CHAPTER III

MEAN VALUE ITERATIONS ON THE CLOSED UNIT N-DISK (N-CELL)

3.1 Introduction. In chapter II, we considered a mapping T which continuously maps the closed interval $E = [0,1]$ into itself and we proved that the iterative scheme

(1) $x_{n+1} = Tv_n$,

(2) $v_n = \frac{1}{n}(x_1 + \dots + x_n)$, $n = 1,2,3,\dots$,

(3) $v_1 = x_1 \in E$

converges to a fixed point of T on E.

In this chapter, we wish to consider the case where E is the closed unit N-cell or the closed unit N-disk of $\mathbb{R}^N$. It will be shown that, with some restrictions on T, the iterative scheme (1)-(3) converges to a fixed point of T.

The statement that the iterative scheme (1)-(3) converges means that each of $\{x_n\}$ and $\{v_n\}$ converges and they converge to the same point. It has been shown, Theorem 2.4, that their common limit is a fixed point of T.

3.2 Remark. In $\mathbb{R}^2$, only continuity of T is not an adequate restriction to guarantee convergence of the iterative scheme (1)-(3) to a fixed point of T on the closed unit disk or the closed square. In fact, we have the following theorems.

3.3 Theorem. Let E be the closed unit disk of $\mathbb{R}^2$. Then there exists a continuous mapping T of E into itself such that the iterative scheme

(1) $$x_{n+1} = Tv_n,$$

(2) $$v_n = \frac{1}{n}(x_1 + \dots + x_n),\ n = 1,2,3,\dots,$$

(3) $$v_1 = x_1 \in E\ ,\ x_1 \neq 0,$$

does not converge.

An analogous result about the closed unit square of $\mathbb{R}^2$, we have

3.4 Theorem. Let E be the closed unit square of $\mathbb{R}^2$. Then there exists a continuous mapping T of E into itself such that the iterative scheme

(1) $$x_{n+1} = Tv_n,$$

(2) $$v_n = \frac{1}{n}(x_1 + \dots + x_n),\ n = 1,2,3,\dots,$$

(3) $$v_1 = x_1 \in E\ ,\ x_1 \neq 0,$$

does not converge.

3.5 Notation. For any $(x,y) \in \mathbb{R}^2$, by the norm $|\ \ |$, we shall mean the usual or Euclidean norm, i.e.,

$$|(x,y)| = (x^2 + y^2)^{\frac{1}{2}}.$$

Proof. (of Theorem 3.3) Let E be the closed unit disk centered at 0 in the complex plane, suppose that $0 < \phi < \pi/4$, and let T be the mapping defined on E by

$$T(re^{i\theta}) = (2r - r^2)e^{i(\theta+\phi)} .$$

Then T is continuous, since

$$\begin{aligned} T(v) = T(re^{i\theta}) &= (2r - r^2)e^{i(\theta+\phi)} \\ &= (re^{i\theta})(2-r)e^{i\phi} \\ &= v(2 - |v|)e^{i\phi} \\ &= f(v)g(v) \end{aligned}$$

where $f(v) = v$ and $g(v) = (2 - |v|)e^{i\phi}$ are continuous, hence T is continuous. In fact, T maps E into itself since $0 \leq 2r - r^2 \leq 1$ for all $r \in [0,1]$. Then

$$0 \leq |T(re^{i\theta})| = |(2r - r^2)e^{i(\theta+\phi)}| \leq 1.$$

Hence T continuously maps E into itself.

The only fixed point of T is 0, since if $v \neq 0$ and $Tv = v$, then

$$v(2 - |v|)e^{i\phi} = v,$$

or

$$(2 - |v|) = e^{-i\phi} = \cos\phi - i\sin\phi.$$

Hence $\sin\phi = 0$, which is impossible since $0<\phi<\pi/4$ and $0 < \sin\phi < \frac{1}{\sqrt{2}}$.

If $v_1 = 0$, then clearly $\{v_n\}$ converges to 0. We will show that, for any $v_1 \neq 0$, the iterative scheme (1)-(3) does not converge.

