CHAPTER II

AN EMBEDDING THEOREM

In this chapter we show how to construct a semilattice of groups with semilattice having a zero element, which is an extension of a given semilattice of groups such that these two semigroups have the same maximum group homomorphic image. Moreover, this extension preserves the properties of having identity, being proper and being F-inverse.

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2.1 Theorem. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then S can be embedded in a semigroup S' which is a semilattice Z of groups such that Z has a zero element, and S and S' have the same maximum group homomorphic image. Moreover S' contains an identity if S contains an identity, S' is proper if S is proper, and if S is F-inverse, then so is S'.

Proof. Let σ be the minimum group congruence on S. If Y has a zero element, let Z = Y and then the theorem is proved. Suppose Y has no zero element. Let $Z = Y^0$ be the semigroup arising from Y by adjoining the zero element 0. Then Z is a semilattice with zero 0. Let $S' = \bigcup_{\alpha \in Z} G_\alpha$ where $G_0 = S/\sigma$. Define an operation $\circ$ on S' as follow : for a, b $\in$ S,

(i) $a \circ b = ab$,

(ii) $(a\sigma) \circ (b\sigma) = (ab)\sigma$, and (iii) $a \circ (b\sigma) = (ab)\sigma$ and $(b\sigma) \circ a = (ba)\sigma$.

To show $\circ$ is well-defined, let $a \in S$, $b\sigma$, $c\sigma \in G_0$ $(= S/\sigma)$ such that $b\sigma = c\sigma$. Then there exist $e, f \in E(S)$ such that $be = ce$ and $fb = fc$ and hence

$$(ab)e = a(be) = a(ce) = (ac)e,$$

and $$f(ba) = (fb)a = (fc)a = f(ca).$$

By Introduction page 5, we have $(ab)\sigma = (ac)\sigma$ and $(ba)\sigma = (ca)\sigma$. Then

$$a \circ (b\sigma) = (ab)\sigma = (ac)\sigma = a \circ (c\sigma)$$

and $$(b\sigma) \circ a = (ba)\sigma = (ca)\sigma = (c\sigma) \circ a.$$

Next we show that the operation $\circ$ is associative on S, let $x, y, z \in S'$. Since S and G_0 are associative, $(x \circ y) \circ z = x \circ (y \circ z)$ if either $x, y, z \in S$ or $x, y, z \in G_0$. If $x, y \in S$ and $z \in G_0$, say $z = a\sigma$ $(a \in S)$, then

$$\begin{aligned} (x \circ y) \circ (a\sigma) &= (xy) \circ (a\sigma) \\ &= ((xy)a)\sigma \\ &= (x(ya))\sigma \\ &= x \circ ((ya)\sigma) \\ &= x \circ (y \circ (a\sigma)). \end{aligned}$$

The proofs are similar for the remaining cases.

Therefore $(S', \circ)$ is a semigroup.

It follows directly from the definition of S' and the operation $\circ$ on S' that S' is a disjoint union of the subgroups G_α of S', and $G_\alpha \circ G_\beta \subseteq G_{\alpha\beta}$ for all $\alpha, \beta \in Z$. Hence $S' = \bigcup_{\alpha \in Z} G_\alpha$ is a semilattice Z of groups G_α.

By Proposition 1.1, $G_0 = S/\sigma$ is the maximum group homomorphic image of S'. But S/σ is also the maximum group homomorphic image of S. Therefore S and S' have the same maximum group homomorphic image.

Assume S has identity 1. Then $1 \in S'$. Let $x \in S'$. If $x \in S$, then $x \circ 1 = x1 = x = 1x = 1 \circ x$. If $x \in G_0$ (= S/σ), then $x = a\sigma$ for some $a \in S$, so that

$$1 \circ (a\sigma) = (1a)\sigma = a\sigma = (a1)\sigma = (a\sigma) \circ 1.$$

Hence 1 is also the identity of S'.

