CHAPTER II

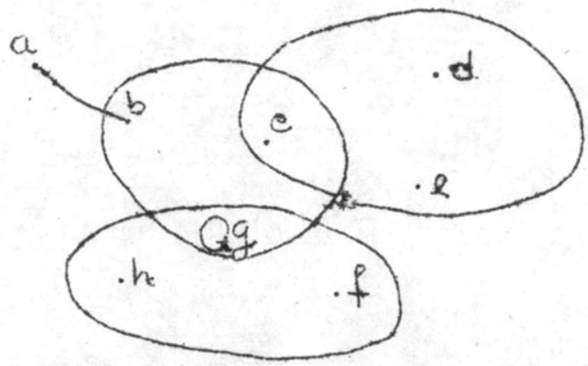
PRELIMINARIES

2.1 Hypergraphs

A hypergraph is a couple $H = (X, \mathcal{E})$ where X is a finite set and $\mathcal{E}$ is a collection of non-empty subsets of X. The set in $\mathcal{E}$ are called hyperedges or simply edges while the elements of X are called vertices. For any set S, here and in the sequel, $|S|$ denote the cardinality of S. A hypergraph $H = (X, \mathcal{E})$ in which $|X| = 1$ and $|\mathcal{E}| = 0$ is called trivial hypergraph. To represent a hypergraph $(X, \mathcal{E})$ by a diagram, we represent each vertex v by a point v and an edge E such that $|E| > 2$ is drawn as a curve encircling all the vertices that belongs to E. An edge E with $|E| = 2$ is drawn as a curve connecting its two vertices. An edge E with $|E| = 1$ is drawn as a loop as in a graph.

For example, let $X = \{a,b,c,d,e,f,g,h\}$ and $\mathcal{E} = \{\{c,d,e\},\{b,c,g\},\{d,f\},\{f,g,h\},\{a,b\},\{g\}\}$.

Then $(X, \mathcal{E})$ is a hypergraph. This hypergraph can be represented by the following diagram



For any positive integer r, an edge that has cardinality r is called r-edges. By rank of a hypergraph we mean the maximum cardinality of edges in the hypergraph. A hypergraph in which every edge has the same cardinality is called a uniform hypergraph. A uniform hypergraph of rank r is also called an r-graph. In a hypergraph any two distinct vertices u, v are said to be adjacent if there is an edge E that contain both u and v. A point v and an edge E are said to be incident if v belongs to the edge E. For example vertices c, d, e in the above example are adjacent and the vertex b and the edge $\{b,c,g\}$ are incident.

2.2 Complex

A complex is any hypergraph $K = (X, \mathcal{E})$ in which every non-empty subset of any edge is also an edge of K. In a complex, the edge are called simplexes, the dimension of a simplex X is the number $|X| - 1$. A simplex of dimension m is also called an m-simplex. The dimension of a complex K is the maximum dimension of any of its simplexes. A complex of dimension n is also called an n-complex. In a complex the number of 0-simplex is the number of 1-edge and also the number of 1-edge equal to the number of vertices.

For any hypergraph $H = (X,\mathcal{E})$ we shall associate a complex $K(H) = (X',\mathcal{E}')$, where $X' = X$ and $\mathcal{E}' = \{M/M$ is a non-empty subset of an edge $E \in \mathcal{E}\}$. We shall called the complex K(H) the associated complex of the hypergraph H or simply the associated complex. We shall denote the number of i-edge of K(H) by P_i. We see that for any r-graph H the number of r-edge in H is equal to the number of r-edge in K(H).

2.3 Degree

In a hypergraph $H = (X, \mathcal{E})$ the degree of a vertex v, written $d_H(v)$, is the cardinality of the set $\{E \in \mathcal{E} \ / \ v \in E\}$ and the r-degree of a vertex v, written $d_H^r(v)$, is the cardinality of the set $\{E \in \mathcal{E} \ / \ v \in E$ and E is an r-edge$\}$.

Remark 2.3.1 Let H be a hypergraph. Let P_i be the number of i-edges in K(H). Then the sum of the i-degree is i times the number of i-edges where i is any positive integer i.e $\Sigma\, d_H^i(v_j) = i.P_i$

Proof Let i be arbitrary but fixed positive integer. For each $v \in X$, let

$$\mathcal{L}_v = \{(v,E) / E \text{ is an i-edge of } K(H)\},$$

for each $E \in \mathcal{E}$, let

$$\mathcal{L}_E = \{(v,E) / v \in E\}.$$

Since $\mathcal{L}_v \cap \mathcal{L}_{v'} = \phi$ for $v \neq v'$ and $\mathcal{L}_E \cap \mathcal{L}_{E'} = \phi$ for $E \neq E'$

and $\bigcup_{v \in X} \mathcal{L}_v = \bigcup_{E \in \mathcal{E}} \mathcal{L}_E$.