Suppose, on the contrary, that the iterative scheme (1)-(3) converges. Since 0 is the only fixed point of T, we have by virtue of Theorem 2.4 that $\{v_n\}$ must converge to 0.

Now, for each n, we have

$$v_{n+1} = \frac{1}{n+1}(x_1+\dots+x_n+x_{n+1})$$

$$= \frac{1}{n+1}(n\cdot v_n + Tv_n)$$

$$= \frac{n}{n+1}v_n + \frac{1}{n+1}Tv_n$$

$$= (1-\frac{1}{n+1})v_n + \frac{1}{n+1}Tv_n$$

$$= (1-c_n)v_n + c_nTv_n, \quad c_n = \frac{1}{n+1}, \; n = 1,2,3,\dots,$$

and

$$Tv_n = v_n(2-|v_n|)e^{i\phi}.$$

Therefore, for each n,

$$(4) \qquad |v_{n+1}| = |v_n| \, |(1-c_n) + c_n(2-|v_n|)e^{i\phi}| .$$

Since $1 > \cos\phi > \cos^2\phi$, so that

$$|(1-c_n)+c_n(2-|v_n|)e^{i\phi}|$$

$$= \left[(1-c_n)^2 + 2c_n(1-c_n)(2-|v_n|)\cos\phi + c_n^2(2-|v_n|)^2\right]^{\frac{1}{2}}$$

$$> \left[(1-c_n)^2+2c_n(1-c_n)(2-|v_n|)\cos\phi+c_n^2(2-|v_n|)^2\cos^2\phi\right]^{\frac{1}{2}}$$

$$= \left[((1-c_n) + c_n(2-|v_n|)\cos\phi)^2\right]^{\frac{1}{2}}$$

$$= (1-c_n) + c_n(2-|v_n|)\cos\phi$$

$$= 1 + c_n((2-|v_n|)\cos\phi - 1).$$

Substituting this last expression into (4), we obtain

(5) $$|v_{n+1}| > |v_n| \,(1 + c_n((2-|v_n|)\cos\phi - 1)).$$

Hence by induction

(6) $$|v_{n+1}| > \prod_{k=1}^{n}(1 + c_k((2-|v_k|)\cos\phi - 1))\,|v_1|.$$

Let $\varepsilon > 0$ be such that $(2-\varepsilon)\cdot\frac{1}{\sqrt{2}} > 1$. Then there exists an N such that $|v_n| < \varepsilon$ for all $n \geq N$. Since $\cos\phi > \frac{1}{\sqrt{2}}$, then

$$(2-|v_n|)\cos\phi > (2-\varepsilon)\frac{1}{\sqrt{2}} > 1$$

for all $n \geq N$.

Now from (6), we have for $n > N$

$$|v_{n+1}| > |v_1| \prod_{k=1}^{N}(1-c_k+c_k(2-|v_k|)\cos\phi) \prod_{k=N+1}^{n}(1+c_k((2-|v_k|)\cos\phi-1))$$

$$> |v_1| \prod_{k=1}^{N}\left(\frac{k}{k+1}+\frac{1}{k+1}(2-|v_k|)\cos\phi\right) \prod_{k=N+1}^{n}\left(1+\frac{1}{k+1}\left((2-\varepsilon)\frac{1}{\sqrt{2}}-1\right)\right)$$

Since each factor in the second product is greater than unity, therefore, for all $n > N$,

$$|v_{n+1}| > |v_1| \prod_{k=1}^{N}\left(\frac{k}{k+1} + \frac{1}{k+1}(2-|v_k|)\cos\phi\right) = m,$$

where m is a positive real number independent of n.

This is contradictory to the supposition that $\{v_n\}$ converges to 0. Consequently, the iterative scheme (1)-(3) does not converge.

Q.E.D.

