CHAPTER V

DEGREE SEQUENCE OF CONNECTED HYPERGRAPHS

5.1 Connected hypergraph

Lemma 5.1.1 Let H be an r-graph with P_1 vertices, P_r edges. If H is connected, then $P_1 - (r-1)P_r \leq 1$.

Proof Let H be a connected r-graph.

Case 1 Assume that H contains no cycle. Thus H is acyclic. Hence H is an r-tree. Thus by theorem 4.1.6 $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$ and $P_i = \binom{r}{i} P_r$; $i = 2,3,\ldots,r-1$, thus by lemma 4.1.4

$$\sum_{i=1}^{r} (-1)^{i+1} P_i = P_1 - (r-1) P_r .$$

Hence $P_1 - (r-1) P_r = 1$.

Case 2 Assume that H contains a cycle. Thus by lemma 4.1.5 $P_1 - (r-1) P_r \leq 0$. Hence $P_1 - (r-1)P_r < 1$. Therefore in any case $P_1 - (r-1) P_r \leq 1$.

Theorem 5.1.2 Let H be an r-graph which has k connected components. Then H contains no cycles iff $P_1 - (r-1)P_r = k$, where, as usual, P_1 and P_r denote the numbers of vertices and edges of H.

<u>Proof</u> First we prove the necessary part. Let the k components of H be $H^1, H^2, \ldots, H^k$. Assume that H has no cycles. Hence each component H^j is an r-tree. Let P_i^j be the number of i-edge of $K(H^j)$. Thus from theorem 4.1.6, $P_i^j = \binom{r}{i} P_r^j$; $i = 2,3,\ldots,r-1$; $j = 1,2,\ldots,k$, and $\sum_{i=1}^{r} (-1)^{i+1} P_i^j = 1$, $j = 1,2,\ldots, k$.

By lemma 4.1.4 $\sum_{i=1}^{r} (-1)^{i+1} P_i^j = P_1^j - (r-1)P_r^j$, $j = 1,2,\ldots, k$,

thus $P_1^j - (r-1) P_r^j = 1.$

Therefore
$$\sum_{j=1}^{k} (P_1^j - (r-1) P_r^j) = k ,$$

$$\sum_{j=1}^{k} P_1^j - (r-1) \sum_{j=1}^{k} P_r^j = k ,$$

we observe that $\sum_{j=1}^{k} P_1^j = P_1$ and $\sum_{j=1}^{k} P_r^j = P_r$.

Hence
$$P_1 - (r-1) P_r = k .$$

Next we prove the sufficiency part

Again, assume the k connected components of H be $H^1, H^2, \ldots, H^k$. Assume that $P_1 - (r-1) P_r = k$. We shall show that H has no cycle. Suppose that H has a cycle. Without loss of generality we may assume H^k has a cycle, thus by lemma 4.1.5

$$P_1^k - (r-1) P_r^k \leqslant 0 \qquad (5.1.2.1)$$

Since $H^1, H^2, \ldots, H^{k-1}$ are connected, thus by lemma 5.1.1

$$P_1^j - (r-1)P_r^j \leqslant 1 \quad ; j = 1,2,\ldots, k-1.$$

Hence $$\sum_{j=1}^{k-1} P_1^j - (r-1) \sum_{j=1}^{k-1} P_r^j \leqslant k-1 \quad \ldots\ldots (5.1.2.2).$$

From (5.1.2.1) and (5.1.2.2), we see that

$$\sum_{j=1}^{k} P_1^j - (r-1) \sum_{j=1}^{k} P_r^j \leqslant k-1 \quad \ldots\ldots (5.1.2.3).$$

Since $$P_1 = \sum_{j=1}^{k} P_1^j \text{ and } P_r = \sum_{j=1}^{k} P_r^j ,$$

hence it follows from (5.1.2.3) that

$$P_1 - (r-1)\, P_r \leqslant k-1.$$

Thus we get a contradiction.

