CHAPTER II

TOPOLOGICAL CONCEPTS

Let S be a given non-empty set of objects called the *points* of S. A *topology* in S is a non-empty collection $\mathcal{T}$ of subsets of S called *open sets* satisfying the following three axioms :

1) ϕ and S are open.
2) The union of any family of open sets is open.
3) The intersection of any finite number of open sets is open.

The pair $(S, \mathcal{T})$ is called a *topological space*. When no confusion seems possible we may forget to mention the topology and write "S is a topological space" or simply "S is a space".

Let $(S, \mathcal{T})$ be a given topological space. By a *neighborhood* of $x \in S$ is meant any subset N of S such that there is $U \in \mathcal{T}$ and $x \in U \subseteq N$. A family $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if each open set is the union of members of $\mathcal{B}$. In other words, for every U in $\mathcal{T}$ and each point x in U, there is a $V \in \mathcal{B}$ such that $x \in V \subseteq U$. A subset $\mathcal{V}$ of $\mathcal{T}$ is called a *subbase* for $\mathcal{T}$ if the set of all intersection of finitely many sets in $\mathcal{V}$ is a base for $\mathcal{T}$. A point p in S is called a *limit point* of $A \subseteq S$ if every neighborhood N of p contains at least one point of $A - \{p\}$. The set of limit points of the set A is denoted by A' and is called the *derived set* of A. Following from the definition of limit point we have :

2.1 Theorem. If A and B are subsets in the space $(S, \mathcal{T})$, then $(A \cup B)' = A' \cup B'$. #

2.2 Corollary. If A and B are subsets in the space $(S, \mathcal{T})$ such that $A \subseteq B$, then $A' \subseteq B'$. #

Let S be a space. The closure of $A \subseteq S$, denoted by $\bar{A}$, is defined to be the set $A \cup A'$. It is obvious that if A and B are subsets of S then $\overline{A \cup B} = \bar{A} \cup \bar{B}$. $A \subseteq S$ is defined to be closed if $A = \bar{A}$. A point p in S is said to be an interior point of $A \subseteq S$ if there exists a neighborhood N of p such that $N \subseteq A$. The interior of A is defined to be the set of all interior points of A. Now, the following theorems are easy to prove.

2.3 Theorem. A set G in a space S is open if and only if G consists entirely of interior points. #

2.4 Theorem. A set F in a space S is closed if and only if the complement of F in S, S - F, is open. #

By using DeMorgan's Laws and the axioms of open sets, we have :

2.5 Theorem. The closed sets of a space S satisfy the following three conditions :

1) ϕ and S are closed.
2) The intersection of any family of closed sets is closed.
3) The union of finite number of closed sets is closed. #

A space S is said to be a T_1-space if for any point p in S, {p} is a closed subset of S. If any two distinct points p and q in S belong to disjoint neighborhoods, then S is called a T_2-space or a Hausdorff space.

A function f from a space S into a space T is said to be continuous if for any open set U in T $f^{-1}(U)$, the set of points of S mapped by f into U, is open in S. Equivalently, f is continuous provided $f^{-1}(F)$ is closed whenever F is closed. Let $f : S \to T$ be a function from a space S into a space T. f is continuous at $p \in S$ if for every neighborhood U of f(p) there is a neighborhood V of p such that $f(V) \subseteq U$. And the following theorem is valid.

2.6 Theorem. Let S and T be topological spaces. Then a function f from S into T is continuous if and only if it is continuous at every point of S. #

If f is a bijection from a topological space onto a topological space such that both f and f^{-1} are continuous, then f is called a homeomorphism. If a homeomorphism $h : S \to T$ exists, then two spaces S and T are said to be homeomorphic, and each space is said to be a homeomorph of the other. A property which when possessed by a space is also possessed by each of its homeomorphs is called a topological property.

Let f and g be functions from topological space to topological space such that the composition $g \circ f$ is defined. It can be proved that the following is valid.

2.7 Theorem. If f and g are continuous, so is $g \circ f$. If f and g are homeomorphisms, then f^{-1}, g^{-1} and $g \circ f$ are also homeomorphisms. #

A function f from a space S into a space T is called open (closed) if the image in T of every open (closed) set in S is open (closed) in T. It is obvious that if f is an open (closed), continuous bijection from a space S onto a space T, then f is a homeomorphism.

Let S be a space with topology $\mathcal{T}$ and X is a non-empty subset of S. It is obvious the collection $\{U \mid U = G \cap X$ for some $G \in \mathcal{T}\}$ is a topology for X. This topology is called the relative topology on X and is denoted by $r\text{-}\mathcal{T}$, and X is called a subspace of S. And we have that a subset F of X is closed in X if and only if there exists F^* a closed set in S such that $F = F^* \cap X$.

2.8 Theorem. Let X be any subspace of a space S and let A be any subset of X. Then a point p in X is a limit point of A in X if and only if p is a limit point of A in S.

Proof. A. Let p in X be a limit point of A in X and let G be any open set in S containing p. $G \cap X$ is open in X and containing p; therefore, $G \cap X$ contains at least one point of A different from p. Hence G contains a point of A different from p and so p is a limit point, in S, of A.