Proof. (of Theorem 3.4) Let E denote the square $[-1,1] \times [-1,1]$ of $\mathbb{R}^2$. For any two points (x,y), $(x',y') \in \mathbb{R}^2$, define

$$(x,y) + (x',y') = (x+x',y+y')$$

and

$$(x,y)(x',y') = (xx'-yy',x'y+xy').$$

Suppose that $0 < \emptyset < \pi/4$, and let T be the mapping defined on E by

$$T(x,y) = (x,y)(2 - \|(x,y)\|)(\cos\emptyset,\sin\emptyset).$$

Here, by the norm $\|(x,y)\|$, $(x,y) \in E$ we mean

$$\|(x,y)\| = \max\{|x|,|y|\}.$$

The mapping T is continuous since

$$T(x,y) = (x,y)(2 - \|(x,y)\|)(\cos\emptyset,\sin\emptyset)$$

$$= f(x,y)g(x,y)$$

where $f(x,y) = (2 - \|(x,y)\|)(x,y)$ and $g(x,y) = (\cos\emptyset,\sin\emptyset)$ are continuous, hence T is continuous.

We now show that T carries E into itself. Since

$$0 \leq \|(x,y)\| \, \|(2- \|(x,y)\|)\| \leq 1,$$

therefore

$$0 \leq \|T(x,y)\| = \|(x,y)\| \, \|(2- \|(x,y)\|)\| \, \|(\cos\emptyset,\sin\emptyset)\| \leq 1.$$

Hence T continuously maps E into itself.

We observe that $0 = (0,0)$ is a fixed point of T. We claim that 0 is a unique fixed point of T. Suppose there exists $(x,y) \neq 0$ such that $T(x,y) = (x,y)$. Then, by definition of T,

$$(x,y)(2-\|(x,y)\|)(\cos\phi,\sin\phi) = (x,y)$$

$$(2-\|(x,y)\|)(\cos\phi,\sin\phi) = (1,0).$$

Hence

$$(2-\|(x,y)\|)\sin\phi = 0,$$

or

$$\sin\phi = 0.$$

But this is absurd, since $0<\phi<\pi/4$ and $0<\sin\phi<\frac{1}{\sqrt{2}}$. .
Therefore $0 = (0,0)$ is the only fixed point of T.

We prove that $\{v_n\}$ does not converge to 0 by contradiction. Suppose that $\{v_n\}$ converges to 0.

For each n, we have

$$\begin{aligned} v_{n+1} &= \frac{1}{n+1}(x_1+\ldots+x_n+x_{n+1}) \\ &= \frac{1}{n+1}(n\cdot v_n + Tv_n) \\ &= \frac{n}{n+1}v_n + \frac{1}{n+1}Tv_n \end{aligned}$$

and

$$Tv_n = v_n(2-\|v_n\|)(\cos\phi,\sin\phi).$$

Therefore, for each n

$$(4)\qquad \begin{aligned} v_{n+1} &= v_n\left(\frac{n}{n+1}(1,0) + \frac{1}{n+1}(2-\|v_n\|)(\cos\phi,\sin\phi)\right) \\ &= v_n\left(\frac{n}{n+1} + \frac{1}{n+1}(2-\|v_n\|)\cos\phi,\ \frac{1}{n+1}(2-\|v_n\|)\sin\phi\right). \end{aligned}$$

Therefore

$$(5)\qquad \|v_{n+1}\| = \|v_n\| \, \Big\| \Big(\frac{n}{n+1} + \frac{1}{n+1}(2-\|v_n\|)\cos\phi, \ \frac{1}{n+1}(2-\|v_n\|)\sin\phi \Big) \Big\| .$$

$$= \|v_n\| \Big(\frac{n}{n+1} + \frac{1}{n+1}(2-\|v_n\|)\cos\phi \Big).$$

Hence by induction

$$(6)\qquad \|v_{n+1}\| = \prod_{k=1}^{n} \Big(\frac{k}{k+1} + \frac{1}{k+1}(2-\|v_k\|)\cos\phi \Big) \|v_1\| .$$

Let $\varepsilon > 0$ be such that $(2-\varepsilon)\frac{1}{\sqrt{2}} > 1$. Then there exists an N such that $\|v_n\| < \varepsilon$ for all $n \geq N$. Since $\cos\phi > \frac{1}{\sqrt{2}}$, therefore

$$(2-\|v_n\|)\cos\phi > (2-\varepsilon)\frac{1}{\sqrt{2}} > 1$$

for all $n \geq N$.

