ldots v^{j_{n+1}}\ \frac{\partial^{n+1}\Phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1}\ldots\partial v^{j_{n+1}}} = \frac{\partial}{\partial\alpha}\left(v^{j_1}\ldots v^{j_n}\ \frac{\partial^n\Phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1}\ldots\partial v^{j_n}}\right)$$

$$= \frac{\partial}{\partial\alpha}\ (\text{RHS of } (13^*))$$

Hence

$$v^{j_1}v^{j_2}\ldots v^{j_{n+1}}\ \frac{\partial^{n+1}\Phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1}\ldots\partial v^{j_{n+1}}} = f_\alpha(\alpha,t)\ \frac{\partial^n\mathring{\Phi}^i}{\partial\alpha^n}\ (\vec{P},\vec{V},f(\alpha,t))$$

$$+\left\{\binom{n-1}{0}\frac{\partial^{n-1}\mathring{\Phi}^i}{\partial\alpha^{n-1}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial f_\alpha(\alpha,t)}{\partial\alpha}+\binom{n-1}{1}\frac{\partial^{n-1}\mathring{\Phi}^i}{\partial\alpha^{n-1}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial f_\alpha(\alpha,t)}{\partial\alpha}\right\}$$

$$+\left\{\binom{n-1}{1}\frac{\partial^{n-2}\mathring{\Phi}^i}{\partial\alpha^{n-2}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^2 f_\alpha(\alpha,t)}{\partial\alpha^2}+\binom{n-1}{2}\frac{\partial^{n-2}\mathring{\Phi}^i}{\partial\alpha^{n-2}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^2 f_\alpha(\alpha,t)}{\partial\alpha^2}\right\}$$

$$+\left\{\binom{n-1}{2}\frac{\partial^{n-3}\mathring{\Phi}^i}{\partial\alpha^{n-3}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^3 f_\alpha(\alpha,t)}{\partial\alpha^3}+\binom{n-1}{3}\frac{\partial^{n-3}\mathring{\Phi}^i}{\partial\alpha^{n-3}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^3 f_\alpha(\alpha,t)}{\partial\alpha^3}\right\}$$

$$\vdots$$

$$+\left\{\binom{n-1}{n-2}\frac{\partial\mathring{\Phi}^i}{\partial\alpha}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^{n-1} f_\alpha(\alpha,t)}{\partial\alpha^{n-1}}+\binom{n-1}{n-1}\frac{\partial\mathring{\Phi}^i}{\partial\alpha}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^{n-1} f_\alpha(\alpha,t)}{\partial\alpha^{n-1}}\right\}$$

$$+\binom{n-1}{n-1}\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^n f_\alpha(\alpha,t)}{\partial\alpha^n}\ .$$

Note that $\binom{n}{k-1}+\binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!}+\frac{n!}{(n-k)!k!}$

$$= \frac{n!k + n!(n-k+1)}{(n-k+1)!\ k!}$$

$$= \frac{n!(n+1)}{(n-k+1)!k!} \qquad = \frac{(n+1)!}{(n-k+1)!k!}$$

Hence

$$\binom{n}{k-1}+\binom{n}{k} = \binom{n+1}{k}$$

Thus,

$$(14)\qquad v^{j_1}\ldots v^{j_{n+1}}\ \frac{\partial^{n+1}\Phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1}\ldots\partial v^{j_{n+1}}}$$

$$= f_\alpha(\alpha,t)\ \frac{\partial^n\mathring{\Phi}^i}{\partial\alpha^n}(\vec{P},\vec{V},f(\alpha,t)) + \binom{n}{1}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial f_\alpha(\alpha,t)}{\partial\alpha}$$

$$+\binom{n}{2}\frac{\partial^{n-2}\mathring{\Phi}^i}{\partial\alpha^{n-2}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^2 f_\alpha(\alpha,t)}{\partial\alpha^2}+\binom{n}{3}\frac{\partial^{n-3}\mathring{\Phi}^i}{\partial\alpha^{n-3}}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^3 f_\alpha(\alpha,t)}{\partial\alpha^3}$$

$$+\ldots+\binom{n}{n-1}\frac{\partial\mathring{\Phi}^i}{\partial\alpha}(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^{n-1} f_\alpha(\alpha,t)}{\partial\alpha^{n-1}}+\binom{n}{n}\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))\ \frac{\partial^n f_\alpha(\alpha,t)}{\partial\alpha^n}$$

Hence (13) is true for k = n+1. Therefore, (13) is true for all k = 2,3,4,.... The proof of the lemma is complete.

Lemma 3-2.2 If $f_\alpha(0,t) \equiv 0$, then $\dfrac{\partial^k f(0,t)}{\partial\alpha^k} \equiv 0$

for all k = 2,3,...

Proof We will prove this lemma by induction on k.

Assume $f_\alpha(0,t) \equiv 0$

k = 2, we get from equation (7) by substituting $\alpha = 0$:

$$v^{j_1} v^{j_2} \frac{\partial^2 \phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \partial v^{j_2}} = \ddot{\phi}^i(\vec{P},\vec{V},0) f_\alpha^2(0,t) + \dot{\phi}^i(\vec{P},\vec{V},0) f_{\alpha\alpha}(0,t).$$

Hence from the assumption we obtain

$$v^{j_1} v^{j_2} \frac{\partial^2 \phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \partial v^{j_2}} = v^i f_{\alpha\alpha}(0,t).$$

By corollary 1-35, we have $\dfrac{\partial^2 \phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \partial v^{j_2}} \equiv 0$ and $f_{\alpha\alpha}(0,t) \equiv 0$

Hence k = 2, $f_{\alpha\alpha}(0,t) = 0$.

Now let us assume that for all k = 2,3,...,n $\dfrac{\partial^k f(0,t)}{\partial\alpha^k} = 0$.

For k = n+1, equation (13) is true; hence

$$v^{j_1} \dots v^{j_{n+1}} \frac{\partial^{n+1} \phi^i(\vec{P},\alpha\vec{V},t)}{\partial v^{j_1} \dots \partial v^{j_{n+1}}} = f_\alpha(\alpha,t) \frac{\partial^n \dot{\phi}^i(\vec{P},\vec{V},f(\alpha,t))}{\partial\alpha^n}$$

$$+ \binom{n}{1} \frac{\partial^{n-1}}{\partial\alpha^{n-1}} \dot{\phi}^i(\vec{P},\vec{V},f(\alpha,t)) \frac{\partial f_\alpha(\alpha,t)}{\partial\alpha} + \binom{n}{2} \frac{\partial^{n-2} \dot{\phi}^i}{\partial\alpha^{n-2}} (\vec{P},\vec{V},f(\alpha,t)) \frac{\partial^2 f_\alpha(\alpha,t)}{\partial\alpha^2}$$

$$+ \dots + \binom{n}{n-1} \frac{\partial \dot{\phi}^i}{\partial\alpha} (\vec{P},\vec{V},f(\alpha,t)) \frac{\partial^{n-1} f_\alpha(\alpha,t)}{\partial\alpha^{n-1}} + \binom{n}{n} \dot{\phi}^i(\vec{P},\vec{V},f(\alpha,t)) \frac{\partial^n f_\alpha(\alpha,t)}{\partial\alpha^n}$$

Substitute $\alpha = 0$ into the above equation, we obtain

$$v^{j_1} \ldots v^{j_{n+1}} \frac{\partial^{n+1} \Phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \ldots \partial v^{j_{n+1}}} = f_\alpha(0,t) \frac{\partial^n \dot{\Phi}^i}{\partial \alpha^n}(\vec{P},\vec{V},f(0,t))$$

$$+ \binom{n}{1} \frac{\partial^{n-1} \dot{\Phi}^i}{\partial \alpha^{n-1}}(\vec{P},\vec{V},f(0,t)) \frac{\partial f_\alpha(0,t)}{\partial \alpha} + \binom{n}{2} \frac{\partial^{n-2} \dot{\Phi}^i}{\partial \alpha^{n-2}}(\vec{P},\vec{V},f(0,t)) \frac{\partial^2 f_\alpha(0,t)}{\partial \alpha^2}$$

$$+ \ldots + \binom{n}{n-1} \frac{\partial \dot{\Phi}^i}{\partial \alpha}(\vec{P},\vec{V},f(0,t)) \frac{\partial^{n-1} f_\alpha(0,t)}{\partial \alpha^{n-1}} + \binom{n}{n} \dot{\Phi}^i(\vec{P},\vec{V},f(0,t)) \frac{\partial^n f_\alpha(0,t)}{\partial \alpha^n}$$

By assumption, $f(0,t) \equiv 0,\ f_\alpha(0,t) \equiv \frac{\partial^2 f(0,t)}{\partial \alpha^2} \equiv \ldots \equiv \frac{\partial^n f(0,t)}{\partial \alpha^n} \equiv 0$

Hence

$$v^{j_1} \ldots v^{j_{n+1}} \frac{\partial^{n+1} \Phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \ldots \partial v^{j_{n+1}}} = v^i \frac{\partial^n f_\alpha(0,t)}{\partial \alpha^n} = v^i \frac{\partial^{n+1} f(0,t)}{\partial \alpha^{n+1}}$$

Thus, $\frac{\partial^{n+1} \Phi^i(\vec{P},\vec{0},t)}{\partial v^{j_1} \ldots \partial v^{j_{n+1}}} \equiv 0$ and $\frac{\partial^{n+1} f(0,t)}{\partial \alpha^{n+1}} \equiv 0$

That is, for $k = n+1$ $\frac{\partial^{n+1} f(0,t)}{\partial \alpha^{n+1}} \equiv 0$

Hence $\frac{\partial^k f(0,t)}{\partial \alpha^k} \equiv 0$ for all $k = 2,3,\ldots$

This completes the proof.

Lemma 3-2.3 If $f(\alpha,t)$ is analytic in a neighbourhood W of $(0,0)$, and $f(0,t) \equiv 0$, $\frac{\partial^n f(0,t)}{\partial \alpha^n} \equiv 0$, $n \in Z^+$, then $f(\alpha,t) \equiv 0$.

Proof Let t_0 be an arbitrary fixed element of $\pi_2(W)$ such that $f(\alpha,t_0)$ is defined in a neighbourhood of $\alpha = 0$.