B. If p, in X, is a limit point, in S, of A contained in X and G is any open set in X which contains p, then there exists G*, open in S, such that $G^* \cap X = G$. Now G* contains at least one point of A different from p by hypothesis. Since $A \subseteq X$, $G^* \cap X = G$ must contain at least one point of A different from p. Hence p is a limit point, in X, of A. #

2.9 <u>Theorem</u>. If X is any subspace of a space S and A is any subset of X, then $\bar{A}_X = \bar{A}_S \cap X$, where $\bar{A}_X$ and $\bar{A}_S$ denote the closures of A in X and in S, respectively.

<u>Proof.</u> Let A'_X and A'_S denote the derived sets of A in X and in S, respectively; then $\bar{A}_X = A \cup A'_X$, by definition of closure. By theorem 2.8, $A \cup A'_X = A \cup (A'_S \cap X)$. A, A'_X and X are all subsets of S; by the distributive law for union and intersection of sets, $A \cup (A'_X \cap X) = (A \cup A'_S) \cap (A \cup X) = \bar{A}_S \cap (A \cup X)$. Since $A \subseteq X$, $A \cup X = X$. Hence, $\bar{A}_X = \bar{A}_S \cap X$. #

The following theorem is obvious :

2.10 <u>Theorem</u>. If $(S, \mathcal{T})$ is a topological space and if $M \subseteq X \subseteq S$, then the relative topology for M from $(X, r\text{-}\mathcal{T})$ is the same as the relative topology for M from $(S, \mathcal{T})$. #

2.11 <u>Theorem</u>. If S is any space such that $S = F_1 \cup F_2 \cup \ldots \cup F_k$, where k is a natural number and each F_i is closed in S, if $\{f_1, f_2, \ldots, f_k\}$ is a set of functions such that

f_i; $i = 1, 2, \ldots, k$ is a continuous function from F_i into a space T, and if $f_i(x) = f_j(x)$ for $x \in F_i \cap F_j$, then the function h from S into T defined by $h(x) = f_i(x)$ for x in F_i, is continuous on S.

Proof. Let F^* be any closed set in T, then $h^{-1}(F^*) = \bigcup_{i=1}^{k} f_i^{-1}(F^*)$ by definition of h. Since f_i is continuous on F_i, $f_i^{-1}(F^*)$ is closed in S. Hence $h^{-1}(F^*)$ is the union of finite number of closed sets in S. Thus, h is continuous on S. #

By a metric in a non-empty set S, we mean a non-negative real-valued function $\rho : S \times S \to \mathbb{R}$ satisfying the following conditions : For all a, b, c $\in$ S,

1) $\rho(a, b) = 0$ if and only if $a = b$.

2) $\rho(a, b) = \rho(b, a)$.

3) $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$.

The pair (S, ρ) is called a metric space. If $p \in S$ and $\epsilon > 0$, $B(p, \epsilon)$ is defined to be $\{s \in S \mid \rho(p, s) < \epsilon\}$. Define a family $\mathcal{T}$ of subsets of S as follows : For arbitrary subset U of S, U is in $\mathcal{T}$ if and only if for any point $p \in U$, there exists a positive real number ϵ such that U contains $B(p, \epsilon)$. One can easily verify that the family $\mathcal{T}$ is a topology on S. The topology $\mathcal{T}$ on S is called the topology determined by the metric ρ. A given topological space S is said to be metrizable if there exists a metric $\rho : S \times S \to \mathbb{R}$ which defines the topology for the space S.

The following theorem is obvious.

2.12 Theorem. If S is a metrizable space and X is a subspace of S, then X is metrizable. #

Let h be a homeomorphism from a space $(S, \mathcal{T})$ onto a metrizable space $(T, \mathcal{T}^*)$. Suppose ρ^* is an admissible metric for T which induces $\mathcal{T}^*$. For any two points p and q in S, let $\rho(p, q) = \rho^*(h(p), h(q))$. It can be shown that ρ is a metric for S which induces $\mathcal{T}$. Thus, $(S, \mathcal{T})$ is metrizable and we have the following :

2.13 Theorem. Metrizability is a topological property. #

Let (S, ρ) be a metric space, A a non-empty subset of S and p is a point in S, then the distance $\rho(p, A)$ between a point p and A is defined by $\rho(p, A) = \inf\{\rho(p, a) \mid a \in A\}$.

2.14 Theorem. Let (S, ρ) be a metric space and let A be a non-empty subset of S. Then $\rho(t, A) = 0$ if and only if $t \in \bar{A}$. #

From theorem 2.14, we have :

2.15 Corollary. If F is a non-empty closed subset of a metrizable space S with metric ρ and if x is any point which is not in F, then $\rho(x, F)$ is positive. #

It is clear that if the topological space S is metrizable then it is also a Hausdorff and T_1-space. Let (S, ρ) be a metric space and let A be a non-empty subset of S. A is said to be <u>bounded</u> if there exists a positive real number M such that $\rho(x, y) \leq M$ for all x and y in A.