Now, from (6), we have for $n > N$,

$$\|v_{n+1}\| = \|v_1\| \prod_{k=1}^{N} \Big(\frac{k}{k+1} + \frac{1}{k+1}(2-\|v_k\|)\cos\phi \Big) \prod_{k=N+1}^{n} \Big(1 + \frac{1}{k+1}((2-\|v_k\|)\cos\phi - 1)\Big)$$

$$> \|v_1\| \prod_{k=1}^{N} \Big(\frac{k}{k+1} + \frac{1}{k+1}(2-\|v_k\|)\cos\phi \Big) \prod_{k=N+1}^{n} \Big(1 + \frac{1}{k+1}\Big((2-\varepsilon)\frac{1}{\sqrt{2}} - 1\Big)\Big).$$

Since each factor in the second product of the last expression is greater than unity, therefore, for $n > N$,

$$\|v_{n+1}\| > \|v_1\| \prod_{k=1}^{N} \Big(\frac{k}{k+1} + \frac{1}{k+1}(2-\|v_k\|)\cos\phi \Big) = m$$

where m is a positive real number independent of n.

This is contradictory to the supposition that $\{v_n\}$ converges to 0.

Therefore $\{v_n\}$ does not converge to 0. Consequently, the iterative scheme (1)-(3) does not converge by virtue of Theorem 2.4 .

Q.E.D.

To give a partial generalization of Theorem 2.3 , we first define some classes of mappings.

3.6 Definition. Let E be a subset of $\mathbb{R}^N$. A mapping $T:E \longrightarrow E$ is said to be nonexpansive provided

$$|Tx - Ty| \leq |x - y| \qquad \ldots.(*)$$

for all $x,y \in E$, and is said to be quasi-nonexpansive provided T has at least one fixed point in E, and if $p \in E$ is any fixed point of T, then

$$|Tx - p| \leq |x - p|$$

for all $x \in E$ (i.e., T is nonexpansive about each of its fixed points).

We note that every nonexpansive mapping is continuous, since it follows from condition (*) that $T(x_n) \longrightarrow T(x)$ whenever $x_n \longrightarrow x$.

It is clear that a nonexpansive mapping which has at least one fixed point in E is quasi-nonexpansive, since if p is a fixed point of T, then

$$|Tx - p| = |Tx - Tp| \leq |x - p|$$

for all $x \in E$.

Furthermore, a linear quasi-nonexpansive mapping is nonexpansive, since if p is a fixed point of T, then

$$\begin{aligned} |Tx - Ty| &= |Tx - Ty + Tp - p| \\ &= |T(x - y + p) - p| \\ &\leq |x - y + p - p| \\ &= |x - y| \end{aligned}$$

for all $x, y \in E$.

However, there exist continuous quasi-nonexpansive mappings which are not nonexpansive.

3.7 Example. Let $\mathbb{R}^1$ be the real line and let T be defined as follows

$$T(0) = 0$$

$$T(x) = (x/2)\sin(1/x),\ x \neq 0.$$

The only fixed point of T is 0, since if $x \neq 0$ and $x = Tx$, then

$$x = (x/2)\sin(1/x),$$

or

$$2 = \sin(1/x)$$

which is impossible.

T is quasi-nonexpansive since if $y \in \mathbb{R}^1$, $p = 0$, then

$$|Ty-p| = |Ty-0| = |(y/2)|\,|\sin(1/y)| \leq |(y/2)| < |y| = |y-p|.$$

However, T is not a nonexpansive mapping. This is seen by choosing $x = 2/5\pi$, $y = 2/7\pi$, for then

$$|Tx - Ty| = |(1/5\pi)\sin(5\pi/2) - (1/7\pi)\sin(7\pi/2)| = 12/35\pi,$$

whereas $|x - y| = 4/35\pi$.

We shall consider a more general class of contractive-type mappings.

3.8 <u>Definition</u>. Let E be a subset of $\mathbb{R}^N$. A mapping $T:E \longrightarrow E$ is said to be <u>strictly pseudo-contractive mapping</u> if there exists a number k satisfying $0 \leq k < 1$ such that

$$|Tx - Ty|^2 \leq |x-y|^2 + k\,|(I-T)x - (I-T)y|^2 \quad \ldots.(*)$$

for all $x, y \in E$. A <u>pseudo-contractive mapping</u> is one satisfying $(*)$ with $k = 1$.