Hence $\sum_v |\mathcal{L}_v| = |\bigcup_v \mathcal{L}_v| = |\bigcup_E \mathcal{L}_E| = \sum_E |\mathcal{L}_E|$.

Thus $\Sigma\, d_H^i(v_j) = i \cdot P_i$. Thus the proof is complete.

Let E be an edge of a hypergraph $H = (X, \mathcal{E})$. By Hypergraph $H \setminus E$ we mean the hypergraph $H' = (X', \mathcal{E}')$ where $X' = X$ and $\mathcal{E}' = \mathcal{E} \setminus \{E\}$.

2.4 Walk, Path and cycle

A walk of a hypergraph H is an alternating sequences P of vertices and edges $x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1}$ in which x_k, $x_{k+1} \in E_k$ for $k = 1,2,3,\ldots, q$. The vertex x_1 is called an initial points of P and x_{q+1} is called a terminal points of P. A vertex (edge) which is a term of a walk is said to be on the walk. A path is a walk in which all the vertices and all the edge are distinct. A cycle is a walk $x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1}$ in which $x_i \neq x_{i+1}$ and all the edge and all the vertices are distinct except $x_1 = x_{q+1}$. The cycle is simple if for any vertex x on the cycle $x \notin E_i \;\; E_j$ for $j \neq i+1$ except $i = 1$ and $j = q+1$.

The length of a cycle is the number of edges on the cycle (or the number of vertices on the cycle). By a path from u to v we mean a path which has u as its initial point and v as its terminal point. The length of a path is the number of edges on the path. A geodesic from u to v is a path from u to v that has the minimum length. In a path $x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1}$, a vertex x_{k+1} is called the successor of vertex x_k.

Remark 2.4.1 Let $H = (X, \mathcal{E})$ be an r-graph. If H contains a cycle, then H contains a simple cycle.

Proof Let P be a cycle of H, say P is $x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1}$. If P is not a simple cycle, there exists a vertex x_i such that $x_i \in E_i \cap E_j$ for $j \neq i+1$ except $i = 1$ and $j = q+1$. Without loss of generality we may assume that $i \neq 1$. Hence $x_1, E_1, x_2, \ldots, E_{i-1}, x_i, E_j, \ldots, E_q, x_{q+1}$ is a cycle. Let $x_1^* = x_1$, $E_1^* = E_1, \ldots, E_{i-1}^* = E_{i-1}$, $x_i^* = x_i$, $E_i^* = E_j$, $x_{i+1}^* = x_{j+1}$, $E_{i+1}^* = E_{j+1}, \ldots, x_{q^*+1}^* = x_{q+1}$, where $q^* = q-j+i$. Let P^* be the cycle $x_1^*, E_1^*, x_2^*, E_2^*, \ldots, x_{q^*+1}^*$. If P^* is not a simple cycle, there exists a vertex x_k^* such that $x_k^* \in E_k^* \cap E_j^*$ for $j \neq k+1$ except $i = 1$ and $j = q^*+1$. By continuing the same process, we shall obtain a new cycle of shorter length. Hence the process must come to an end. Therefore we must obtain a simple cycle.

2.5 Connected hypergraph, hypertree

A hypergraph H is connected if for every distinct vertices u and v there is a path from u to v. We also agree to say that the trivial hypergraph is connected. A hypergraph H is acyclic if H contains no cycles. A hypertree is a connected acyclic hypergraph.

2.6 Subhypergraph, component.

Let $H = (X, \mathcal{E})$ be a hypergraph. A hypergraph $H' = (X', \mathcal{E}')$ is said to be a subhypergraph of H if X' is a subset of X and $\mathcal{E}'$ is a subset of $\mathcal{E}$. If $X' = X$ we say that H' is a spanning subhypergraph. It can be seen that the relation of being a subhypergraph is a partial ordering in any class of hypergraphs. For any hypergraph H, a maximal connected subhypergraph of the class of connected subhypergraph is called a connected component or simply a component, of H.