The most important metric space is the <u>Euclidean n-space</u> $\mathbb{R}^n$ where $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}$ and $\mathbb{R}$ is the set of real numbers$\}$ and the metric ρ on $\mathbb{R}^n$ is defined by

$$\rho((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}}.$$

In particular, if n = 1 the metric ρ on $\mathbb{R}$ is defined as $\rho(x, y) = |x - y|$ for x and y in $\mathbb{R}$. The metric ρ on $\mathbb{R}^n$ as defined above is called the <u>usual metric</u>.

A topological space $(S, \mathcal{T})$ is <u>connected</u> if S contains no subset, except S and ϕ, which is both open and closed. A space is called <u>disconnected</u> if it is not connected. It is immediate from the definition that a space S is connected if and only if S is not the union of two non-empty, disjoint, open sets. From now on we use the notation "$S = A \cup B$ separation" when 1) $S = A \cup B$ and 2) $A \cap B = \phi$ and 3) $A \neq \phi \neq B$ and 4) A and B are both open in S.

2.16 <u>Theorem</u>. If f is a continuous function from a space $(S, \mathcal{T})$ onto a space $(T, \mathcal{T}^*)$ and if $(S, \mathcal{T})$ is connected, then $(T, \mathcal{T}^*)$ is connected.

<u>Proof</u>. Let A be any non-empty, proper, open subset in the space $(T, \mathcal{T}^*)$. f is continuous; therefore, $f^{-1}(A)$ is open in S. Since A is non-empty and proper, $\phi \neq f^{-1}(A) \neq S$. Since S is

connected, $f^{-1}(A)$ is not also closed. Thus A can not be closed. Hence, $(T, \mathcal{T}^*)$ is connected. #

2.17 Corollary. Connectedness is a topological property. #

A subset M of a space $(S, \mathcal{T})$ is called a connected (disconnected) subset in $(S, \mathcal{T})$ if $(M, r-\mathcal{T})$, where $r-\mathcal{T}$ denotes the relative topology, is a connected (disconnected) space.

2.18 Theorem. Let $(S, \mathcal{T})$ be a space; let $S = A \cup B$ separation, and let C be a connected subset of S. Then $C \subseteq A$ or $C \subseteq B$.

Proof. Since A and B are open in S, $C \cap A$ and $C \cap B$ are open in the subspace $(C, r-\mathcal{T})$. Furthermore, $(C \cap A) \cup (C \cap B) = C$ and $(C \cap A) \cap (C \cap B) = \phi$. Since C is connected, $C \cap A = \phi$ or $C \cap B = \phi$. Hence, $C \subseteq B$ or $C \subseteq A$. #

2.19 Lemma. If $(S, \mathcal{T})$ is a space, $M \subseteq S$, M is connected in $(S, \mathcal{T})$ and $(X, r-\mathcal{T})$ is a subspace of $(S, \mathcal{T})$ such that $M \subseteq X$, then M is a connected subset of $(X, r-\mathcal{T})$ where $r-\mathcal{T}$ denotes the relative topology.

Proof. By theorem 2.10, the relative topology for M from $(S, \mathcal{T})$ is the same as the relative topology for M from $(X, r-\mathcal{T})$. Thus, there is just one subspace $(M, r-\mathcal{T})$ based on M an one relative topology. By hypothesis, this topology contains no proper, non-empty, open sets. #

2.20 Theorem. If M is a connected subset of a space $(S, \mathcal{T})$ and if $M \subseteq N \subseteq \bar{M}$, then N is a connected subset of $(S, \mathcal{T})$.

Proof. By lemma 2.19, M is a connected subset of N. Suppose $N = A \cup B$ separation. By theorem 2.18, $M \subseteq A$, say. A is closed in N; hence the closure of M in the subspace $(N, r-\mathcal{T})$ is contained in A. By theorem 2.9, the closure of M in $(N, r-\mathcal{T})$ is $\bar{M} \cap N$, where as usual $\bar{M}$ denotes the closure of M in $(S, \mathcal{T})$. However, since $N \subseteq \bar{M}$ by hypothesis, $\bar{M} \cap N = N$. Hence, $N \subseteq A$ and $B = \phi$ which is a contradiction. Thus, there is no separation of N. #

2.21 Corollary. If M is a connected subset of $(S, \mathcal{T})$, then so is $\bar{M}$. #

A finite sequence $\{M_1, M_2, \ldots, M_k\}$ of distinct sets in a set X is called a bridge between M_1 and M_k if $M_i \cap M_{i+1} \neq \phi$ for all $i = 1, 2, \ldots, k-1$. A collection K of sets in a set X is called bridged or a bridged system if for any two sets A and B in K, there is a bridge between A and B whose sets are all in K.

2.22 Theorem. Let $(S, \mathcal{T})$ be a space and let $\{C_v\}$ be a collection of connected subsets of S which form a bridged system. Then $\bigcup_v C_v$ is connected.