The class of nonexpansive mappings is a proper subclass of strictly pseudo-contractive mappings. This is seen by taking $k = 0$ in $(*)$.

However, the class of quasi-nonexpansive mappings and the class of strictly pseudo-contractive mappings are independent.

The T in Example 3.7 is quasi-nonexpansive but not strictly pseudo-contractive. To show that T does not belong to the class of strictly pseudo-contractive mappings, we pick $x = 2/(4n+1)\pi$ and $y = 2/(4n+3)\pi$, $n \geq 1$. Then

$$\begin{aligned} |Tx-Ty|^2 &= |(1/(4n+1)\pi)\sin(4n+1)\pi/2 - (1/(4n+3)\pi)\sin(4n+3)\pi/2|^2 \\ &= |1/(4n+1)\pi + 1/(4n+3)\pi|^2 \\ &= (8n+4)^2/[(4n+1)(4n+3)\pi]^2 , \end{aligned}$$

whereas

$$|x-y|^2 + |(I-T)x - (I-T)y|^2$$

$$= |x-y|^2 + |(x-y) - (Tx-Ty)|^2$$

$$= \left| \frac{2}{(4n+1)\pi} - \frac{2}{(4n+3)\pi} \right|^2 + \left| \frac{2}{(4n+1)\pi} - \frac{2}{(4n+3)\pi} - \frac{8n+4}{(4n+1)(4n+3)\pi} \right|^2$$

$$= \left| \frac{4}{(4n+1)(4n+3)\pi} \right|^2 + \left| \frac{4}{(4n+1)(4n+3)\pi} - \frac{8n+4}{(4n+1)(4n+3)\pi} \right|^2$$

$$= \frac{16 + 64n^2}{\left[(4n+1)(4n+3)\pi\right]^2} .$$

Hence

$$|Tx-Ty|^2 > |x-y|^2 + |(I-T)x - (I-T)y|^2$$

$$> |x-y|^2 + k|(I-T)x - (I-T)y|^2$$

for all $k \in [0,1)$.

Now consider the following example.

3.9 Example. Let $\mathbb{R}^1$ be the real line and let T be defined as follows:

$$T(x) = -x/a + 1, \quad 0 \leq x \leq a, \quad \frac{1}{3} < a < \frac{1}{2},$$

$$T(x) = 0, \quad a < x \leq 1.$$

The mapping T is strictly pseudo-contractive but not quasi-nonexpansive. We first show that T is not quasi-nonexpansive. Note that $\frac{a}{1+a}$ is a fixed point of T, since

$$T\left(\frac{a}{1+a} \right) = -\frac{1}{a} \cdot \frac{a}{1+a} + 1 = \frac{a}{1+a} .$$

At the point $p = \frac{a}{1+a}$, we have, for $x < a$,

$$|Tx-p| = \left| -\frac{x}{a} + 1 - \frac{a}{1+a} \right| = \left| \frac{1}{1+a} - \frac{x}{a} \right|$$

$$= \frac{1}{a} \left| \frac{a}{1+a} - x \right| = \frac{1}{a} \left| x - \frac{a}{1+a} \right|$$

$$> \left| x - \frac{a}{1+a} \right| = |x - p|$$

since $\frac{1}{a} > 2$.

Next, we show that T is a strictly pseudo-contractive mapping.

For $0 \leq x, y \leq a$, we have

$$|Tx-Ty|^2 = \left| -\frac{x}{a} + 1 + \frac{y}{a} - 1 \right|^2 = \left| \frac{x-y}{a} \right|^2 ,$$