We now show that S' is proper if S is proper. Assume S is proper. Let $\psi'_{\alpha,\beta}$ ($\alpha \geq \beta$) be the corresponding homomorphisms of S'. If $\alpha, \beta \in Y$, then by Proposition 1.5, we have $\psi'_{\alpha,\beta}$ is one-to-one (since S is proper). Therefore it remains to show that, for each $\alpha \in Y$, $\psi'_{\alpha,0}$ is one-to-one. Assume $\alpha \in Y$. Let $a \in G_\alpha$ such that $a\psi'_{\alpha,0}$ is the identity of G_0. Let $e \in E(S)$. Since σ is a group congruence, $e\sigma$ is the identity of S/σ (= G_0) [Lemma 1.3]. Therefore $a\psi'_{\alpha,0} = e\sigma$. Then by the definition of $\psi'_{\alpha,0}$ we obtain $a \circ (e\sigma) = e\sigma$ which implies $(ae)\sigma = e\sigma$. Hence $aef = ef$ for some $f \in E(S)$. Since S is a proper inverse semigroup and $ef \in E(S)$ [Introduction page 1], we have $a \in E(S)$. Because $a \in G_\alpha$, a is the identity of G_α. Hence $\psi'_{\alpha,0}$ is one-to-one.

Let σ', $\leq'$, e_α denote the minimum group congruence on S', the natural partial order on S' and the identity of G_α for all $\alpha \in Y$; respectively.

To prove the remainder of the theorem, we need the following

Lemmas.

2.1.1 Lemma. From the above notation, for any $a, b \in S$, $a\sigma b$ if and only if $a\sigma' b$.

Proof. The " $\Longrightarrow$ " part is clear.

To prove the converse, let $a, b \in S$ and $a\sigma' b$. Then $a \circ e = b \circ e$ for some $e \in E(S')$. If $e \in E(S)$, then $ae = be$ which implies $a\sigma b$.

Assume $e \in E(G_0)$. Let $f \in E(S)$. Then $f\sigma$ is the identity of S/σ so that $e = f\sigma$. Hence $a \circ (f\sigma) = b \circ (f\sigma)$, and so $(af)\sigma = (bf)\sigma$. Therefore $afh = bfh$ for some $h \in E(S)$. Since $fh \in E(S)$, we have $a\sigma b$. #

2.1.2 Lemma. For any $a \in S$, $b\sigma \in S/\sigma$, if $a\sigma'(b\sigma)$, then $b\sigma \leq' a$.

Proof. Assume $a\sigma' = (b\sigma)\sigma'$. Then $a \circ e = (b\sigma) \circ e$ for some $e \in E(S')$. If $e \in E(S)$, then $a \circ e = (be)\sigma = (b\sigma)\circ(e\sigma) = b\sigma$ because $e\sigma$ is the identity of G_0 (= S/σ). Hence, by Introduction page 3, $b\sigma \leq' a$. If $e \in E(G_0)$, then e is the identity of G_0 so that $a \circ e = b\sigma$ which implies $b\sigma \leq' a$. #

2.1.3 Lemma. For any $a, b \in S$, $a \leq b$ if and only if $a \leq' b$.

Proof. The " $\Longrightarrow$ " part is clear.

To prove the converse, let $a, b \in S$ and $a \leq' b$. Then $a \circ a^{-1} = a \circ b^{-1}$. Since $a, b \in S$ and S is an inverse semigroup, we get $a^{-1}, b^{-1} \in S$, so that $aa^{-1} = ab^{-1}$. Hence $a \leq b$. #

2.1.4 <u>Lemma</u>. (i) $S'/\sigma' = \{a\sigma'/a \in S\}$.

(ii) For each $a \in S$, $a\sigma \subseteq a\sigma'$ and $a\sigma' \setminus a\sigma \subseteq G_0$ ($= S/\sigma$).

(iii) For each $a \in S$, if m is the maximum element of $a\sigma$, then m is also the maximum element of $a\sigma'$.

<u>Proof</u>. (i) Let $x\sigma' \in S'/\sigma'$. If $x \in S$, then $x\sigma' \in \{a\sigma'/a \in S\}$. Assume $x \in G_0$, then $x = a_0\sigma$ for some $a_0 \in S$. Let $b\sigma$ be the identity of G_0. Therefore

$$x\circ(b\sigma) = (a_0\sigma)\circ(b\sigma) = (a_0b)\sigma = a_0\circ(b\sigma).$$

Hence $x\sigma' = a_0\sigma'$ so that $x\sigma' \in \{a\sigma'/a \in S\}$.