Let $f(\alpha,t_0) = F_{t_0}(\alpha)$. In this case the subscript does not mean differentiate with respect to t_0. Hence $F_{t_0}(\alpha)$ is analytic.

Now, we apply the Taylor theorem for function of one variable to F_{t_0}.

This yields

$$F_{t_0}(\alpha) = F_{t_0}(0) + F'_{t_0}(0)\frac{\alpha}{1!} + F''_{t_0}(0)\frac{\alpha^2}{2!} + \dots + F^{(n)}_{t_0}(0)\frac{\alpha^n}{n!} + \dots$$

Hence, $f(\alpha,t_0) = f(0,t_0) + \frac{\partial f}{\partial\alpha}(0,t_0)\frac{\alpha}{1!} + \frac{\partial^2 f}{\partial\alpha^2}(0,t_0)\frac{\alpha^2}{2!} + \dots +$

$\frac{\partial^n f}{\partial\alpha^n}(0,t_0)\frac{\alpha^n}{n!} + \dots$

Thus, $f(\alpha,t_0) = 0$ for all α in neighbourhood of 0.

Since t_0 is arbitrary fixed, hence $f(\alpha,t) \equiv 0$.

This complete the proof.

Next, we will show that there exists t_0 such that $f_\alpha(0,t_0) \neq 0$.

Suppose $f_\alpha(0,t) \equiv 0$.

By lemma 3-2.2 and lemma 3-2.3, we conclude that $f(\alpha,t) \equiv 0$

Since $\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t))$.

Hence $\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},0) = p^i$

Thus $\dot{\Phi}^i(\vec{P},\alpha\vec{V},t) = 0$

Substituting $t = 0$, yields

$$\dot{\Phi}^i(\vec{P},\alpha\vec{V},0) = 0$$

Hence $\alpha v^i = 0 \qquad \forall\alpha \quad \forall\vec{V}$

This is a contradiction.

Therefore, our assumption that $f_\alpha(0,t) \equiv 0$ is false.

Hence there exists t_0 such that $f_\alpha(0,t_0) \neq 0$

We claim that this t_0 is not zero.

Let $\vec{U}_i$ be a unit vector in the direction of i, denoted by $\vec{U}_i = (0,0,\ldots,1^{i\text{ th}}, 0,\ldots,0)$.

Since from equation (8), we have $\phi^i(\vec{P},\vec{0},t) = p^i$

Hence

$$\phi^i(\vec{P},h\vec{U}_i,t)-\phi^i(\vec{P},\vec{0},t) = p^i - p^i = 0 \qquad \text{if } t = 0$$

Thus, $$\lim_{h\to 0} \frac{\phi^i(\vec{P},h\vec{U}_i,t)-\phi^i(\vec{P},\vec{0},t)}{h} = 0 \qquad \text{if } t = 0$$

Hence, $$\frac{\partial \Phi^i}{\partial v^i}(\vec{P},\vec{0},t) = 0 \qquad \text{if } t = 0$$

But from equation (9), we have $f_\alpha(0,t) = \frac{\partial \Phi^i}{\partial v^i}(\vec{P},\vec{0},t)$

Hence $$f_\alpha(0,t) = 0 \qquad \text{if } t = 0 .$$

But $f_\alpha(0,t_0) \neq 0$. Therefore, there exists $t_0 \neq 0$ such that $f_\alpha(0,t_0) \neq 0$.

Then, (c) is proved.

Substitute (a), (b) and (c) into equation (10.1), we obtain

$$J^i_{j_1}(\vec{P})t_0^{j_1} + J^i_{j_1 j_2}(\vec{P})t_0^{j_2}v^{j_1} + J^i_{j_1 j_2 j_3}(\vec{P})t_0^{j_3}v^{j_1}v^{j_2} + \ldots$$

$$+ J^i_{j_1\ldots j_{k+1}}(\vec{P})t_0^{j_{k+1}}v^{j_1}\ldots v^{j_k} + \ldots \equiv 0$$

Hence $$J^i_{j_1\ldots j_k}(\vec{P}) \equiv 0 \qquad \forall\ k = 1,2,3,\ldots$$

Thus, for case $c \neq 0$ we obtain

$$\ddot{\psi}^i = c\dot{\psi}^i + G^i_{jk}(\vec{\psi})\,\dot{\psi}^j\dot{\psi}^k .$$

Case 2 $c = 0$

Substitute $c = 0$ into equation (10.1), we obtain

$$(10.2)\qquad v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = f^2_\alpha(0,t) J^i_{j_1}(\vec{P}) t^{j_1}$$

$$+ [f_{\alpha\alpha}(0,t)\delta^i_{j_1} + f^2_\alpha(0,t) t^{j_2} J^i_{j_1 j_2}(\vec{P})]\, v^{j_1}$$

$$+ [f^2_\alpha(0,t)(G^i_{j_1 j_2}(\vec{P}) + J^i_{j_1 j_2 j_3}(\vec{P}) t^{j_3})]\, v^{j_1} v^{j_2}$$

$$+ [f^2_\alpha(0,t) J^i_{j_1 \cdots j_4}(\vec{P}) t^{j_4}]\, v^{j_1} v^{j_2} v^{j_3}$$

$$\vdots$$

$$+ [f^2_\alpha(0,t) J^i_{j_1 \cdots j_{k+1}}(\vec{P}) t^{j_{k+1}}]\, v^{j_1} \cdots v^{j_k}$$

$$\vdots$$

Since from equation (12) when $c = 0$, we have

$$v^j v^k \frac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t) = f_{\alpha\alpha}(0,t) v^i + G^i_{j_1 j_2}(\vec{P}) f^2_\alpha(0,t) v^{j_1} v^{j_2}$$

Hence $\quad f_{\alpha\alpha}(0,t) = 0, \quad G^i_{j_1 j_2}(\vec{P}) f^2_\alpha(0,t) = \dfrac{\partial^2 \Phi^i}{\partial v^j \partial v^k}(\vec{P},\vec{0},t)$

Substitute these two equations into (10.2), and since we have already proved that $\exists t_0 \neq 0 \ni f_\alpha(0,t_0) \neq 0$, so we get

$$f_\alpha^2(0,t_0)J^i_{j_1}(\vec{P})t_0^{j_1} + f_\alpha^2(0,t_0)t_0^{j_2}J^i_{j_1j_2}(\vec{P})v^{j_1} + f_\alpha^2(0,t_0)J^i_{j_1\cdots j_3}(\vec{P})t_0^{j_3}v^{j_1}v^{j_2}$$

$$+ f_\alpha^2(0,t_0)J^i_{j_1\cdots j_4}(\vec{P})t_0^{j_4}v^{j_1}v^{j_2}v^{j_3} + \ldots + f_\alpha^2(0,t_0)J^i_{j_1\cdots j_{k+1}}(\vec{P})t_0^{j_{k+1}}v^{j_1}\ldots v^{j_k} + \ldots$$

$$\equiv 0$$

Hence $\quad 0 \equiv J^i_{j_1}(\vec{P}) = J^i_{j_1j_2}(\vec{P}) = \ldots = J^i_{j_1\cdots j_k}(\vec{P}) = \ldots$

Thus, $\quad \ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$.

To complete the proof of the theorem, we must show the following :

If Φ^i satisfies the second order ordinary differential equation of the form $\ddot{\Phi}^i = c\dot{\Phi}^i + G^i_{j_1j_2}(\vec{\Phi})\dot{\Phi}^{j_1}\dot{\Phi}^{j_2}$, $c \neq 0$. and the functional equation

$$\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t))$$

where $\Phi^i(\vec{P},\vec{V},0) = p^i$, $\dot{\Phi}^i(\vec{P},\vec{V},0) = v^i$, then

$$f(\alpha,t) = \frac{1}{c}\ln(1 - \alpha + \alpha e^{ct}), \quad c \neq 0.$$

Proof Assume $\ddot{\Phi}^i = c\dot{\Phi}^i + G^i_{j_1j_2}(\vec{\Phi})\dot{\Phi}^{j_1}\dot{\Phi}^{j_2}$ for all $i = 1,2,\ldots,n$

From equation (4), we have

$$(4) \quad \ddot{\Phi}^i(\vec{P},\alpha\vec{V},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t) + \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t)$$

Hence

$$\text{LHS of (4)} = c\mathring{\Phi}^i(\vec{P},\alpha\vec{V},t) + G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\mathring{\Phi}^{j_1}(\vec{P},\alpha\vec{V},t)\mathring{\Phi}^{j_2}(\vec{P},\alpha\vec{V},t)$$

$$= c\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t(\alpha,t) + G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\mathring{\Phi}^{j_1}(\vec{P},\vec{V},f(\alpha,t))\dot{\Phi}^{j_2}(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t).$$

$$\text{RHS of (4)} = c\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t) + G^i_{j_1 j_2}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))(\dot{\Phi}^{j_1}\dot{\Phi}^{j_2})(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)$$

$$+ \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t).$$

$$\text{LHS} - \text{RHS of (4)} \equiv 0$$

Hence

$$c\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t(\alpha,t)(1-f_t(\alpha,t)) - \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t) \equiv 0$$

$$[\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))][cf_t(\alpha,t)(1-f_t(\alpha,t)) - f_{tt}(\alpha,t)] \equiv 0$$

Suppose $\mathring{\Phi}^i(\vec{P},\vec{V},f(\alpha,t)) \equiv 0$

$t = 0$, $\quad \mathring{\Phi}^i(\vec{P},\vec{V},0) = 0$

Hence, $\quad v^i = 0$ for all $\vec{V}$.

This is a contradiction .

Thus, $\quad cf_t(\alpha,t)(1-f_t(\alpha,t)) - f_{tt}(\alpha,t) = 0 \qquad$ (by theorem 1-3.1)

Now, we are solving for $f(\alpha,t)$ from the initial value problem :

$$f_{tt}(\alpha,t) - cf_t(\alpha,t) + cf_t^2(\alpha,t) \equiv 0, \quad c \neq 0$$

where $f(\alpha,0) = 0$, $f_t(\alpha,0) = \alpha \quad \forall\, (\alpha,0) \in W$.