and

$$|x-y|^2 + k|(I-T)x - (I-T)y|^2$$

$$= |x-y|^2 + k\left|x-y+\frac{x}{a} - 1 - \frac{y}{a} + 1\right|^2$$

$$= |x-y|^2 + k\left|(x-y)\left(\frac{1+a}{a}\right)\right|^2$$

$$= |x-y|^2\left(1 + k\left(\frac{1+a}{a}\right)^2\right) .$$

Let $k = \frac{1-a}{1+a}$. Then

$$|Tx-Ty|^2 = \left|\frac{x-y}{a}\right|^2 = |x-y|^2 + k|(I-T)x - (I-T)y|^2 .$$

For $0 \leq x \leq a$, $a < y \leq 1$, we have

$$0 \leq a - x < y - x$$

and

$$|a - x|^2 < |y - x|^2 .$$

Therefore

$$k|(I-T)x - (I-T)y|^2 + |x-y|^2$$

$$= k\left| x - y + \frac{x}{a} - 1\right|^2 + |x - y|^2$$

$$= k\left|(y-x) + \frac{a-x}{a}\right|^2 + |y - x|^2$$

$$> k\left|(a-x) + \frac{a-x}{a}\right|^2 + |a - x|^2$$

$$= \frac{1-a}{1+a}\,|a-x|^2 \left(\frac{1+a}{a} \right)^2 + |a - x|^2$$

$$= \left|\frac{a-x}{a}\right|^2 = \left|- \frac{x}{a} + 1\right|^2$$

$$= |Tx-Ty|^2.$$

Finally, for $a < x \leq 1$, $a < y \leq 1$, we have

$$0 = |Tx-Ty|^2 \leq |x-y|^2 + \frac{1-a}{1+a}|x-y|^2 .$$

Hence T is a strictly pseudo-contractive mapping.

Our main theorems are the following.

3.10 Theorem. Let E be a closed unit N-disk or a closed unit N-cell of $\mathbb{R}^N$ and let T be a continuous quasi-nonexpansive mapping of E into itself. Then the iterative scheme

(1) $\quad x_{n+1} = Tv_n$;

(2) $\quad v_n = \frac{1}{n}(x_1+\ldots+x_n)$, $n = 1,2,3,\ldots$,

(3) $\quad v_1 = x_1 \in E$

converges to a fixed point of T.

3.11 Corollary. Let E be a closed unit N-disk or a closed unit N-cell of $\mathbb{R}^N$, and let $T:E \longrightarrow E$ be continuous. If T satisfies either

$$\text{(A)} \quad |Tx - Ty| \leq \alpha\big[|x - Tx| + |y - Ty|\big] \;, \quad 0 \leq \alpha \leq \tfrac{1}{2} \,,$$

or

$$\text{(B)} \quad |Tx - Ty| \leq \beta\big[|x - Ty| + |y - Tx|\big] \;, \quad 0 \leq \beta \leq \tfrac{1}{2},$$

for all $x, y \in E$, then the iterative scheme (1)-(3) of Theorem 3.10 converges to a fixed point of T.

For a strictly pseudo-contractive mapping we have the following.

3.12 Theorem. Let E be a closed unit N-disk or a closed unit N-cell of $\mathbb{R}^N$ and let T be a continuous strictly pseudo-contractive mapping of E into itself. Then the iterative scheme

(1) $\quad x_{n+1} = Tv_n \,,$

(2) $\quad v_n = \frac{1}{n}(x_1 + \ldots + x_n), \; n = 1,2,3,\ldots,$

(3) $\quad v_1 = x_1 \in E$

converges to a fixed point of T.

To prove the above two theorems we need the following lemmas.

3.13 Lemma. Let x, y and z be any three points in $\mathbb{R}^N$ and let t be a real number. Then

$$|tx+(1-t)y-z|^2 = t|x-z|^2 + (1-t)|y-z|^2 - t(1-t)|x-y|^2.$$

roof. For any $x = (x_1,\ldots,x_N)$, $y = (y_1,\ldots,y_N)$ in $\mathbb{R}^N$, we set

$$|x| = \sqrt{\langle x,x\rangle} = \sqrt{x_1^2 + \ldots + x_N^2} .$$

Then

$$\begin{aligned} |x-y|^2 &= \langle x-y,x-y\rangle \\ &= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= |x|^2 - 2\langle x,y\rangle + |y|^2 . \end{aligned}$$