Proof. Suppose $\bigcup_v C_v$ is not connected. Let $\bigcup_v C_v = A \cup B$ separation. Let C_v be any set in $\{C_v\}$. Since C_v is connected, by theorem 2.18, $C_v \subseteq A$, say. Let C_α be any other set in $\{C_v\}$. Then, there exists a bridge $C_v = C_{i_1}, C_{i_2}, \ldots, C_{i_k} = C_\alpha$ in $\{C_v\}$. Let P consist of the subset of natural numbers which are greater than k along with the subset of natural numbers m such that $C_{i_m} \subseteq A$. 1 belongs to P since $C_{i_1} = C_v \subseteq A$. Assume j is in P for $j < k$. Then $C_{i_j} \subseteq A$. Hence, since $C_{i_j} \cap C_{i_{j+1}} = \phi$, $C_{i_{j+1}} \subseteq A$. If $j > k$, $j + 1$ is in P, by the definition of P. Hence all natural numbers are in P and $C_\alpha \subseteq A$. Since C_α was any other set in $\{C_v\}$, $\bigcup_v C_v \subseteq A$ and $B = \phi$. This contradicts the definition of $A \cup B$ separation. Hence $\bigcup_v C_v$ is connected. #

2.23 <u>Theorem</u>. Let $\{C_v\}$ be a collection of connected subsets in a space $(S, \mathfrak{T})$ such that $\bigcap_v C_v \neq \phi$. Then $\bigcup_v C_v$ is connected.

<u>Proof</u>. Let C_α and C_β be any two sets in $\{C_v\}$. $\{C_\alpha, C_\beta\}$ is a bridge from C_α to C_β. Hence, $\{C_v\}$ is a bridged system of connected sets, and so by the previous theorem, $\bigcup_v C_v$ is connected. #

2.24 <u>Theorem</u>. Let $(S, \mathfrak{T})$ be a space such that each pair of points in S is contained in a connected subset of S. Then $(S, \mathfrak{T})$ is connected.

<u>Proof</u>. Let s be a given point of S and let x be any point of S. There exists a connected subset Z_x containing s and x. $S = \bigcup_x Z_x$, $\bigcap_x Z_x \neq \phi$ and each Z_x is connected. Hence, $(S, \mathfrak{T})$ is connected by the previous theorem.

2.25 <u>Theorem</u>. The closed interval $[a, b] = \{x \mid a \leq x \leq b$ where $a < b\}$ is connected in $\mathbb{R}$, the space of real numbers.

<u>Proof</u>. Suppose $[a, b] = A \cup B$ where A and B are closed subsets of $[a, b]$. Let b be in B. Since A is a bounded subset of real numbers, if A is not empty, then A has a least upper bound γ, and $a \leq \gamma \leq b$. Since A is closed, γ is in A. If $\gamma = b$, then γ is in B and $A \cap B \neq \phi$. If $\gamma < b$, then $\{x \mid \gamma < x \leq b\} \subseteq B$ by definition of γ, and since B is closed, γ is in B. Thus, γ is in $A \cap B$ and hence no separation of $[a, b]$ exists. #

2.26 <u>Theorem</u>. The open interval $(a, b) = \{x \mid a < x < b$, for $a < b\}$ is connected in $\mathbb{R}$.

<u>Proof</u>. Let p and q be any two distinct points in (a, b) and let $p < q$. Then $[p, q] \subsetneq (a, b)$. Since $[p, q]$ is connected by theorem 2.25, (a, b) is connected by theorem 2.24. #

2.27 <u>Corollary</u>. The "half-open" interval $[a, b) = \{x \mid a \leq x < b\}$ and $(a, b] = \{x \mid a < x \leq b\}$ are connected in $\mathbb{R}$.

<u>Proof</u>. The sets [a, b) and (a, b] lie between (a, b) and its closure [a, b], and so, by theorem 2.20, are connected. #

2.28 <u>Theorem</u>. Let M be any connected subset of $\mathbb{R}$. If a and b belong to M such that $a < b$, then $\{x \mid a \leq x \leq b\}$ is a subset of M.

<u>Proof</u>. Let $a < \gamma < b$. Assume γ is not in M. The sets

$A = \{x \mid x < \gamma\}$ and $B = \{x \mid x > \gamma\}$ are open subsets of $\mathbb{R}$. Hence, $M \cap A$ and $M \cap B$ are open in the subspace M and $M = (M \cap A) \cup (M \cap B)$ separation. This is a contradiction since M is connected. It follows that γ belongs to M and hence $\{x \mid a \leq x \leq b\} \subseteq M$. #

Let $(S, \mathcal{T})$ be a topological space. A subset C of S is a component of S; provided that C is connected, but is not a proper subset of another connected subset of S. It is obvious that if S is connected, then S itself is the only component of S.

2.29 Theorem. In any topological space S, components are closed.

Proof. Let S be a topological space and let C be a component of S. Since C is connected, by corollary 2.21, $\bar{C}$ is connected. Since C is a component, $C = \bar{C}$ and hence C is closed. #

2.30 Theorem. In a topological space S, any non-empty connected subset which is both open and closed is a component.

Proof. Let A be a non-empty connected subset of S which is both open and closed. Furthermore, assume $A \subseteq B$. By the definition of relative topology, A is both open and closed in B. Hence, B is not connected unless A = B. #

Let S be a connected space. If p is a point of S such that $S-\{p\}$ is disconnected, then p is called a cut point of S; otherwise, p is a non-cut point. If A is a subset of a connected space S such that S-A is not connected A is said to separate S. It is obvious that the property of being either a cut point or a non-cut point is a topological property.