Fix $\alpha_0 \neq 0$ and $\alpha_0 \neq 1$

Case 1 $0 < \alpha_0 < 1$

Let $\eta_{\alpha_0}(t) = f_t(\alpha_0,t)$ where again η_{α_0} in this case does not mean differentiate with respect to α_0 .

Since $f_{tt}(\alpha_0,t) - cf_t(\alpha_0,t) + cf_t^2(\alpha_0,t) = 0, \quad c \neq 0$

Hence

$$\dot{\eta}_{\alpha_0}(t) - c\eta_{\alpha_0}(t) + c\eta_{\alpha_0}^2(t) = 0 \quad c \neq 0$$

$$\dot{\eta}_{\alpha_0}(t) = c\eta_{\alpha_0}(t) - c\eta_{\alpha_0}^2(t)$$

$$\frac{d\eta_{\alpha_0}(t)}{c\eta_{\alpha_0}(t) - c\eta_{\alpha_0}^2(t)} = dt$$

(15) $$\frac{1}{c}\left(\frac{1}{\eta_{\alpha_0}} + \frac{1}{1-\eta_{\alpha_0}}\right)d\eta_{\alpha_0} = dt$$

It is clear that $\eta_{\alpha_0} \neq 0$ and $1-\eta_{\alpha_0} \neq 0$.

Since $f_t(\alpha_0,t)$ is continuous in t and for all sufficient small t. Hence $f_t(\alpha_0,t)$ is continuous at 0. Thus, $\lim_{t\to 0} f_t(\alpha_0,t) = f_t(\alpha_0,0)$ $= \alpha_0 \neq 0, 1.$

If $f_t(\alpha_0,t) = 0 \ \forall t$ sufficiently small, then $\lim_{t\to 0} f_t(\alpha_0,t) = 0.$ A contradiction .

If $f_t(\alpha_0,t) = 1 \ \forall t$ sufficiently small, then $\lim_{t\to 0} f_t(\alpha_0,t) = 1.$ A contradiction .

Therefore, $\eta_{\alpha_0}(t) = f_t(\alpha_0,t) \neq 0,1$.

Integrate (15) both sides :

$$\int \frac{1}{c}\left(\frac{1}{\eta_{\alpha_0}} + \frac{1}{1-\eta_{\alpha_0}}\right) d\eta_{\alpha_0} = \int dt$$

$$\frac{1}{c}\,[\ell n\ \eta_{\alpha_0}(t) - \ell n(1-\eta_{\alpha_0}(t))] = t + g(\alpha_0).$$

$$\ell n\left(\frac{\eta_{\alpha_0}(t)}{1-\eta_{\alpha_0}(t)}\right) = ct + cg(\alpha_0)$$

$$\frac{\eta_{\alpha_0}(t)}{1-\eta_{\alpha_0}(t)} = e^{ct}.\ e^{cg(\alpha_0)}$$

$$\eta_{\alpha_0}(t) = e^{ct}.\ e^{cg(\alpha_0)} - \eta_{\alpha_0}(t)e^{ct}e^{cg(\alpha_0)}$$

$$\eta_{\alpha_0}(t)(1+e^{ct}\ e^{cg(\alpha_0)}) = e^{ct}\ e^{cg(\alpha_0)}$$

$$\eta_{\alpha_0}(t) = \frac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$$

Hence

$$(16) \qquad f_t(\alpha_0,t) = \frac{e^{ct}\ e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$$

$$f_t(\alpha_0,0) = \frac{e^{cg(\alpha_0)}}{1+e^{cg(\alpha_0)}}$$

$$(17) \qquad \alpha_0 = \frac{e^{cg(\alpha_0)}}{1+e^{cg(\alpha_0)}}$$

$$\alpha_0+\alpha_0 e^{cg(\alpha_0)} - e^{cg(\alpha_0)} = 0$$

$$\alpha_0 + (\alpha_0-1)\ e^{cg(\alpha_0)} = 0$$

$$(18) \qquad e^{cg(\alpha_0)} = \frac{\alpha_0}{1-\alpha_0}$$

From equation (16), $\frac{d}{dt} f(\alpha_0,t) = \frac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$

$$\int d\, f(\alpha_0,t) = \int \frac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}\, dt$$

$$= \frac{1}{c}\int\frac{d(e^{ct}e^{cg(\alpha_0)})}{1+e^{ct}e^{cg(\alpha_0)}}$$

Hence

(19) $$f(\alpha_0,t) = \frac{1}{c}\ln(1+e^{ct}e^{cg(\alpha_0)}) + h(\alpha_0)$$

$$0 = f(\alpha_0,0) = \frac{1}{c}\ln(1+e^{cg(\alpha_0)}) + h(\alpha_0)$$

Thus,

(20) $$h(\alpha_0) = -\frac{1}{c}\ln(1+e^{cg(\alpha_0)})$$

Substitute (20) into (19) we obtain,

$$f(\alpha_0,t) = \frac{1}{c}\ln(1+e^{ct}e^{cg(\alpha_0)}) - \frac{1}{c}\ln(1+e^{cg(\alpha_0)})$$

$$= \frac{1}{c}\ln\left(\frac{1+e^{ct}e^{cg(\alpha_0)}}{1+e^{cg(\alpha_0)}}\right)$$

Hence

(21) $$f(\alpha_0,t) = \frac{1}{c}\ln\left[\frac{1}{1+e^{cg(\alpha_0)}} + e^{ct}\frac{e^{cg(\alpha_0)}}{1+e^{cg(\alpha_0)}}\right]$$

Substitute (17),(18) into (21), we obtain

$$f(\alpha_0,t) = \frac{1}{c}\ln\left[\frac{1}{1+\frac{\alpha_0}{1-\alpha_0}} + e^{ct}\alpha_0\right]$$

Thus, $$f(\alpha_0,t) = \frac{1}{c}\ln\,[1-\alpha_0+\alpha_0 e^{ct}]$$

Case 2 $-1 < \alpha_0 < 0$

By continuity of $\eta_{\alpha_0}(t)$ at $t = 0$, we have $\eta_{\alpha_0}(t) < 0$ in some small neighbourhood of $t = 0$.

Let $\eta_{\alpha_0}(t) = -\beta(t)$ where $\beta(t) > 0 \quad \forall t$

Hence $$\mathring{\eta}_{\alpha_0}(t) = -\dot{\beta}(t)$$

Substitute η_{α_0} and $\mathring{\eta}_{\alpha_0}$ into the equation $f_{tt}(\alpha,t)-cf_t(\alpha,t)+cf_t^2(\alpha,t) = 0$ we obtain a first order ordinary differential equation

$$-\dot{\beta}(t)+c\beta(t)+c\beta^2(t) = 0$$

$$-\frac{d\beta}{dt} + c\beta(1+\beta) = 0$$

$$\frac{d\beta}{\beta(1+\beta)} = c\ dt$$

$$[\frac{1}{\beta} - \frac{1}{1+\beta}]\ d\beta = c\ dt$$

Integrate both sides we get

$$\ln \beta - \ln(1+\beta) = ct + c_1$$

$$\ln \frac{\beta}{1+\beta} = ct + c_1$$

$$\frac{\beta}{1+\beta} = e^{ct+c_1}$$

$$\beta = c_2e^{ct} + \beta c_2e^{ct}$$

$$\beta(1-c_2e^{ct}) = c_2e^{ct}$$

$$-\eta_{\alpha_0}(t) = \beta = \frac{c_2e^{ct}}{1-c_2e^{ct}}$$

Hence

$$(22) \qquad -f_t(\alpha_0,t) = \frac{c_2e^{ct}}{1-c_2e^{ct}}$$

$$t = 0, \quad -\alpha_0 = -f_t(\alpha_0,0) = \frac{c_2}{1-c_2}$$

$$c_2 = -\alpha_0 + \alpha_0 c_2$$

$$c_2(1-\alpha_0) = -\alpha_0$$

$$c_2 = \frac{-\alpha_0}{1-\alpha_0} > 0$$

From (22), $$\frac{df}{dt}(\alpha_0,t) = -\frac{c_2e^{ct}}{1-c_2e^{ct}}$$

Hence $$\int df(\alpha_0,t) = \int -\frac{c_2e^{ct}}{1-c_2e^{ct}}\,dt$$

$$\int df(\alpha_0,t) = \frac{1}{c}\int \frac{d\,(-c_2e^{ct})}{1-c_2e^{ct}}, \qquad c \neq 0$$

$$(23) \qquad f(\alpha_0,t) = \frac{1}{c}\ln(1-c_2e^{ct}) + c_3$$

Since $$0 < c_2 = \frac{-\alpha_0}{1-\alpha_0} < 1$$

Hence $0 > -c_2 > -1$

And since the exponential map e^{ct} is continuous at $t = 0$ that is as $t \to 0$, $e^{ct} \to e^0 = 1$

Hence $0 > -c_2e^{ct} > -1$ for all t in some small neighbourhood of 0.

Thus, $1 > 1-c_2e^{ct} > 0$

Therefore, $\ell n(1-c_2e^{ct})$ is defined .

Substitute $c_2 = \frac{-\alpha_0}{1-\alpha_0}$ into (23) we obtain

$$f(\alpha_0,t) = \frac{1}{c}\,\ell n(1+\frac{\alpha_0}{1-\alpha_0}\,e^{ct})+ c_3$$

$$t = 0, \quad 0 = f(\alpha_0,0) = \frac{1}{c}\,\ell n(1+\frac{\alpha_0}{1-\alpha_0})+ c_3$$

Hence,

$$c_3 = -\frac{1}{c}\,\ell n(1+\frac{\alpha_0}{1-\alpha_0})$$

(24)

$$c_3 = -\frac{1}{c}\,\ell n(\frac{1}{1-\alpha_0})$$

Substitute (24) into (23), we get

$$f(\alpha_0,t) = \frac{1}{c}\,\ell n[1+\frac{\alpha_0}{1-\alpha_0}\,e^{ct}]-\frac{1}{c}\,\ell n(\frac{1}{1-\alpha_0})$$

$$= \frac{1}{c}\,\ell n\left[\frac{1+\frac{\alpha_0}{1-\alpha_0}\,e^{ct}}{\frac{1}{1-\alpha_0}}\right]$$

Hence

$$f(\alpha_0,t) = \frac{1}{c}\,\ell n[1-\alpha_0+\alpha_0\,e^{ct}], \quad c \neq 0.$$

Consider $\alpha_0 = 0$, since $f(0,t) = \frac{1}{c}\,\ell n\,[1] = 0$

hence f is also defined at $\alpha_0 = 0$.