Therefore

$$\begin{aligned} |tx+(1-t)y-z|^2 &= |t(x-y) + (y-z)|^2 \\ &= t^2|x-y|^2 + |y-z|^2 + 2t\langle x-y,y-z\rangle \\ &= t^2|x-y|^2 + |y-z|^2 + t\left[(|x|^2 - 2\langle x,z\rangle + |z|^2)\right. \\ &\qquad \left.-(|x|^2 - 2\langle x,y\rangle + |y|^2) - (|z|^2 - 2\langle z,y\rangle + |y|^2)\right] \\ &= t^2|x-y|^2 + |y-z|^2 \\ &\qquad + t(|x-z|^2 - |x-y|^2 - |y-z|^2) \\ &= t|x-z|^2 + (1-t)|y-z|^2 - t(1-t)|x-y|^2. \end{aligned}$$

Q.E.D.

3.14 Lemma. Let $\{|x_n-p|\}$ be a nonincreasing sequence of real numbers. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to p, then the sequence $\{x_n\}$ also converges to the point p.

Proof. Since $\{|x_n - p|\}$ is nonincreasing, we see that if $n \geq N$, then

$$|x_{n+1} - p| \leq |x_n - p| \qquad \ldots. (*)$$

Let $\varepsilon > 0$ be given. Then there is an n_{k_0} such that

$$|x_{n_{k_0}} - p| < \varepsilon$$

and $n_{k_0} \geq N$. Hence from (*)

$$|x_n - p| < \varepsilon$$

for $n \geq n_{k_0}$.

Q.E.D.

Proof. (of Theorem 3.10) We first show that $v_n \in E$ by induction on n.

Since $v_1 = x_1 \in E$, we have finished the first step.

Assume $v_n \in E$. Then

$$\begin{aligned} v_{n+1} &= \frac{1}{n+1}(x_1 + \ldots + x_n + x_{n+1}) \\ &= \frac{1}{n+1}(n \cdot v_n + Tv_n) \\ &= \frac{n}{n+1} v_n + \frac{1}{n+1} Tv_n \in E \end{aligned}$$

by induction hypothesis. (Recall that if x and y are points in E, a convex set, then the point

$$z(t) = tx + (1-t)y, \quad 0 \leq t \leq 1,$$

lies on the straight line segment joining x and y with $z(0) = y$, $z(1) = x$.)

Hence by induction, $v_n \in E$ for all positive integral values of n.

If we set $c_n = \frac{1}{n+1}$, then

$$(4) \qquad v_{n+1} = (1-c_n)v_n + c_nTv_n \ , \ n = 1,2,3,\ldots .$$

If for some n, $v_n = Tv_n$, then clearly $\{v_n\}$ converges to v_n. Hence we may assume that $Tv_n \neq v_n$ for each n.

From Brouwer's fixed point theorem, F(T), the set of fixed point of T, is not empty.

Let q denote any point of F(T).

For any $v_n \in E$, we have

$$(5) \qquad |v_{n+1} - q| = |(1-c_n)v_n + c_nTv_n - q| .$$

The Lemma 3.18 and (5), then gives

$$(6) \qquad |v_{n+1} - q|^2 = |(1-c_n)v_n + c_nTv_n - q|^2$$

$$= (1-c_n)|v_n-q|^2 + c_n|Tv_n-q|^2 - c_n(1-c_n)|Tv_n-v_n|^2$$

Since T is quasi-nonexpansive, so that

$$|Tv_n - q|^2 \leq |v_n - q|^2 .$$

This inequality, along with (6), gives

$$(7) \qquad |v_{n+1} - q|^2 \leq |v_n-q|^2 - c_n(1-c_n)|Tv_n-v_n|^2 .$$

Therefore adding these inequalities with m,m+1,...,n for n, we derive the following inequality

$$|v_{n+1}-q|^2 \leq |v_m-q|^2 - \sum_{k=m}^{n} c_k(1-c_k)|Tv_k-v_k|^2 ,$$

from which we have

$$(8) \qquad \sum_{k=m}^{n} c_k(1-c_k)|Tv_k-v_k|^2 \leq |v_m-q|^2 - |v_{n+1}-q|^2 .$$