2.31 Theorem. Let $(S, \mathcal{T})$ be a connected space and let M be a connected subset of S such that M separates S, i.e., $S - M = A \cup B$ separation. Then $A \cup M$ and $B \cup M$ are both connected.

Proof. Suppose $A \cup M$ is not connected; let $A \cup M = A_1 \cup A_2$ separation. M is connected; therefore, $M \subseteq A_1$, say. Thus, A_2 is contained in A. Now, consider $S = (A \cup M) \cup B = A_2 \cup (A_1 \cup B)$. $\bar{A}_2 \cap A_1 = \phi$ and since $A_2 \subseteq A$ and $\bar{A} \cap B = \phi$, $\bar{A}_2 \cap B = \phi$. Hence $\bar{A}_2 \cap (A_1 \cup B) = \phi$. Furthermore, $\bar{A}_1 \cap A_2 = \phi$ and since $\bar{B} \cap A = \phi$, $\bar{B} \cap A_2 = \phi$. Hence, $A_2 \cap (\bar{A}_1 \cup \bar{B}) = A_2 \cap \overline{(A_1 \cup B)} = \phi$. Therefore, $S = A_2 \cup (A_1 \cup B)$ separation which is a contradiction. Thus, $A \cup M$ is connected. The same proof is valid for $B \cup M$. #

2.32 Corollary. If p is a cut point of a connected space S such that $S - \{p\} = A \cup B$ separation, then $A \cup \{p\}$ and $B \cup \{p\}$ are connected. #

A family G of sets is a cover of a set X if each point of X belongs to some member of G. The family is an open cover of X if each member of G is an open set. A subcover of G is a subfamily which is also a cover. A topological space S is said to be compact if every open cover of S has a finite subcover. A subset X of a topological space S is said to be compact if, with the relative topology, the subspace X is compact. A topological space S is called countably compact if every countable open covering of S contains a finite subcovering. A collection of closed sets in a topological space is said to have the finite intersection property if the intersection of any finite number of sets in the collection is not empty.

2.33 Theorem. A space S is compact if and only if every family of closed sets with the finite intersection property has a non-empty intersection.

Proof. A. Let S be compact; let $\mathcal{F} = \{F_v \mid v \in I$ where I is an index set$\}$ denote a family of closed sets in S with the finite intersection property. Consider the collection $G = \{\sim F_v \mid v \in I\}$ of all complements of sets in $\mathcal{F}$. Each $\sim F_v$ is open in S. Assume $\bigcap_v F_v = \phi$. Then $\sim \bigcap_v F_v = S$. However, $\sim \bigcap_v F_v = \bigcup_v \sim F_v$. Hence, the collection G is an open covering of S. Since S is compact, a finite number, say $\sim F_{v_1}, \sim F_{v_2}, \ldots, \sim F_{v_k}$ of sets in G cover S. Therefore, $\bigcup_{i=1}^{k} \sim F_{v_i} = S$ and so, by De Morgan's law, $\sim \bigcap_{i=1}^{k} F_{v_i} = S$. Hence, $\bigcap_{i=1}^{k} F_{v_i} = \sim S = \phi$. This is a contradiction since the family $\mathcal{F}$ was supposed to have the finite intersection property. Thus the assumption that $\bigcap_v F_v = \phi$ is false.

B. Let S have the property that any family $\mathcal{F} = \{F_v \mid v \in I\}$ of closed sets with the finite intersection property has a non-empty intersection. Then, let $G = \{G_v \mid v \in I\}$

be an open covering of S. Assume that no finite subset of G covers S. Then, if $\{G_{v_1}, G_{v_2}, \ldots, G_{v_k}\}$ is any finite subset of G, $\bigcup_{i=1}^{k} G_{v_i} \neq S$. Hence, $\sim\bigcup_{i=1}^{k} G_{v_i} \neq \phi$. Therefore, $\bigcap_{i=1}^{k} \sim G_{v_i} \neq \phi$. Hence, if $\mathfrak{F}$ denotes the family $\{\sim G_v \mid v \in I\}$ of complements of sets in G, $\mathfrak{F}$ satisfies the finite intersection property. Hence, $\bigcap_v \sim G_v$ is not empty; therefore, $\sim\bigcup_v G_v \neq \phi$. Thus, $\bigcup_v G_v$ is not equal to S. This is a contradiction since $\{G_v \mid v \in I\}$ was a covering of S. Hence, the assumption that no finite subset of G covers S is false and so S is compact. #

The proof of the following theorem is an analogue of the proof of theorem 2.33.

2.34 <u>Theorem</u>. A space S is countably compact if and only if every countable family of closed sets with the finite intersection property has a non-empty intersection. #

2.35 <u>Theorem</u>. If a space S is countably compact, then every infinite subset of S has a limit point in S.