5.2 Degree sequence of connected r-graph

Theorem 5.2.1 Let $\pi = (d_1, d_2, \ldots, d_{P_1})$ be a sequence of non-negative integer with $d_1 \geqslant d_2 \geqslant d_3 \geqslant \ldots \geqslant d_{P_1}$. Then there exist a connected r-graph with degree sequence π iff

(5.2.1.1) π is a degree sequence of an r-graph ,

(5.2.1.2) $d_i \geqslant 1$ for all i ,

(5.2.1.3) $\Sigma d_i \geqslant \dfrac{r(P_1-1)}{r-1}$.

Proof First we prove the sufficiency part.

Let H be an r-graph with degree sequence π.

Suppose H is not connected, thus H has $k \geqslant 2$ components, from remark (2.3.1) $P_r = \frac{\Sigma d_i}{r}$,

and from (5.2.1.3) $\frac{\Sigma d_i}{r} \geqslant \frac{P_1 - 1}{r-1}$,

thus
$$P_r \geqslant \frac{P_1 - 1}{r-1} ,$$

$$(r-1)P_r \geqslant P_1 - 1 ,$$

thus
$$(r-1)P_r \geqslant (r-1)\frac{P_1 - 1}{r-1} ,$$

$$P_1 - (r-1)P_r \leqslant P_1 - (r-1)\frac{P_1 - 1}{r-1} ,$$

$$P_1 - (r-1)P_r \leqslant 1 ,$$

since $k \geqslant 2$. Thus $P_1 - (r-1)P_r \neq k$. Thus from Lemma 5.1.1 H has a cycle, say P. Assume the cycle P is $x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1}$, since $x_2 \in E_2$ and $x_2 \in E_1$, thus $d_H(x_2) \geqslant 2$. Thus there exist a vertex x on the cycle P such that $d_H(x) \geqslant 2$. Let E be an edge on the cycle that contains x, say $E = \{x_1, x_2, \ldots, x_{r-1}, x\}$ and let $\{a_1, a_2, \ldots, a_r\}$ be an edge of the other component. Construct a new hypergraph $H' = (X', \mathcal{E}')$ where $X' = X$ and $\mathcal{E}' = (\mathcal{E} \setminus \{\{x_1, x_2, \ldots, x_{r-1}, x\}, \{a_1, a_2, \ldots, a_r\}\}) \cup \{\{x_1, x_2, \ldots, x_{r-1}, a_r\}, \{a_1, a_2, \ldots, a_{r-1}, x\}\}$.

In H' the number of connected components is reduced in which

$d_{H'}(x) = d_H(x)$ for every $x \varepsilon X = X'$. If H' is not connected, thus H' has $k \geqslant 2$ components thus $P_1 - (r-1)P_r \leqslant 1$, in which $P_1 - (r-1)P_r \neq k$, therefore H' has a cycle. By repeating the same process in H , as many times as needed, we obtained a connected hypergraph.

Next we prove the necessary part.

Suppose there exist a connected r-graph H with degree sequence π. Then conditions (5.2.1.1) satisfied. Since H is connected and if H has no cycle, then H is a r-tree, thus by theorem 4.1.6 $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$. If H has a cycle, then by lemma 4.1.5 $\sum_{i=1}^{r} (-1)^{i+1} P_i < 0$. Hence in any case $\sum_{i=1}^{r} (-1)^{i+1} P_i \leqslant 1$,

and by lemma 4.1.4 $\sum_{i=1}^{r} (-1)^{i+1} P_i = P_1 - (r-1) P_r$,

thus
$$P_1 - (r-1) P_r \leqslant 1$$
$$P_1 - 1 \leqslant (r-1) P_r$$
$$\frac{P_1 - 1}{r-1} = P_r .$$

By remark 2.3.1
$$P_r = \frac{\Sigma d_i}{r},$$

thus
$$\frac{\Sigma d_i}{r} \geqslant \frac{P_1 - 1}{r-1},$$

i.e.
$$\Sigma d_i \geqslant \frac{r(P_1 - 1)}{r-1}.$$

The theorem is prove.