CHAPTER IV

MEAN VALUE ITERATIONS OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

4.1 Introduction. In this chapter we apply a mean value iterative method to obtain a theorem on the determination of a fixed point of a nonexpansive mapping of a Banach space.

We first give the following definitions.

4.2 Definition. A subset E of a vector space X is said to be convex if whenever $x_1, x_2 \in E$, then

$$x(t) = tx_2 + (1-t)x_1 \in E, \quad 0 \leq t \leq 1.$$

4.3 Definition. A normed vector space X is said to be uniformly convex provided for each $\varepsilon > 0$ there corresponds a $\delta(\varepsilon) > 0$ such that if $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$, then

$$\left\| \tfrac{1}{2}(x+y) \right\| \leq 1 - \delta(\varepsilon).$$

This definition is a geometric property of the unit sphere of the space: if the midpoint of a line segment with the end points on the surface of the sphere approaches the surface, then the end points must come closer together.

We remark that Euclidean spaces of all dimensions and

Hilbert spaces are all uniformly convex. This follows, for example, from the identity

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

which is known to be characteristic of such spaces. In fact, for each ε, $0 < \varepsilon \leq 2$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$, we see that

$$\begin{aligned} \|x+y\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 \\ &\leq 4 - \varepsilon^2 \\ &= 4(1 - \tfrac{\varepsilon^2}{4}). \end{aligned}$$

Thus

$$\frac{1}{4}\|x+y\|^2 \leq 1 - \frac{\varepsilon^2}{4},$$

or

$$\|\tfrac{1}{2}(x+y)\| \leq (1 - \frac{\varepsilon^2}{4})^{\frac{1}{2}}.$$

Let $\delta(\varepsilon) = 1 - (1 - \frac{\varepsilon^2}{4})^{\frac{1}{2}}$. Then

$$\|\tfrac{1}{2}(x+y)\| \leq 1 - \delta(\varepsilon).$$

F.E. Browder has proved that each nonexpansive mapping which maps a closed bounded convex subset E of a uniformly convex Banach space into itself has a fixed point in E.

If such a mapping satisfies one additional requirement, we may approximate one of its fixed points using the mean value iterations.

4.4 <u>Definition</u>. A mapping $T:E \to X$ of a subset E of a Banach space X is said to be <u>demicompact</u> provided whenever $\{x_n\} \subset E$ is bounded and $\{x_n - Tx_n\}$ converges then there is a subsequence $\{x_{n_k}\}$ which converges.

4.5 Theorem. Let X be a uniformly convex Banach space, E be a closed bounded convex subset of X. If T is a nonexpansive mapping of E into itself and if T is demicompact, then the iterative scheme

(1) $$x_{n+1} = Tv_n ,$$

(2) $$v_n = \frac{1}{n} (x_1+\dots+x_n),\ n = 1,2,3,\dots,$$

(3) $$v_1 = x_1 \in E$$

converges to a fixed point of T.

To prove this theorem we need a lemma.

4.6 Lemma. Let X be a uniformly convex Banach space and let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon > 0$, then

$$\|tx+(1-t)y\| \leq 1 - 2t(1-t)\delta(\varepsilon)$$

for $0 \leq t \leq 1$.

Proof. We first assume that $0 \leq t \leq \frac{1}{2}$. But this implies $(1-2t) \geq 0$. We then have

$$\begin{aligned} \|tx+(1-t)y\| &= \|2t\cdot\tfrac{1}{2}(x+y) + (1-2t)y\| \\ &\leq 2t\|\tfrac{1}{2}(x+y)\| + (1-2t)\|y\| \\ &\leq 2t(1-\delta(\varepsilon)) + (1-2t) \\ &= 1 - 2t\,\delta(\varepsilon) \\ &\leq 1 - 2t(1-t)\,\delta(\varepsilon) \end{aligned}$$

since $t \geq t(1-t)$.

Next suppose that $1 \geq t \geq \frac{1}{2}$. Then $2t-1 \geq 0$.