Therefore, $f(\alpha,t) = \frac{1}{c}\,\ell n[1-\alpha+\alpha e^{ct}]\ \forall\,\alpha \in (-1,1)\ \forall\, t$ sufficiently small

We then claim that this function f is the unique solution of

$$f_{tt}(\alpha,t)- cf_t(\alpha,t)+ cf_t^2(\alpha,t) = 0$$

Since $\eta_{\alpha_0} = f_t$, $\mathring{\eta}_{\alpha_0} = f_{tt}$

$$\mathring{\eta}_{\alpha_0} = c\eta_{\alpha_0} - c\eta^2_{\alpha_0}$$

Let $H(t) = c\eta_{\alpha_0} - c\eta^2_{\alpha_0}$

Since f is analytic, hence f_t is analytic.

So $c\eta_{\alpha_0}$, $c\eta^2_{\alpha_0}$, $c\eta_{\alpha_0} - c\eta^2_{\alpha_0}$ are analytic.

Hence, H must be an analytic function.

Therefore, the initial value problem

$$\mathring{\eta}_{\alpha_0} = H(t)$$

$$\eta_{\alpha_0}(0) = \alpha_0 \qquad \forall\ \alpha_0 \in (-1,1)$$

has a unique solution $\eta_{\alpha_0}(t) = \dfrac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$

Hence $f_t(\alpha_0,t) = \dfrac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$ which is analytic in t

So we have I.V.P. $f_t(\alpha_0,t) = \dfrac{e^{ct}e^{cg(\alpha_0)}}{1+e^{ct}e^{cg(\alpha_0)}}$

$$f(\alpha_0,0) = 0 \qquad \forall\ \alpha_0 \in (-1,1)$$

By fundamental theorem, this I.V.P. must have a unique solution which is

$$f(\alpha_0,t) = \frac{1}{c}\,\ell n[1-\alpha_0+\alpha_0\ e^{ct}] \quad \text{for all}$$

$\alpha_0 \in (-1,1)$ and for all t in some small neighbourhood of 0.

This completes the proof.

If Φ^i satisfies the 2^{nd} order ordinary differential equation

$$\ddot{\Phi}^i = G^i_{jk}(\vec{\Phi})\dot{\Phi}^j\dot{\Phi}^k \qquad \text{for all } i = 1,2,\ldots,n$$

and the functional equation $\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t))$

then $f(\alpha,t) = \alpha t$

<u>Proof</u>. Since $\Phi^i(\vec{P},\alpha\vec{V},t) = \Phi^i(\vec{P},\vec{V},f(\alpha,t))$

(a) $$\dot{\Phi}^i(\vec{P},\alpha\vec{V},t) = \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t(\alpha,t)$$

(b) $$\ddot{\Phi}^i(\vec{P},\alpha\vec{V},t) = \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)+\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t).$$

$$\begin{aligned}\text{LHS.(b)} &= \ddot{\Phi}^i(\vec{P},\alpha\vec{V},t)\\ &= G^i_{jk}(\vec{\Phi}(\vec{P},\alpha\vec{V},t))\dot{\Phi}^j(\vec{P},\alpha\vec{V},t)\dot{\Phi}^k(\vec{P},\alpha\vec{V},t)\\ &= G^i_{jk}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\dot{\Phi}^j(\vec{P},\vec{V},f(\alpha,t))\dot{\Phi}^k(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)\end{aligned}$$

$$\begin{aligned}\text{RHS.(b)} &= \ddot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)+\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t)\\ &= G^i_{jk}(\vec{\Phi}(\vec{P},\vec{V},f(\alpha,t)))\dot{\Phi}^j(\vec{P},\vec{V},f(\alpha,t))\dot{\Phi}^k(\vec{P},\vec{V},f(\alpha,t))f_t^2(\alpha,t)\\ &\quad + \dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t)\end{aligned}$$

$$\text{RHS} - \text{LHS of (b)} = 0$$

$$\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))f_{tt}(\alpha,t) = 0$$

We have already proved that $\dot{\Phi}^i(\vec{P},\vec{V},f(\alpha,t))$ is not identically equal zero. So we conclude that $f_{tt}(\alpha,t) \equiv 0$

$$f_{tt}(\alpha,t) = 0$$

$f_t(\alpha,t) = g(\alpha)$ where g is arbitrary function of α

$t = 0, \quad \alpha = f_t(\alpha,0) = g(\alpha)$

Hence $\quad f_t(\alpha,t) = \alpha$

$f(\alpha,t) = \alpha t + h(\alpha)$ where h is arbitrary

function of α

$t = 0, \quad 0 = f(\alpha,0) = h(\alpha)$

Hence, $\quad f(\alpha,t) = \alpha t$

<u>Converse to theorem 3-2</u>

For each $i = 1,2,\ldots,n$ if ψ^i satisfies the second order ordinary differential equation of the type $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i$ where the function G^i_{jk} is c^1 on open subset D of R^n for $j,k = 1,2,\ldots,n$, then for all $\vec{P} \in D$, $\vec{V} \in R^n$, $\alpha \in R$,
the solution ψ^i must satisfy the functional equation

$$\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},f(\alpha,t)) \qquad \forall t \in J(0) \text{ which}$$

is an open interval of zero in R when $f(\alpha,t) = \frac{1}{c}\ln(1-\alpha+\alpha e^{ct})$ when $c \neq 0$ or $\psi^i(\vec{P},\alpha\vec{V},t)$ exists if and only if $\psi^i(\vec{P},\vec{V},\alpha t)$ exists and

$$\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t) \qquad \text{when } c = 0 \text{ and } G^i_{jk} \text{ is}$$

analytic on D.

<u>Proof</u> Let $H^i(\vec{\psi},\dot{\vec{\psi}},t) = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i$ where G^i_{jk} is c^1 on $D \subsetneq R^n$, $c \neq 0$.

Hence H^i is defined on $D \times R^n \times R$ and H^i is c^1 on $D \times R^n \times R$.

Fix $\vec{P}_0 \in D$, $\vec{V}_0 \in R^n$ and $\alpha_0 \in R$.

Since $(\vec{P}_0,\alpha_0\vec{V}_0,0)$ is in the domain of difinition of H^i, hence by the fundamental theorem for 2^{nd} order o.d.e. there exists a neighbourhood $I_1(0)$ of zero in R such that $\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$ satisfies the differential equation $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i$ with initial conditions

$$\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = p^i_0 \ , \quad \mathring{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = \alpha_0 v^i_0$$

(1) Let $F^i(t) = \psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$. Then F^i is defined on $I_1(0)$ Since $(\vec{P}_0,\vec{V}_0,0)$ is in the domain of definition of H^i, hence by the fundamental theorem for 2^{nd} order o.d.e there exists a neighbourhood $I_2(0)$ of zero in R such that $\psi^i(\vec{P}_0,\vec{V}_0,t)$ satisfies the differential equation for all $t \in I_2(0)$ with initial conditions

$$\psi^i(\vec{P}_0,\vec{V}_0,0) = p^i_0, \quad \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,0) = v^i_0$$

(2) Let $G^i(t) = \psi^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))$ where t is such that $f(\alpha_0,t) \in I_2(0)$.

Since $f(\alpha_0,t) = \frac{1}{c}\ln(1-\alpha_0+\alpha_0 e^{ct})$, hence $f(\alpha_0,t)$ is continuous at $t = 0$ and $f(\alpha_0,0) = 0$.

Therefore, there is a neighbourhood $I_3(0)$ of zero in R such that $f(\alpha_0,t) \in I_2(0)$ for all $t \in I_3(0)$.

Hence $G^i(t)$ is defined for all $t \in I_3(0)$

Substitute $t = 0$ into (1) and (2), we get

$$F^i(0) = \psi^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = p^i_0$$

$$G^i(0) = \psi^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,0)) = \psi^i(\vec{P}_0,\vec{V}_0,0) = p^i_0$$

Then $$F^i(0) = G^i(0) = p^i_0 \qquad (3)$$

Differentiate (1), (2) with respect to t, we get

$$\mathring{F}^i(t) = \mathring{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$$

$$\mathring{G}^i(t) = \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0 t))f_t(\alpha_0,t)$$

Substitute t = 0 into the above equations, we get

$$\mathring{F}^i(0) = \mathring{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = \alpha_0 v_0^i$$

$$\mathring{G}^i(0) = \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,0))f_t(\alpha_0,0)$$

$$= \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,0)\alpha_0 = \alpha_0 v_0^i$$

Then $$\mathring{F}^i(0) = \mathring{G}^i(0) = \alpha_0 v_0^i \qquad (4)$$

Since

$$\ddot{F}^i(t) = \ddot{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,t) \quad \text{for all } t \in I_1(0)$$

Hence

$$\ddot{F}^i(t) = G^i_{jk}(\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t))(\mathring{\psi}^j\mathring{\psi}^k)(\vec{P}_0,\alpha_0\vec{V}_0,t)+c\mathring{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$$

for all $t \in I_1(0)$

Thus $$\ddot{F}^i(t) = G^i_{jk}(\vec{F}(t))(\mathring{F}^j\mathring{F}^k)(t)+c\mathring{F}^i(t)$$

We will prove that G^i satisfies the same type of the 2nd order o.d.e and there are two cases $\alpha_0 \neq 0$ or $\alpha_0 = 0$ to be considered

Case 1 $\alpha_0 \neq 0$

Differentiate $\mathring{G}^i(t) = \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t)f_t(\alpha_0,t)$ with respect to t, we get

$$\forall t\in I_3(0),\ \ddot{G}^i(t) = \ddot{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))f_t^2(\alpha_0,t)+\mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0 t))f_{tt}(\alpha_0,t)$$

$$= [G^i_{jk}(\vec{\psi}(\vec{P}_0,\vec{V}_0,f(\alpha_0,t)))(\mathring{\psi}^j\mathring{\psi}^k)(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))+c\mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))]f_t^2(\alpha_0,t)$$

$$+ \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))f_{tt}(\alpha_0,t)$$

$$= G^i_{jk}(\vec{G}(t))(\mathring{G}^j\mathring{G}^k)(t)+c\mathring{G}^i(t)f_t(\alpha_0,t)+c\mathring{G}^i(t)\ \frac{f_{tt}(\alpha_0,t)}{cf_t(\alpha_0,t)}$$

(since $f_t(\alpha_0,t) = 0$ only when $\alpha_0 = 0$, so we can divide by $f_t(\alpha_0,t)$)

$$\ddot{G}^i(t) = G^i_{jk}(\vec{G}(t))(\mathring{G}^j\mathring{G}^k)(t)+c\mathring{G}^i(t)\left(f_t(\alpha_0,t)+ \frac{f_{tt}(\alpha_0,t)}{cf_t(\alpha_0,t)}\right)$$

Since we have already proved in page 45 that

$$f_{tt}(\alpha,t)+cf_t^2(\alpha,t) = cf_t(\alpha,t)$$

Hence
$$f_t(\alpha_0,t) + \frac{f_{tt}(\alpha_0,t)}{cf_t(\alpha_0,t)} = 1$$

Therefore, $\ddot{G}^i(t) = G^i_{jk}(\vec{G}(t))(\mathring{G}^j\mathring{G}^k)(t)+c\mathring{G}^i(t) \qquad \forall\ t \in I_3(0).$

Case 2 $\alpha_0 = 0$

It is easily shown that $f_t(0,t) = 0$ for all $t \in I_3(0)$.