(ii) The first part follows from the fact that $E(S) \subseteq E(S')$. Let $a \in S$. To show that $a\sigma' \setminus a\sigma \subseteq G_0$, let $x \in a\sigma' \setminus a\sigma$. Suppose $x \notin G_0$. Then $x \in b\sigma$ for some $b \in S$ so that $a\sigma \neq b\sigma$. Hence by Lemma 2.1.1, $x \in b\sigma'$ and so $a\sigma' = x\sigma' = b\sigma'$. Again, by Lemma 2.1.1, we have $a\sigma = b\sigma$ which is a contradiction. Hence $x \in G_0$.

(iii) Let $a \in S$ and m be the maximum element of $a\sigma$. By (ii), we have $m \in a\sigma'$. To show m is the maximum element of $a\sigma'$, let $x \in a\sigma'$. If $x \in a\sigma$, then $x \leq m$ and hence $x \leq' m$.

Assume $x \notin a\sigma$. By (ii), $x \in G_0$ and so $x = y\sigma$ for some $y \in S$. Therefore, from the assumption, we get $(y\sigma)\sigma' = a\sigma'$. Hence by Lemma 2.1.2, $y\sigma \leq' a$. Because $a \in a\sigma$, $a \leq m$ and so by Lemma 2.1.3, $a \leq' m$. Hence by the transivity of $\leq'$, we obtain $x = y\sigma \leq' m$.

Thus m is the maximum element of $a\sigma'$. #

The remaining part of this theorem follows directly from Lemma 2.1.4.

Hence the theorem is completely proved. #

The following remark shows that the semigroup we have constructed gives the converse of the last part of the theorem.

2.2 Remark. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups and S' be constructed from S as in Theorem 2.1. Then the following hold :

(i) If S' contains an identity, then so does S.

(ii) If S' is proper, then S is also proper.

(iii) If S' is F-inverse, then so is S.

Proof. (i) Assume S' contains an identity e_{α_1}. Then by Introduction page 6, α_1 is the identity of $Z = Y^0$. If $\alpha_1 = 0$, then for all $\alpha \in Y$ we have $\alpha = \alpha\alpha_1 = \alpha 0 = 0$ which implies $Y = \{0\}$ so that $Z = Y = \{0\}$ and hence $S = S'$ which has an identity. If $\alpha_1 \neq 0$, then $\alpha_1 \in Y$ and so α_1 is the identity of Y. Thus e_{α_1} is the identity of S [Introduction page 6].

(ii) Since S is an inverse subsemigroup of S' which is proper, it follows that S is proper.

(iii) Let σ and σ' be the minimum group congruences on S and S'; respectively. By Lemma 2.1.4 (ii), $a\sigma \subseteq a\sigma'$ and $a\sigma' \setminus a\sigma \subseteq G_0$ for all $a \in S$.

Let $a\sigma \in S/\sigma$ $(a \in S)$. Therefore $a\sigma \subseteq a\sigma'$. Since S' is F-inverse, $a\sigma'$ has a maximum element, say m. We claim that $m \in a\sigma$. Suppose not. Then $m \in a\sigma' \setminus a\sigma$ which implies $m \in G_0$. Thus $m = b\sigma$ for some $b \in S$. Since $m \in a\sigma'$, $m\sigma' = a\sigma'$ and so $(b\sigma)\sigma' = a\sigma'$. Hence by Lemma 2.1.2, we have $b\sigma \leq' a$ which implies $m \leq' a$. But $a \leq' m$. Therefore a = m, so that $a \in G_0$ which is a contradiction. Hence $m \in a\sigma$. Since $a\sigma \subseteq a\sigma'$, for each $x \in a\sigma$ implies $x \in a\sigma'$ and so $x \leq' m$, hence it follows from Lemma 2.1.3 that $x \leq m$. Thus m is the maximum element of $a\sigma$. #