Hence

$$\begin{aligned} \|tx+(1-t)y\| &= \|2(1-t)\cdot\tfrac{1}{2}(x+y) + (2t-1)x\| \\ &\leq 2(1-t)\,\|\tfrac{1}{2}(x+y)\| + (2t-1)\|x\| \\ &\leq 2(1-t)(1-\delta(\varepsilon)) + (2t-1) \\ &= 1 - 2(1-t)\,\delta(\varepsilon) \\ &\leq 1 - 2t(1-t)\,\delta(\varepsilon) \end{aligned}$$

since $(1-t) \geq t(1-t)$. Therefore

$$\|tx+(1-t)y\| \leq 1 - 2t(1-t)\delta(\varepsilon)$$

for $0 \leq t \leq 1$.

Q.E.D.

Proof. (of Theorem 4.5) It can be shown, by induction on n, that $v_n \in E$ for all n.

For each n, we have

$$\begin{aligned} v_{n+1} &= \frac{1}{n+1}(x_1+\dots+x_n+x_{n+1}) \\ &= \frac{1}{n+1}(n\cdot v_n + Tv_n) \\ &= (1-\frac{1}{n+1})v_n + \frac{1}{n+1}Tv_n . \end{aligned}$$

We set $c_n = \frac{1}{n+1}$, $n = 1,2,3,\dots$. Then

$$v_{n+1} = (1-c_n)v_n + c_nTv_n , \quad n = 1,2,3,\dots . \tag{4}$$

If for some n, $v_n = Tv_n$, then clearly $\{v_n\}$ converges to v_n. Hence we may suppose that $Tv_n \neq v_n$ for each n.

From Browder's fixed point theorem, $F(T)$, the set of fixed point of T, is not empty. Let q denote any point of $F(T)$.

For any $v_n \in E$, we have

$$(5) \qquad \|v_{n+1} - q\| = \|(1-c_n)v_n + c_nTv_n - q\|$$
$$= \|(1-c_n)(v_n-q) + c_n(Tv_n-q)\|$$
$$\leq (1-c_n)\|v_n-q\| + c_n\|Tv_n-q\| .$$

Since T is nonexpansive and q is a fixed point of T, therefore

$$\|Tv_n-q\| = \|Tv_n-Tq\| \leq \|v_n-q\| .$$

Substituting this last expression into (5), we obtain

$$(6) \qquad \|v_{n+1} - q\| \leq \|v_n - q\| .$$

Thus $\{\|v_n-q\|\}$ is nonincreasing.

Furthermore, the sequence $\{\|v_n-Tv_n\|\}$ is also nonincreasing since, by (4),

$$\|v_{n+1} - v_n\| = c_n\|Tv_n-v_n\|$$

and since T is nonexpansive, therefore

$$\|v_{n+1}-Tv_{n+1}\| = \|(1-c_n)v_n + c_nTv_n - Tv_{n+1}\|$$
$$= \|(1-c_n)(v_n-Tv_n) + (Tv_n - Tv_{n+1})\|$$
$$\leq (1-c_n)\|v_n-Tv_n\| + \|Tv_n - Tv_{n+1}\|$$
$$\leq (1-c_n)\|v_n-Tv_n\| + \|v_n - v_{n+1}\|$$
$$= (1-c_n)\|v_n-Tv_n\| + c_n\|Tv_n-v_n\|$$
$$= \|v_n - Tv_n\| .$$

We want to show that $\lim_{n\to\infty}\|Tv_n-v_n\| = 0$. Suppose $\{\|v_n-Tv_n\|\}$ does not converge to 0. Then, by Theorem 1.5, there is an $\varepsilon > 0$ such that $\|v_n-Tv_n\| \geq \varepsilon$ for all n.

Also since

$$\|v_n - Tv_n\| = \|(v_n-q)-(Tv_n-q)\| \leq 2\|v_n-q\| ,$$

we may assume that there is an $a > 0$ such that $\|v_n-q\| \geq a$ for all n.