Since $\mathring{G}^i(t) = \mathring{\psi}^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t))f_t(\alpha_0,t)$

Then $\mathring{G}^i(t) = 0 \qquad \forall\ t \in I_3(0)$

$\ddot{G}^i(t) = 0 \qquad \forall\ t \in I_3(0)$

$G^i_{jk}(\vec{G}(t))(\mathring{G}^j\mathring{G}^k)(t)+c\mathring{G}^i(t) = 0 \qquad \forall\ t \in I_3(0)$

Hence $\ddot{G}^i(t) = G^i_{jk}(\vec{G}(t))(\mathring{G}^j\mathring{G}^k)(t)+c\mathring{G}^i(t) \qquad \forall\ t \in I_3(0)$

Therefore, for all $\alpha_0 \in R$, $F^i(t)$ and $G^i(t)$ satisfy the same 2^{nd} order ordinary differential equation in a neighbourhood $I(0) = I_1 \cap I_3\ (0)$ and also satisfy the same initials conditions (3), (4) .

Since for each $i = 1,2,\ldots,n$ $F^i(t)$ and $G^i(t)$ satisfy the same 2^{nd} order o.d.e

$$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i \quad \text{where } G^i_{jk} \text{ is } c^1 \text{ for all } j,k = 1,\ldots,n$$

which is a c^1 differentiable equation.

Hence by theorem 1-1.11, $F^i(t) = G^i(t) \quad \forall t \in I\ (0)$

That is $\quad \psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t) = \psi^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t)) \quad \forall t \in I\ (0)$.

Then the proof for the case $c \neq 0$ is complete.

Now, we will prove that if ψ^i is a solution to the o.d.e. $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ for $1 \le i \le n$ where G^i_{jk} is analytic on D then we have $\psi^i(\vec{P},\alpha\vec{V},t)$ exists if and only if $\psi^i(\vec{P},\vec{V},\alpha t)$ exists and $\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t)$.

In order to prove this, we have to prove :

if ψ^i is a solution to the o.d.e $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ then

$$\overset{(n)i}{\psi} = X^i_{j_1\ldots j_n}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_n} \qquad n = 2,3,\ldots \text{ for some function}$$

$X^i_{j_1\ldots j_n}(\vec{\psi})$.

<u>Proof</u> Using induction on n.

Since ψ^i is a solution of $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$

Hence when $n = 2$, $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ is true.

Assume $\quad \overset{(n)i}{\psi} = X^i_{j_1\ldots j_n}(\vec{\psi})\,\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_n}$.

$$\text{Hence} \quad \overset{(n+1)i}{\psi} = \frac{\partial X^i_{j_1\ldots j_n}(\vec{\psi})}{\partial x^{j_{n+1}}}\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}} + X^i_{\ell_1 j_2\ldots j_n}(\vec{\psi})\ddot{\psi}^{\ell_1}\dot{\psi}^{j_2}\ldots\dot{\psi}^{j_n}$$

$$+ X^i_{j_1\ell_2\ldots j_n}(\vec{\psi})\dot{\psi}^{j_1}\ddot{\psi}^{\ell_2}\dot{\psi}^{j_3}\ldots\dot{\psi}^{j_n} + \ldots + X^i_{j_1\ldots j_{n-1}\ell_n}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n-1}}\ddot{\psi}^{\ell_n}$$

Substitute $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ where $i = 1,2,\ldots,n$ into the above equation we obtain

$$\overset{(n+1)i}{\psi} = \frac{\partial X^i_{j_1\ldots j_n}}{\partial x^{j_{n+1}}}(\vec{\psi})\,\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}} + X^i_{\ell_1 j_2\ldots j_n}(\vec{\psi})G^{\ell_1}_{j_1 j_{n+1}}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}} +$$

$$X^i_{j_1\ell_2 j_3\ldots j_n}(\vec{\psi})G^{\ell_2}_{j_2 j_{n+1}}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}} + \ldots + X^i_{j_1\ldots j_{n-1}\ell_n}(\vec{\psi})G^{\ell_n}_{j_n j_{n+1}}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}}$$

$$= \left(\frac{\partial X^i_{j_1\ldots j_n}}{\partial x^{j_{n+1}}}(\vec{\psi}) + X^i_{\ell_1 j_2\ldots j_n}(\vec{\psi})G^{\ell_1}_{j_1 j_{n+1}}(\vec{\psi}) + \ldots + X^i_{j_1\ldots j_{n-1}\ell_n}(\vec{\psi})G^{\ell_n}_{j_n j_{n+1}}(\vec{\psi})\right)\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}}$$

$$= Y^i_{j_1\ldots j_{n+1}}(\vec{\psi})\,\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_{n+1}}$$

Therefore $\overset{(n)i}{\psi} = X^i_{j_1\ldots j_n}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_n}$ for all $n = 2,3,\ldots$

Since ψ^i is the solution of $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ where G^i_{jk} is analytic on $D \subseteq R^n$. Then $G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ is analytic on $D \times R^n \times R$.

Fix $\vec{P}_0 \in D$, $\vec{V}_0 \in R^n$, $\alpha_0 \in R$.

Since $(\vec{P}_0,\alpha_0\vec{V}_0,0) \in D \times R^n \times R$, then the fundamental theorem implies that there exists an interval $I(0)$ of zero in R such that $\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$ exists on $I(0)$ and $\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = p^i_0$, $\dot{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,0) = \alpha_0 v^i_0$ and also ψ^i is an analytic function in t on $I(0)$.

Therefore, $\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t) = \psi^i(\vec{P}_0,\alpha_0\vec{V}_0,0) + \dot{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,0)t +$
$\ddot{\psi}^i(\vec{P}_0,\alpha_0\vec{V}_0,0)\frac{t^2}{2!} + \ldots + \overset{(n)i}{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,0)\frac{t^n}{n!} + \ldots$

$$= p_0^i+\alpha_0 v_0^i t+G^i_{jk}(\vec{P})(\alpha_0 v_0^j)(\alpha_0 v_0^k)\frac{t^2}{2!}+\dots+X^i_{j_1\dots j_n}(\vec{P})(\alpha_0 v_0^{j_1})\dots(\alpha_0 v_0^{j_n})\frac{t^n}{n!}+\dots$$

(since $\overset{(n)i}{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t) = X^i_{j_1\dots j_n}(\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t))(\mathring{\psi}^{j_1}\dots\mathring{\psi}^{j_n})(\vec{P}_0,\alpha_0\vec{V}_0,t))$

$$= p_0^i+v_0^i(\alpha_0 t)+G^i_{jk}(\vec{P})(v_0^j v_0^k)\frac{(\alpha_0 t)^2}{2!}+\dots+X^i_{j_1\dots j_n}(\vec{P})(v_0^{j_1}\dots v_0^{j_n})\frac{(\alpha_0 t)^n}{n!}+\dots$$

$$= \psi^i(\vec{P}_0,\vec{V}_0,0)+\mathring{\psi}^i(\vec{P}_0,\vec{V}_0,0)(\alpha_0 t)+\ddot{\psi}^i(\vec{P}_0,\vec{V}_0,0)\frac{(\alpha_0 t)^2}{2!}+\dots+$$

$$\overset{(n)i}{\psi}(\vec{P}_0,\vec{V}_0,0)\frac{(\alpha_0 t)^n}{n!}+\dots$$

$$= \psi^i(\vec{P}_0,\vec{V}_0,\alpha_0 t)$$

That is, $\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t)$ exists if and only if $\psi^i(\vec{P}_0,\vec{V}_0,\alpha_0 t)$ exists and they are equal.

Then the proof for case $c = 0$ is complete.

<u>Corollary 3-2.1</u> Let the hypothesis to this corollary be the same as in theorem 3-2 except for each $i = 1,2,\dots,n$ H^i is an analytic function of $\vec{\psi}$ and $\mathring{\vec{\psi}}$ and we do not assume that $f(0,t) = 0$, wherever defined. Then the differential equation must be either

(a) $$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i , \quad c \neq 0$$

and $f(\alpha,t) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct})$ for all (α,t) in some small neighbourhood of $(0,0)$

or (b) $\quad \ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k \quad$ and $f(\alpha,t) = \alpha t$

for all (α,t) in some small neighbourhood of $(0,0)$ for all $i = 1,2,\ldots,n$.

Proof For each $i = 1,2,\ldots,n$ H^i is analytic in a neighbourhood of $(\vec{P}_0,0)$. Thus H^i is represented by a Taylor's series

$$H^i(\vec{\psi},\dot{\vec{\psi}}) = G^i(\vec{\psi})+G^i_{j_1}(\vec{\psi})\dot{\psi}^{j_1}+ G^i_{j_1j_2}(\vec{\psi})\dot{\psi}^{j_1}\dot{\psi}^{j_2}+\ldots+G^i_{j_1\ldots j_k}(\vec{\psi})\dot{\psi}^{j_1}\ldots\dot{\psi}^{j_k}+\ldots$$

Then using the same type of proof which we used in theorem 3-2, we get either result (a) or (b).