<u>Proof</u>. Let A be any infinite subset of S and let $\{x_1, x_2, \ldots.\}$ denote any countable infinite subset of A. Let $x_i \neq x_j$ for $i \neq j$. Assume that the set $\{x_1, x_2, \ldots.\}$ has no limit point in S. Then, by corollary 2.2, no subset of $\{x_1, x_2, \ldots..\}$ has a limit point in S. In particular, the sets $F_n = \{x_n, x_{n+1}, \ldots\}$ are all closed sets in S. Furthermore, the countably family $\{F_n \mid n \text{ is a natural number}\}$ has the finite intersection property

since $\{x_1, x_2, \ldots\}$ is infinite. However, $\bigcap_{n=1}^{\infty} F_n$ is empty which contradicts theorem 2.34. Thus, the assumption that $\{x_1, x_2, \ldots\}$ has no limit point is false. Hence A has a limit point in S. #

2.36 Theorem. If f is a continuous function from a compact space S onto a space T, then T is compact.

Proof. Let $G = \{G_v\}$ be any open covering of T. The collection $\{f^{-1}(G_v)\}$ constitutes an open covering of S. Since S is compact, a finite number of these sets, say $f^{-1}(G_{v_1}), f^{-1}(G_{v_2}), \ldots, f^{-1}(G_{v_k})$ cover S. Then $S = \bigcup_{i=1}^{k} f^{-1}(G_{v_i}) = f^{-1}(\bigcup_{i=1}^{k} G_{v_i})$ and hence $\bigcup_{i=1}^{k} G_{v_i}$ is a cover of T; so T is compact. #

2.37 Corollary. Compactness is a topological property. #

Let P be a set and let $\leq$ be a binary relation in P such that for any x, y and z in P the following conditions hold :

1) $x \leq x$, and
2) $x \leq y$ and $y \leq x$ imply $x = y$, and
3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

Then $\leq$ is called a partial ordering for P and the pair $(P, \leq)$ is said to be a partially ordered set. Let Y be a subset of $(P, \leq)$, Y is simply ordered by $\leq$ if for every x and y in Y, $x \leq y$ or $y \leq x$.

Hausdorff's Maximum Principle. Every partially ordered set P contains a maximal (relative to inclusion) simply ordered subset $\mathcal{L}$, i.e., $\mathcal{L}$ is not contained properly in any other simply ordered subset of P. We use Hausdorff's Maximum Principle to prove the next theorem.

2.38 Theorem. A compact, connected T_1-space S with more than one point contains at least two non-cut points.

Proof. A. Let N denote the set of non-cut points of S. Assume $N = \phi$ or $N = \{s\}$. Then, since S contains more than one point, there exists a cut point c in S. Let $S-\{c\} = A \cup B$ separation. Let $s \in B$, then $N \subseteq B$ and so $N \cap A = \phi$. This means that every point of A is a cut point of S. For each x in A, let $A_x \cup B_x$ denote a separation of $S-\{x\}$. Let $c \in B_x$. By corollary 2.32, $A_x \cup \{x\}$ is connected in S. Now, since $c \in B_x$ and $x \neq c$, $A_x \cup \{x\} \subseteq S-\{c\} = A \cup B$. Also, since $A_x \cup \{x\}$ is connected and $x \in A$, $A_x \cup \{x\} \subseteq A$, by theorem 2.18. Thus, the set $P = \{A_x \cup \{x\}$ such that $x \in A\}$ is a set of subsets of A partially ordered by inclusion.

B. The Hausdorff's Maximum Principle will be invoked on the partially ordered set P, but first it must be shown that if $q \in A$ and $p \in A_q$, as defined above, then $A_p \cup \{p\} \subseteq A_q$ and $q \notin A_p \cup \{p\}$. So, let $q \in A$; q is then a cut point of S and $S-\{q\} = A_q \cup B_q$ separation. Now let $p \in A_q$. $(A_p \cup \{p\}) \cap (B_p \cup \{p\}) = \{p\}$. Hence, $q \notin A_p \cup \{p\}$ or $q \notin B_p \cup \{p\}$. Now, $p \notin B_q$ since $p \in A_q$. Therefore, $A_p \cup \{p\} \nsubseteq B_q$ and $B_p \cup \{p\} \nsubseteq B_q$. Since $c \in B_p$ and $c \notin A_q$, $B_p \cup \{p\} \nsubseteq A_q$.

Furthermore, since $B_p \cup \{p\} \nsubseteq A_q$ and $B_p \cup \{p\} \nsubseteq B_q$, and since $B_p \cup \{p\}$ is connected, $B_p \cup \{p\} \nsubseteq A_q \cup B_q$. Therefore, $q \notin B_p \cup \{p\}$. Since $p \in A_q$, $p \neq q$. Hence, $q \notin A_p \cup \{p\}$, and so $A_p \cup \{p\} \subseteq A_q$.