Since

$$\begin{aligned}\|v_{n+1}-q\| &= \|(1-c_n)v_n + c_nTv_n - q\| \\ &= \|(1-c_n)(v_n-q) + c_n(Tv_n-q)\| ,\end{aligned}$$

thus

$$(7) \qquad \|v_{n+1}-q\| = \|v_n-q\|\left[\left\|(1-c_n)\frac{v_n-q}{\|v_n-q\|} + c_n\frac{Tv_n-q}{\|v_n-q\|}\right\|\right] .$$

Now

$$\left\|\frac{(v_n-q)}{\|v_n-q\|}\right\| \leq 1, \qquad \left\|\frac{(Tv_n-q)}{\|v_n-q\|}\right\| \leq 1$$

and

$$\left\|\frac{Tv_n-q}{\|v_n-q\|} - \frac{v_n-q}{\|v_n-q\|}\right\| = \frac{\|Tv_n-v_n\|}{\|v_n-q\|} \geq \frac{\varepsilon}{\|v_{n-1}-q\|} \geq \dots \geq \frac{\varepsilon}{\|v_1-q\|} > 0.$$

Let $b = 2\delta\left(\frac{\varepsilon}{\|v_1-q\|}\right)$. The Lemma 4.6 and (7), gives

$$\begin{aligned}\|v_{n+1}-q\| &\leq \|v_n-q\|(1-c_n(1-c_n)b) \\ &= \|v_n-q\| - \|v_n-q\| c_n(1-c_n)b \\ &\leq \|v_{n-1}-q\| - \|v_{n-1}-q\| c_{n-1}(1-c_{n-1})b \\ &\qquad - \|v_n-q\| c_n(1-c_n)b \\ &\leq \|v_{n-1}-q\| - \|v_n-q\| b\left[c_{n-1}(1-c_{n-1}) + c_n(1-c_n)\right] ,\end{aligned}$$

since $\|v_n-q\| \leq \|v_{n-1}-q\|$.

By induction we have

$$a \leq \|v_{n+1}-q\| \leq \|v_1-q\| - \|v_n-q\|\, b \sum_{k=1}^{n} c_k(1-c_k) .$$

Therefore

$$a(1 + b \sum_{k=1}^{n} c_k(1-c_k)) \leq \|v_1-q\| .$$

This gives a contradiction since the series on the left diverges.

Therefore

$$\lim_{n\to\infty} \|v_n-Tv_n\| = 0 .$$

By the hypothesis T is demicompact, then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ which is convergent.

Suppose that $\lim_{k\to\infty} v_{n_k} = p$.

Since T is a nonexpansive mapping, T is continuous and (I-T) is also continuous. Then

$$\lim_{k\to\infty} (I-T)v_{n_k} = (I-T)p$$

But $\lim_{n\to\infty} (I-T)v_n = 0$. Therefore

$$(I-T)p = 0$$

i.e., p is a fixed point of T.

It follows from (5) and (6) that $\{\|v_n-p\|\}$ is nonincreasing.

The conditions $\lim_{k\to\infty} v_{n_k} = p$ and $\{\|v_n-p\|\}\downarrow$ in n yield $\lim_{n\to\infty} v_n = p$.

Q.E.D.

We note that $\mathbb{R}^N$ is a uniformly convex Banach space and every closed and bounded subset E of $\mathbb{R}^N$ is compact. The following corollary, which the demicompactness of T is relaxed, is a consequence of the Theorem 4.5 .

4.7 Corollary. Let E be a compact convex subset of $\mathbb{R}^N$, and let T be a nonexpansive mapping which maps E into itself. Then the iterative scheme (1)-(3) of Theorem 4.5 converges to a fixed point of T.

Proof. We can see in the proof of Theorem 4.5 that

(i) $$\lim_{n\to\infty} \|v_n - Tv_n\| = 0$$

and, for any $q \in F(T)$,

(ii) $$\{\|v_n - q\|\} \text{ is nonincreasing.}$$

Since E is compact and $\{v_n\} \subset E$, hence there is a subsequence of $\{v_n\}$, $\{v_{n_k}\}$ say, which is convergent. Suppose that $\lim_{k\to\infty} v_{n_k} = p$.

Since T is a nonexpansive mapping, T is continuous and (I-T) is also continuous. Therefore

$$\lim_{k\to\infty} (I-T)v_{n_k} = (I-T)p.$$

This last expression, along with (i), gives

$$(I-T)p = 0, \text{ or } Tp = p.$$

It follows from (ii) that $\{\|v_n - p\|\}$ is nonincreasing.

The conditions $\lim_{k\to\infty} v_{n_k} = p$ and $\{\|v_n - p\|\}\downarrow$ in n yield $\lim_{n\to\infty} v_n = p$.

Q.E.D.