Definition 3-3.1 $(G,*)$ is said to be a semigroup where $*$ is a binary operation on $G \neq \emptyset$ if and only if

$$\forall\ a,b,c \in G, \quad a * (b * c) = (a * b) * c$$

Example : the binary operation of addition on the set N of all natural numbers, $(N,+)$ is a semi group.

The binary operation of ordinary multiplication on the set R of all real numbers, $(R, \times)$ is a semi group.

The binary operation of composition on the set of all continuous function from R into R, $(\mathcal{C}_R,\circ)$ is a semigroup.

Definition 3-3.2 Let $(X,*),(Y,\circ)$ be two semigroups. Then a homomorphism of a set X onto or into a set Y is a transformation H of X onto or into Y such that, for all $x, y \in X$, $H(x*y) = H(x) \circ H(y)$

<u>Definition 3-3.3</u> Let G be a semigroup, S be a set. An <u>action</u> of G on S is a map $\Phi : G \times S \to S$ such that

$$\Phi(h,\Phi(g,s)) = \Phi(hg,s) \text{ whenever both sides are defined.}$$

<u>Definition 3-3.4</u> Let X be a topological space. A <u>local 1-parameter semigroup of local continuous maps</u> is a map $\Phi : U \to X$ where $U \neq \phi$(**empty set**) is an open subset of $R \times X$ such that $\Phi(st,x) = \Phi(s,\Phi(t,x))$ whenever both sides are defined

<u>Proposition 3-4</u> 1) $f(\alpha,t) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct})$, $c \neq 0$ is a local 1-parameter semigroup of local continuous maps.

2) $f(\alpha,t) = \alpha t$ is an action of the semigroup $(R,\times)$ on R.

<u>Proof 1</u> Let f be a map defined on some open neighbourhood W of (0,0) in R^2 such that $f(\alpha,t) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct})$, $c \neq 0$ $\forall (\alpha,t) \in W$.

Hence,

$$\begin{aligned}
f(\beta,f(\alpha,t)) &= \frac{1}{c}\,\ell n(1-\beta+\beta e^{cf(\alpha,t)}) \\
&= \frac{1}{c}\,\ell n(1-\beta+\beta e^{c(\frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct}))}) \\
&= \frac{1}{c}\,\ell n(1-\beta+\beta(1-\alpha+\alpha e^{ct})) \\
&= \frac{1}{c}\,\ell n(1-\beta+\beta-\beta\alpha+\beta\alpha e^{ct}) \\
&= \frac{1}{c}\,\ell n(1-\beta\alpha+\beta\alpha e^{ct}) \\
&= f(\beta\alpha,t)
\end{aligned}$$

By definition 3-3.4, implies that f is a local 1-parameter semigroup of local continuous map.

<u>Proof 2</u> Let $(R,\times)$ be a semigroup.

Let f be a map defined on $R \times R$ into R such that $f(\alpha,t) = \alpha t$

$$\begin{aligned} \text{Hence} \quad f(\beta,f(\alpha,t)) &= \beta(f(\alpha,t)) \\ &= \beta(\alpha t) \\ &= (\beta\alpha)t \\ &= f(\beta\alpha,t) \end{aligned}$$

By definition 3-3.3, f is an action of $(R,\times)$ on R.

Then the proof is complete.

$$\text{Let} \quad f_\alpha(t) = f(\alpha,t)$$

$$\text{Hence} \quad f(\beta,f(\alpha,t)) = f_\beta(f(\alpha,t)) = f_\beta(f_\alpha(t)) = (f_\beta \circ f_\alpha)(t)$$

$$\text{and} \quad f(\beta\alpha,t) = f_{\beta\alpha}(t).$$

Since we have already proved that if f is defined in either case (1) or (2), then $f(\beta,f(\alpha,t)) = f(\beta\alpha,t)$.

Thus $(f_\beta \circ f_\alpha)(t) = f_{\beta\alpha}(t)$ whenever defined.

Therefore, $f_\beta \circ f_\alpha = f_{\beta\alpha}$

Let $\mathcal{C}_R$ be a set of all continuous functions from R into R.

Then $(\mathcal{C}_R,\circ)$ is a semigroup under binary operation of composition.

Also, it is obvious that $(R,\times)$ is a semigroup under multiplication.

Let $\Phi : (R,\times) \to (\mathcal{C}_R,\circ)$ be a map defined by $\Phi(\alpha) = f_\alpha$ when $f_\alpha(t) = \alpha t$.

Therefore, $\Phi(\alpha\beta) = f_{\alpha\beta} = f_\alpha \circ f_\beta = \Phi(\alpha)\circ\Phi(\beta) \ \forall\, \alpha, \beta \in R$.

Let local continuous (R,R) denoted by $\mathcal{L}_R$ be a set of all continuous maps $\psi: U_\psi \to R$ where U_ψ is an open neighbourhood of 0 and $\psi(0) = 0$. Thus $(\mathcal{L}_R, \circ)$ is a semigroup under operation of composition. It is clear that $((-1,1),\times)$ is a semigroup under operation of multiplication. Let $\Phi: ((-1,1),\times) \to (\mathcal{L}_R, \circ)$ be defined by $\Phi(\alpha) = f_\alpha$ where $f_\alpha(t) = \frac{1}{c}\ln(1-\alpha+\alpha e^{ct})$.

Therefore, for any α, $\beta \varepsilon (-1,1)$, $\Phi(\alpha\beta) = f_{\alpha\beta} = f_\alpha \circ f_\beta = \Phi(\alpha) \circ \Phi(\beta)$.

Thus we see that the action f gives a homomorphism of R into the semigroup of all continuous maps of R into itself under composition or the semigroup of local continuous maps under composition.

<u>Theorem 3-5</u> For each i = 1,2,...,n let ψ^i be solution of $\ddot{\psi}^i = H^i(\vec{\psi}, \mathring{\vec{\psi}})$ where H^i is analytic on Ω defined in theorem 3-2. Suppose there exists an open neighbourhood W^* of (0,0) in R^2 and an analytic function $g : W^* \to R$ be such that $\psi^i(\vec{P},\vec{V},\alpha t) = \psi^i(\vec{P}, g(\alpha,t)\vec{V}, t)$ whenever $(\vec{P},\vec{V},\alpha t) \varepsilon V$ where V is mentioned at the beginning of this chapter on page 28 and $(\alpha,t) \varepsilon W^*$. Furthermore, we assume that $g(\alpha,0) = \alpha$ and $g(0,t) = 0$ whenever defined. Then the differential equation that ψ^i satisfies must be either

(a) $$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i, \qquad i = 1,2,\dots, n$$

and $$g(\alpha,t) = \begin{cases} \dfrac{1-e^{c\alpha t}}{1-e^{ct}}, & t \neq 0 \\ \alpha, & t = 0 \end{cases}$$

or

(b) $$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k \text{ and } g(\alpha,t) = \alpha \quad \text{whenever } (\alpha,t) \varepsilon W^*.$$

Conversely, if the solution ψ^i satisfies the second order ordinary differential equation of type (a) where G^i_{jk} is c^1 on $D \subseteq R^n$ or type (b) where G^i_{jk} is analytic on $D \subseteq R^n$ then ψ^i must satisfy the functional equation $\psi^i(\vec{P},\vec{V},\alpha t) = \psi^i(\vec{P},g(\alpha,t)\vec{V},t)$ for all t sufficiently small where $g(\alpha,t)$ is the function defined above or $\psi^i(\vec{P},\alpha\vec{V},t)$ exists iff $\psi^i(\vec{P},\vec{V},\alpha t)$ exists and $\psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t)$.

<u>Proof</u> Assume that ψ^i satisfies the functional equation

(1) $$\psi^i(\vec{P},g(\alpha,t)\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t)$$

where $g(\alpha,t)$ is an analytic function on some open neighbourhood W* of (0,0) in R^2 such that $g(\alpha,0) = \alpha$ and $g(0,t) = 0$ whenever defined.

(2) Let $F(t,\beta,\alpha) = g(\alpha,t) - \beta$ then F is defined in some neighbourhood of $(0,0,0) \in R^3$, says N

Since $g(\alpha,t)$ is analytic, hence F is analytic on N.

When $(t,\beta,\alpha) = (0,0,0)$, equation (2) becomes

$$F(0,0,0) = g(0,0) = 0 \text{ (by the assumption of g)}$$

Differentiate (2) with respect to α we have

$$F_\alpha(t,\beta,\alpha) = g_\alpha(\alpha,t)$$

Hence $$F_\alpha(0,0,0) = g_\alpha(0,0) = \lim_{h\to 0} \frac{g(h,0)-g(0,0)}{h}$$

By the assumption of g, we get

$$F_\alpha(0,0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1 \neq 0$$

By the implicit function theorem 1-2.3 implies there exists an open neighbourhood W_0 of (0,0) such that for any connected open neighbourhood $W \subseteq W_0$ of (0,0), there is a unique real-valued function ℓ **analytic** on W and such that $\ell(0,0) = 0$ and $F(t,\beta,\ell(\beta,t)) = 0$ for any $(\beta,t) \in W$.

Defined $h(\beta,t)$ such that $h(\beta,t) = t\ell(\beta,t)$.

Thus h is analytic on W of (0,0) and $h(\beta,0) = 0 \quad \forall (\beta,0) \in W$.

Since for $1 \le i \le n$, $\psi^i(\vec{P},g(\alpha,t)\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t)$ whenever defined.

Hence $\psi^i(\vec{P},g(\ell(\beta,t),t)\vec{V},t) = \psi^i(\vec{P},\vec{V},\ell(\beta,t)t)$

Since $F(t,\beta,\ell(\beta,t)) = g(\ell(\beta,t),t) - \beta = 0$ (Implicit theorem)

Hence $g(\ell(\beta,t),t) = \beta$.

Therefore $\psi^i(\vec{P},\beta\vec{V},t) = \psi^i(\vec{P},\vec{V},h(\beta,t))$ for all $(\beta,t) \in W$.