C. Now by the Hausdorff's maximum principle, there exists a maximal simply order subset $\mathcal{L}$ of the set $P = \{M \mid M = A_x \cup \{x\}$ for $x \in A\}$. Since S is a T_1-space, $A_x \cup B_x$ is open in S and so is B_x. Thus, $A_x \cup \{x\}$ is closed in S for every x in A. Since $\mathcal{L}$ is simply ordered and $A_x \cup \{x\} \neq \phi$ for all x in A, $\mathcal{L}$ satisfies the finite intersection property. Since S is compact, $\cap\mathcal{L} \neq \phi$ by theorem 2.33. Let $\alpha \in \cap\mathcal{L}$. Then $\alpha \in A$. Hence, by the original assumption, α is a cut point of S. Let $S-\{\alpha\} = A_\alpha \cup B_\alpha$ separation. As in part A above, $A_\alpha \cup \{\alpha\} \subseteq A$. Let $y \in A_\alpha$. By part B, $A_y \cup \{y\} \subseteq A_\alpha$ and $\alpha \notin A_y \cup \{y\}$. Thus, $A_y \cup \{y\} \subsetneq A_\alpha \cup \{\alpha\}$. Since $\alpha \in A_x \cup \{x\}$ for every $A_x \cup \{x\}$ in $\mathcal{L}$, if $x \neq \alpha$, $\alpha \in A_x$. Therefore, by part B, $A_\alpha \cup \{\alpha\} \subseteq A_x$ for $x \neq \alpha$. Hence $A_y \cup \{y\}$ is properly contained in every $A_x \cup \{x\}$ in $\mathcal{L}$. Therefore, $\mathcal{L} \subset \mathcal{L} \cup \{A_y \cup \{y\}\}$ and $\mathcal{L} \cup \{A_y \cup \{y\}\}$ is simply ordered by inclusion. This contradicts the maximality of $\mathcal{L}$. Thus, the existence of the set A_α leads to a contradiction. This means that α can not be a cut point. However, $\alpha \in A$ and, by the original assumption that $N = \phi$ or $N = \{s\}$, every point of A is a cut point. Thus $N \neq \phi$ and $N \neq \{s\}$. #

If S is a connected space and M is a subset of S, then S is said to be <u>irreducibly connected about M</u> if no proper connected subset of S contains M.

2.39 <u>Lemma</u>. Every closed subset of a compact space is compact.

<u>Proof</u>. Let F be a closed subset of a compact space S. Let G be any open covering of F. $G \cup \{\sim F\}$ is an open covering of S. Hence, a finite subcovering $\{\sim F, G_1, G_2, \ldots, G_k\}$ covers S where $G_i \in G$; $i = 1, 2, \ldots, k$. Hence, $\{G_1, G_2, \ldots, G_k\}$ covers F. #

2.40 <u>Theorem</u>. A compact, connected T_1-space is irreducibly connected about its set of non-cut points.

<u>Proof</u>. Let N be the set of non-cut points of the compact, connected T_1-space S. Assume that S is not irreducibly connected about N. Then, there exists a proper, connected subspace X of S such that $N \subseteq X$. Let $\alpha \in S - X$. Then α is a cut point of S and $S - \{\alpha\} = A \cup B$ separation. Since X is connected, $X \subseteq A$ or $X \subseteq B$, by theorem 2.18. Let $X \subseteq A$. Now, $B \cup \{\alpha\}$ is connected by corollary 2.32. Furthermore, since S is a T_1-space, $S - \{\alpha\}$ is open in S and hence A is open in S. Thus $B \cup \{\alpha\}$ is closed in S; so $B \cup \{\alpha\}$ is compact by lemma 2.39. Therefore, $B \cup \{\alpha\}$ is then a compact, connected T_1-space. Furthermore, since $B \neq \phi$ and $\alpha \notin B$, $B \cup \{\alpha\}$ contains more than one point. Thus, by the theorem 2.38, $B \cup \{\alpha\}$ contains at least two non-cut points. Let β be a non-cut point of $B \cup \{\alpha\}$ such that $\beta \neq \alpha$. Thus $\beta \in B$ and $B \cup \{\alpha\} - \{\beta\}$ is connected. Since $A \cup \{\alpha\}$ is connected and $(A \cup \{\alpha\}) \cap (B \cup \{\alpha\} - \{\beta\}) \neq \phi$, $(A \cup \{\alpha\}) \cup (B \cup \{\alpha\} - \{\beta\}) = S - \{\beta\}$ is also connected by theorem 2.23 and hence β is a non-cut point of S. However,

$\beta \in B'$ and $N \subseteq A$, where N is the set of non-cut points of S. Thus, the existence of the proper connected subset X of S containing N leads to a contradiction. Hence, S is irreducibly connected about N. #

2.41 <u>Corollary</u>. If S is a compact, connected T_1-space and if N is the set of non-cut points of S and if $S - \{\alpha\} = A \cup B$ separation, then $N \cap A \neq \phi \neq N \cap B$.

<u>Proof</u>. If $N \cap A = \phi$, then $N \subseteq B$. Since by corollary 2.32, $B \cup \{\alpha\}$ is connected, S would not be irreducibly connected about N. This contradicts the previous theorem. That is $N \cap A \neq \phi$. The same proof is valid for $N \cap B \neq \phi$. #

Now we will state a very important theorem which characterizes compact sets in $\mathbb{R}^n$, Euclidean n-space. The proof of this theorem can be found in [1] pp. 284-285.

2.42 <u>Theorem</u>. A subset A of Euclidean n-space, $\mathbb{R}^n$, is compact if and only if A is closed and bounded. #

2.43 <u>Theorem</u>. Any compact subset of a Hausdorff space is closed.