By corollary 3-2.1, for each i = 1,2,..., n ψ^i must satisfy either differential equation

(a) $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i$ and $h(\beta,t) = \frac{1}{c}\ln(1-\beta+\beta e^{ct})$, $c \ne 0$

or

(b) $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k$ and $h(\beta,t) = \beta t$

<u>Case a</u> $\quad h(\beta,t) = \frac{1}{c}\ln(1-\beta+\beta e^{ct}), \; c \ne 0 \quad \forall (\beta,t) \in W.$

Since $\quad h(\beta,t) = t\,\ell(\beta,t) = t\alpha$.

Hence
$$\alpha t = \frac{1}{c}\ln(1-\beta+\beta e^{ct}), \quad c \ne 0$$
$$e^{c\alpha t} = 1 - \beta(1-e^{ct})$$
$$\beta = \frac{1 - e^{c\alpha t}}{1 - e^{ct}}, \quad t \ne 0$$

Since $g(\ell(\beta,t),t) = \beta$, hence $g(\alpha,t) = \beta$.

Thus $$g(\alpha,t) = \begin{cases} \dfrac{1-e^{c\alpha t}}{1-e^{ct}} , & t \neq 0 \\ \alpha , & t = 0 \end{cases} \quad \text{for all } (\alpha,t) \in W^*$$

<u>Case b</u> $h(\beta,t) = \beta t$

Hence $\alpha t = \beta t$

$(\alpha-\beta)t = 0$

$\alpha-\beta = 0 \quad \text{if } t \neq 0$

Thus $\alpha = \beta \quad \text{if } t \neq 0$

Since $g(\alpha,t) = \beta$, hence $g(\alpha,t) = \alpha$, $t \neq 0$

From the assumption of g, $g(\alpha,0) = \alpha$.

Then $g(\alpha,t) = \alpha \quad \forall (\alpha,t) \in W^*$

Now, we conclude that ψ^i must satisfy either

(a) $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i$, $c \neq 0$ and $g(\alpha,t) = \begin{cases} \dfrac{1-e^{c\alpha t}}{1-e^{ct}} , & t \neq 0 \\ \alpha , & t = 0 \end{cases}$

or

(b) $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ and $g(\alpha,t) = \alpha \quad \forall (\alpha,t) \in W^*$

for all $i = 1,2,\ldots, n$.

Then the proof is complete.

Now, we shall prove the converse to this theorem. Let us first assume that for each $i = 1,2,\ldots,n$, the solution ψ^i satisfies the 2nd order ordinary differential equation

$$\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\mathring{\psi}^j\mathring{\psi}^k + c\mathring{\psi}^i, \; c \neq 0 \text{ where } G^i_{jk} \text{ is } c^1 \text{ function}$$

Define $g(\alpha,t) = \begin{cases} \dfrac{1-e^{c\alpha t}}{1-e^{ct}}, & t \neq 0 \\ \alpha, & t = 0 \end{cases}$

To show ψ^i satisfies the functional equation $\psi^i(\vec{P},g(\alpha,t)\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t)$.

Let $F(t,\beta,\alpha) = g(\alpha,t) - \beta$. Then F is defined in some neighbourhood of (0,0,0). It follows from definition of g that $F(0,0,0) = 0$

and $F_\alpha(t,\beta,\alpha) = g_\alpha(\alpha,t)$

$$= \begin{cases} -\dfrac{ct\, e^{c\alpha t}}{1-e^{ct}}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Thus $F_\alpha(0,0,0) = 1 \neq 0.$

The implicit function theorem 1-2.3 implies, there is an open neighbourhood W of (0,0) and a unique function h define on W such that $h(0,0) = 0$ and $F(t,\beta,h(\beta,t)) = 0$, moreover h is analytic on W. Let $f(\beta,t) = th(\beta,t)$. Hence f is well-defined and analytic on W since h is .

Since $F(t,\beta,\alpha) = g(\alpha,t) - \beta$,

hence $F(t,\beta,h(\beta,t)) = g(h(\beta,t),t) - \beta = 0$.

Thus $g(h(\beta,t),t) = \beta$

For $t \neq 0$, $\dfrac{1-e^{ch(\beta,t)t}}{1-e^{ct}} = \beta$

$$\beta - \beta e^{ct} = 1 - e^{ch(\beta,t)t}$$

$$1 - \beta + \beta e^{ct} = e^{ch(\beta,t)t}$$

$$th(\beta,t) = \frac{1}{c}\ln(1-\beta+\beta e^{ct}), \; t \neq 0$$

Thus $\quad f(\beta,t) = \frac{1}{c}\,\ell n(1-\beta+\beta e^{ct}), \quad t \neq 0$

Since $\quad f(\beta,0) = 0$ (from definition of f)

and $\quad \frac{1}{c}\,\ell n(1-\beta+\beta e^{ct}) = 0 \quad$ for $t = 0$

Hence $\quad f(\beta,t) = \frac{1}{c}\,\ell n(1-\beta+\beta e^{ct}) \quad \forall\,(\beta,t) \in W$

By the converse to theorem 3-2, we conclude that for each $1 \leq i \leq n$ the solution ψ^i must satisfy the functional equation

$$\psi^i(\vec{P},\beta\vec{V},t) = \psi^i(\vec{P},\vec{V},f(\beta,t)) \text{ for all } t \text{ sufficiently small}$$

Hence

$$\psi^i(\vec{P},g(h(\beta,t),t)\vec{V},t) = \psi^i(\vec{P},\vec{V},th(\beta,t)) \text{ for all } t \text{ sufficiently small}$$

$$\psi^i(\vec{P},g(\alpha,t)\vec{V},t) = \psi^i(\vec{P},\vec{V},t\alpha) \text{ for all } t \text{ sufficiently small}$$

Second, assume that ψ^i satisfies the 2^{nd} order ordinary differential equation $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k$ for $1 \leq i \leq n$ and G^i_{jk} is analytic function.

Define $\quad g(\alpha,t) = \alpha \quad$ for all $(\alpha,t) \in W^*$, where W^* is a neighbourhood of $(0,0)$.

Let $\quad f(\alpha,t) = tg(\alpha,t)$. Hence $f(\alpha,t) = t\alpha = \alpha t$

By the converse of theorem 3-2, we will conclude that

$$\psi^i(\vec{P},\alpha\vec{V},t) \text{ exists} \leftrightarrow \psi^i(\vec{P},\vec{V},f(\alpha,t)) \text{ exists}$$

and $\quad \psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},f(\alpha,t)), \quad 1 \leq i \leq n$

That is, $\psi^i(\vec{P},\alpha\vec{V},t)$ exists $\leftrightarrow \psi^i(\vec{P},\vec{V},\alpha t)$ exists

and $\quad \psi^i(\vec{P},\alpha\vec{V},t) = \psi^i(\vec{P},\vec{V},\alpha t), \quad 1 \leq i \leq n$

Then the proof is complete.

We have already proved that if $\vec{\psi}$ satisfies the differential equation

(1) $\ddot{\psi}^i = G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i$, $c \neq 0$ where G^i_{jk} is c^1 on open subset D of R^n then $\vec{\psi}$ must satisfy the functional equation below for all $\vec{P} \in D$, $\vec{V} \in R^n$, $\alpha \in R$

(2) $\vec{\psi}(\vec{P},\alpha\vec{V},t) = \vec{\psi}(\vec{P},\vec{V},f(\alpha,t))$ for all t in some open interval J of zero in R where $f(\alpha,t) = \frac{1}{c}\ \ell n(1-\alpha+\alpha e^{ct})$, $c \neq 0$.

The functional equation (2) gives many geometrical properties for the generalized geodesic curve.

Property 1 Given t_0 in R, $\vec{P}_0$ in D there exists a neighbourhood U of the zero vector at $\vec{P}_0$ such that for all $\vec{V}$ in U, $\vec{\psi}(\vec{P}_0,\vec{V},t_0)$ is defined.

I conjecture that this property is true, but still have no proof.

Property 2 Given any compact neighbourhood U of the zero vector at $\vec{P}_0$, there exists a neighbourhood V of zero in R such that for all $t \in V$, for all $\vec{V} \in U$, $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists.

Proof It is enough to prove that for any ball $B(\vec{0}_{\vec{P}_0},r_1)$ there exists a neighbourhood V of zero in R such that $\vec{\psi}(\vec{P}_0,\vec{V},t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0},r_1)$ and for all t in V.

The fundamental theorem for 2^{nd} order o.d.e implies there exists a ball $B(\vec{0}_{\vec{P}_0}, r_2)$ and a neighbourhood V of zero in R such that

(1) $\vec{\psi}(\vec{P}_0, \vec{V}, t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_2)$ for all $t \in V$.

Let $B(\vec{0}_{\vec{P}_0}, r_1)$ be given , we may assume that the ball $B(\vec{0}_{\vec{P}_0}, r_2)$ is a proper subset of the ball $B(\vec{0}_{\vec{P}_0}, r_1)$.

Choose $\alpha_0 \in R - \{0\}$ such that $|\alpha_0| < \frac{r_2}{r_1}$

For any $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$,

$$|\alpha_0 \vec{V}| = |\alpha_0| \, |\vec{V}| < |\alpha_0| \, |r_1| < r_2 .$$

Therefore, $\alpha_0 \vec{V} \in B(\vec{0}_{\vec{P}_0}, r_2)$ for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$.

We have from (1) above that, there exists a neighbourhood V of zero in R such that $\vec{\psi}(\vec{P}_0, \alpha_0 \vec{V}, t)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in V(0)$.

Since $\vec{\psi}$ satisfies the functional equation $\vec{\psi}(\vec{P}_0, \alpha_0 \vec{V}, t) = \vec{\psi}(\vec{P}_0, \vec{V}, f(\alpha_0, t))$ for all t in a sufficiently small neighbourhood of zero in R, hence there exists a neighbourhood V* of zero in R such that $\vec{\psi}(\vec{P}_0, \vec{V}, f(\alpha_0, t))$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all $t \in V^*$. That is, $\vec{\psi}(\vec{P}_0, \vec{V}, t^*)$ exists for all $\vec{V} \in B(\vec{0}_{\vec{P}_0}, r_1)$ for all t* of the form $t^* = f(\alpha_0, t)$ where $t \in V^*$.

Fix such a t* such that $t^* = f(\alpha_0, t) = f_{\alpha_0}(t) = \frac{1}{c} \ln(1 - \alpha_0 + \alpha_0 e^{ct})$, $c \neq 0$. Since f_{α_0} is analytic in t, hence f_{α_0} is analytic on V*. Also, there is an open subset G of V* containing zero such that f_{α_0} restricted to G is a homeomorphism onto its image. We must show that f_{α_0} is a one to one function.

Suppose $$f_{\alpha_0}(t_1) = f_{\alpha_0}(t_2)$$

Hence

$$\frac{1}{c}\,\ell n(1-\alpha_0+\alpha_0 e^{ct_1}) = \frac{1}{c}\,\ell n(1-\alpha_0+\alpha_0 e^{ct_2})$$

Since logarithm is a one to one mapping, hence

$$1-\alpha_0+\alpha_0 e^{ct_1} = 1-\alpha_0+\alpha_0 e^{ct_2}\ .$$

Thus

$$\alpha_0(e^{ct_1} - e^{ct_2}) = 0$$

But $\alpha_0 \neq 0$, therefore, we conclude that $e^{ct_1} - e^{ct_2} = 0$.

That is,

$$e^{ct_1} = e^{ct_2}$$

Since $c \neq 0$ and the exponential map is a one to one function hence $t_1 = t_2$. Thus f_{α_0} is a 1-1 function.

It is clear that f_{α_0} is onto $f_{\alpha_0}[G]$ and f_{α_0} is continuous on G, so it is only to show that $f_{\alpha_0}^{-1}$ is continuous on the image of f.

Let $$t^* = f_{\alpha_0}(t)$$

Since $$f_{\alpha_0}(t) = \frac{1}{c}\,\ell n(1-\alpha_0+\alpha_0 e^{ct})\ ,$$

hence

$$f_{\alpha_0}^{-1}(t^*) = t$$

and

$$e^{ct^*} = 1-\alpha_0+\alpha_0 e^{ct}$$

$$\alpha_0 e^{ct} = e^{ct^*}+\alpha_0-1$$

$$e^{ct} = \frac{1}{\alpha_0}(e^{ct^*}+\alpha_0-1)$$

$$t = \frac{1}{c}\ell n(\frac{1}{\alpha_0}(e^{ct^*}+\alpha_0-1))$$ which is defined

since f is 1-1, onto its image, hence f^{-1} exists on the image.

Thus,

$$f_{\alpha_0}^{-1}(t^*) = \frac{1}{c}\ell n \frac{1}{\alpha_0}(e^{ct^*}+\alpha_0-1) .$$

Since logarithm is a continuous function we have that $f_{\alpha_0}^{-1}$ is a continuous map.

We then conclude that f_{α_0} is a homeomorphism of G onto $f_{\alpha_0}[G]$. Thus f_{α_0} is an open map. Since G is an open set containing zero, hence $f_{\alpha_0}[G]$ is an open subset containing zero since $0 \varepsilon G$ implies $f_{\alpha_0}(0) \varepsilon f_{\alpha_0}[G]$ and $f_{\alpha_0}(0) = 0$.

Therefore, we conclude that there exists an open neighbourhood $f_{\alpha_0}[G]$ of zero in R such that for all $\vec{V} \varepsilon B(\vec{0}_{\vec{P}_0}, r_1)$, for all $t^* \varepsilon f_{\alpha_0}[G]$, $\vec{\psi}(\vec{P}_0,\vec{V},t^*)$ is defined. Now the proof is complete.

<u>Property 3</u> (Exponential property)

Given initial point $\vec{P}_0 \varepsilon D$ and $t_0 \varepsilon R-\{0\}$ such that $\vec{\psi}(\vec{P}_0,\vec{V},t_0)$ exists for all $\vec{V}$ in some neighbourhood U of the zero vector and $\psi^i(\vec{P}_0,\alpha\vec{V},t_0) = \psi^i(\vec{P}_0,\vec{V},f(\alpha,t_0))$ for all α sufficiently small for $1 \leq i \leq n$ then $\vec{V} \to \vec{\psi}(\vec{P}_0,\vec{V},t_0)$ is a bidifferential map of some open set V of the zero vector onto an open set W.

<u>Proof</u> Let $\vec{G}$ be a map such that $\vec{G}(\vec{V}) = \vec{\psi}(\vec{P}_0,\vec{V},t_0)$

Then $\vec{G}$ is defined on V (by hypothesis) and $\vec{G}$ is a c^1 function on V (since $\vec{\psi}(\vec{P}_0,\vec{V},t_0)$ is c^1 function in $\vec{V}$ by the fundamental theorem) Therefore, it is enough to prove that the jacobian of $\vec{G}$ at $\vec{0}$ is not zero by the inverse function theorem.

Since ψ^i satisfies the functional equation

$$\psi^i(\vec{P}_0,\alpha\vec{V},t_0) = \psi^i(\vec{P}_0,\vec{V},f(\alpha,t_0)) \quad \forall\ \alpha \text{ sufficiently small}$$

$i = 1,2,\ldots,n.$

Differentiate the above equation with respect to α, we get

$$v^j \frac{\partial\psi^i}{\partial v^j}(\vec{P}_0,\alpha\vec{V},t_0) = \mathring{\psi}^i(\vec{P}_0,\vec{V},f(\alpha,t_0))\ f_\alpha(\alpha,t_0)$$

Substitute $\alpha = 0$, we obtain

$$v^j \frac{\partial\psi^i}{\partial v^j}(\vec{P}_0,\vec{0},t_0) = \mathring{\psi}^i(\vec{P}_0,\vec{V},f(0,t_0))\ f_\alpha(0,t_0)$$

Since $\qquad f(\alpha,t_0) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct_0}),$

hence $\qquad f(0,t_0) = 0$, and $f_\alpha(\alpha,t_0) = \frac{1}{c}\left[\frac{-1+e^{ct_0}}{1-\alpha+\alpha e^{ct_0}}\right];$

$f_\alpha(0,t_0) = \frac{1}{c}(e^{ct_0}-1).$

Thus $\qquad v^j \frac{\partial\psi^i}{\partial v^j}(\vec{P}_0,\vec{0},t_0) = v^j\delta^i_j\,\frac{e^{ct_0}-1}{c}$

Therefore, $\frac{\partial\psi^i}{\partial v^j}(\vec{P}_0,\vec{0},t_0) = \frac{e^{ct_0}-1}{c}\,\delta^i_j \ , \quad c \neq 0$

Since

$$J_{\vec{G}}(\vec{0}) = \det \begin{vmatrix} \frac{\partial\psi^1}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^1}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \cdots & \frac{\partial\psi^1}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \\ \frac{\partial\psi^2}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^2}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \cdots & \frac{\partial\psi^2}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \\ \vdots & & & \\ \frac{\partial\psi^n}{\partial v^1}(\vec{P}_0,\vec{0},t_0) & \frac{\partial\psi^n}{\partial v^2}(\vec{P}_0,\vec{0},t_0) & \cdots & \frac{\partial\psi^n}{\partial v^n}(\vec{P}_0,\vec{0},t_0) \end{vmatrix}$$

hence

$$J_{\vec{G}}(\vec{0}) = \det \begin{vmatrix} \frac{e^{ct_0}-1}{c} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \frac{e^{ct_0}-1}{c} \end{vmatrix}$$

$$= \left(\frac{e^{ct_0}-1}{c}\right)^n \neq 0 .$$

Therefore, the inverse function theorem 1-2.2 implies that there are two open set $V \subseteq U$ of the zero vector and W such that $\vec{G}$ is 1-1, differentiable on V onto W and $\vec{G}^{-1}$ exists and also differentiable. Thus $\vec{G}$ is a bidifferential map of V onto W. Then the proof is complete.

<u>Property 4</u> Given $\vec{P}_0 \in D$, $\vec{V}_0$ at $\vec{P}_0$ and any real number α_0, then the solution curve $\vec{\psi}(\vec{P}_0,\vec{V}_0,t)$ with initial conditions $\vec{P}_0,\vec{V}_0$ agree as points set as the solution curve $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t)$ having initial conditions $\vec{P}_0,\alpha_0\vec{V}_0$.

<u>Proof</u> By the fundamental theorem for 2[nd] order o.d.e, $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t)$ is defined on some open interval I_1 of zero in R and $\vec{\psi}(\vec{P}_0,\vec{V}_0,t)$ defined on I_2 of zero in R.

Let $c_1 = \{\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t) \mid \vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t)$ is defined on $I_1 \cap J\}$

where J is an open interval of zero in R such that

$$\psi^i(\vec{P}_0,\alpha_0\vec{V}_0,t) = \psi^i(\vec{P}_0,\vec{V}_0,f(\alpha_0,t)) \qquad \forall\, t \in J .$$

Let $c_2 = \{\vec{\psi}(\vec{P}_0,\vec{V}_0,t) \mid \vec{\psi}(\vec{P}_0,\vec{V}_0,t)$ is defined on $I_2\}$

To show $c_1 \subseteq c_2$, let Q be any point in c_1.

Then there exists $t_1 \in I_1 \cap J$ such that $Q = \vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t_1)$.

Since $t_1 \in I_1 \cap J$, hence $t_1 \in J$. Therefore,

$$\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t_1) = \vec{\psi}(\vec{P}_0,\vec{V}_0,f(\alpha_0,t_1)) .$$

Let $t_2 = f(\alpha_0,t_1) = \frac{1}{c}\,\ell n(1-\alpha_0+\alpha_0 e^{ct_1})$ which is defined, (since $t_1 \in J$).

By fundamental theorem, $\vec{\psi}(\vec{P}_0,\vec{V}_0,t)$ is defined on I_2.

Then $t_2 \in I_2$. Therefore, there exists $t_2 \in I_2$ such that

$Q = \vec{\psi}(\vec{P}_0,\vec{V}_0,t_2)$ i.e. $Q \in c_2$.

Thus $\vec{\psi}(\vec{P}_0,\alpha_0\vec{V}_0,t)$ and $\vec{\psi}(\vec{P}_0,\vec{V}_0,t)$ agree as point set locally.

The proof is complete.