<u>Proof</u>. Let F be a compact subset of a Hausdorff space S. Let p be any point in S - F. Since S is Hausdorff, for each x in F, there exists an open neighborhood U_x of x and V_x of P such that $U_x \cap V_x = \phi$. The collection $\{U_x \mid x \in F\}$ form an open covering of F. F is compact; hence, a finite number of the sets, say

$U_{x_1}, U_{x_2}, \ldots, U_{x_k}$, cover F. The corresponding neighborhoods of p are $V_{x_1}, V_{x_2}, \ldots, V_{x_k}$ and $U_{x_i} \cap V_{x_i} = \phi$ for i = 1, 2, ..., k. Hence, $\bigcap_{i=1}^{k} V_{x_i}$ is a neighborhood of p which contains no point of F. Then, p is an interior point of $\sim F$; so $\sim F$ is open and hence F is closed. #

2.44 Theorem. If f is a continuous function from a compact space S into a Hausdorff space T, then f is closed.

Proof. Let F be any closed set in a compact space S. Then F is compact by lemma 2.39; hence, by theorem 2.36, f(F) is compact in T. Since T is Hausdorff, by theorem 2.43, f(F) is closed. #

2.45 Corollary. Any one-one continuous function from a compact space S onto a Hausdorff space T is a homeomorphism. #

2.46 Theorem. Let $(S, \mathcal{T})$ be a compact space and let $(S, \mathcal{T}^*)$ be a Hausdorff space. If $\mathcal{T}^* \subseteq \mathcal{T}$, then $\mathcal{T}^* = \mathcal{T}$.

Proof. Let $f : (S, \mathcal{T}) \rightarrow (S, \mathcal{T}^*)$ be a function defined by $f(x) = x$. It is obvious that f is one-one and onto. Since $\mathcal{T}^* \subseteq \mathcal{T}$, f is continuous and hence is a homeomorphism by corollary 2.45; so $\mathcal{T}^* = \mathcal{T}$. #

It follows from the theorem 2.46 that a compact, Hausdorff topology on a set S is a minimal element (by inclusion) in the set of Hausdorff topologies for S.

2.47 Theorem. If $(S, \mathcal{T})$ is a compact, Hausdorff space and if $\mathcal{T}^* \subset \mathcal{T}$, then $(S, \mathcal{T}^*)$ is compact but not Hausdorff.

Proof. The case $(S, \mathcal{T}^*)$ is compact is obvious since $\mathcal{T}^* \subset \mathcal{T}$. The case $(S, \mathcal{T}^*)$ is not Hausdorff follows from the theorem 2.46. #

A subset M of a space S is called dense in S if $\bar{M} = S$ and a space S is called separable if it contains a countable dense subset. A finite subset γ of a metric space (S, ρ) is called an ϵ-dense set if for every point p of S, there exists at least one point $p_i \in \gamma$ such that $\rho(p, p_i) < \epsilon$. A metric space (S, ρ) is called totally bounded if for every positive real number ϵ, (S, ρ) has an ϵ-dense set.

2.48 Theorem. If a metric space (S, ρ) is countably compact, then it is totally bounded.

Proof. Assume that there exists a positive real number ϵ such that (S, ρ) has no ϵ-dense subset. Let p_1 be any point in (S, ρ). Then there is a p_2 in S such that $\rho(p_1, p_2) \geq \epsilon$. Similarly, there exists p_3 in S such that $\rho(p_2, p_3) \geq \epsilon$. Assume that for any natural number k the set $\{p_1, p_2. \ldots, p_k\}$ of S has been defined such that $\rho(p_i, p_j) \geq \epsilon$ for $i \neq j$ and $1 \leq i, j \leq k$. Since, by assumption, (S, ρ) is not totally bounded, there exists a point p_{k+1} in S such that $\rho(p_i, p_{k+1}) \geq \epsilon$ for $1 \leq i \leq k$. Thus, a countably infinite set $\{p_1, p_2, \ldots\}$ is defined with the property that $\rho(p_i, p_j) \geq \epsilon$ for $i \neq j$. Since S is countably compact, $\{p_1, p_2, ..\}$

has a limit point q in S by theorem 2.35. Then there exists a p_j such that $p_j \in B(q, \epsilon/2)$. Let $\delta = \rho(q, p_j)$. Then $\delta < \epsilon/2$. Similarly, there exists $p_i \neq p_j$ such that $p_i \in B(q, \delta)$. Hence $\rho(p_i, p_j) \leq \rho(p_i, q) + \rho(q, p_j) < \epsilon/2 + \epsilon/2 = \epsilon$. This is a contradiction since $\rho(p_i, p_j) \geq \epsilon$. Hence (S, ρ) is totally bounded. #

2.49 Theorem. If (S, ρ) is totally bounded, then S is separable.

Proof. For any natural number n, let A_n denote a 1/n-dense set for the totally bounded space (S, ρ). Let $D = \bigcup_n A_n$. D, as the union of a countable set of finite sets, is countable. Let p be any point in S. By the archimedean order on the reals, if ϵ is any positive real, there exists a positive integer n such that $1/n < \epsilon$. Hence, there exists n_i in the 1/n-dense A_n such that $\rho(p, n_i) < 1/n$, i.e., $B(p, \epsilon) \cap D \neq \phi$. Hence $p \in \bar{D}$. Thus $S = \bar{D}$. #