We claim that $\liminf_{n\to\infty}|Tv_n-v_n| = 0$. We prove by contradiction. Suppose that $\liminf_{n\to\infty}|Tv_n-v_n| = b \neq 0$. Then, by Remark 1.18, for any given $\varepsilon > 0$, there exists N such that

$$|Tv_n-v_n| > b-\varepsilon \quad \text{for all } n > N.$$

Since (8) is true for all n, then if $n > m > N$, we get

$$\sum_{k=m}^{n} c_k(1-c_k)(b-\varepsilon)^2 \leq |v_m-q|^2 - |v_{n+1}-q|^2 .$$

or

$$(b-\varepsilon)^2 \sum_{k=m}^{n} c_k(1-c_k) \leq |v_m-q|^2 - |v_{n+1}-q|^2 .$$

This gives a contradiction since the series on the left hand side is unbounded whereas the last member is bounded (in fact, it is bounded by 4). Therefore

$$\liminf_{n\to\infty}|Tv_n-v_n| = 0,$$

i.e., $\lim_{k\to\infty}|Tv_{n_k}-v_{n_k}| = 0$ for some subsequence $\{v_{n_k}\}$. Since E is compact, there exists a subsequence of $\{v_{n_k}\}$, $\{v_m\}$ say, such that $\lim_{m\to\infty} v_m = p$.

Since T is continuous on E, (I-T) is also continuous on E. Therefore

$$\lim_{m \to \infty} (I-T)v_m = (I-T)p,$$

but $\lim_{k \to \infty} (I-T)v_{n_k} = 0$, so that

$$(I-T)p = 0, \text{ or } Tp = p$$

i.e., p is a fixed point of T.

Now from (4), we have

$$\begin{aligned} |v_{n+1} - p| &= |(1-c_n)v_n + c_n Tv_n - p| \\ &= |(1-c_n)(v_n-p) + c_n(Tv_n-p)| \\ &\leq (1-c_n)|v_n-p| + c_n|Tv_n-p| \\ &\leq (1-c_n)|v_n-p| + c_n|v_n-p| \end{aligned}$$

since T is quasi-nonexpansive. Therefore

$$|v_{n+1}-p| \leq |v_n-p|.$$

Hence $\{|v_n-p|\}$ is nonincreasing.

The two conditions $\lim_{m \to \infty} v_m = p$ and $\{|v_n-p|\} \downarrow$ in n yield $\lim_{n \to \infty} v_n = p$, by Lemma 3.14.

Q.E.D.

<u>Proof</u>. (of Corollary 3.11) We need only to prove that if T satisfies either (A) or (B), then T is quasi-nonexpansive.

Since T is continuous, F(T), the set of fixed points of T, is nonempty by the Brouwer's fixed point theorm. Let $p \in F(T)$.

If T satisfies (A) then, with $y = p$,

$$|Tx-p| = |Tx-Tp| \leq \alpha|x-Tx| \leq \alpha\left[|x-p| + |p-Tx|\right].$$

Therefore

$$|Tx-p| \leq \frac{\alpha}{1-\alpha}|x-p| \leq |x-p| ,$$

since $\frac{\alpha}{1-\alpha} \leq 1$.

In view of (B), we have, for $y = p$,

$$|Tx-p| = |Tx-Tp| \leq \beta\left[|x-Tp| + |p-Tx|\right] = \beta\left[|x-p| + |Tx-p|\right].$$

Therefore

$$|Tx-p| \leq \frac{\beta}{1-\beta}|x-p| \leq |x-p|$$

since $\frac{\beta}{1-\beta} \leq 1$.

Q.E.D.

Proof. (of Theorem 3.12) For each n, we have

(4) $$v_{n+1} = (1-c_n)v_n + c_m Tv_n$$

where $c_n = \frac{1}{n+1}$, $n = 1,2,3,\ldots$.

From Brouwer's fixed point theorem, F(T), the set of fixed point of T, is nonempty. Let q denote any point of F(T).

The Lemma 3.13 and (4) then gives

(5) $$|v_{n+1}-q|^2 = (1-c_n)|v_n-q|^2 + c_n|Tv_n-q|^2 - c_n(1-c_n)|Tv_n-v_n|^2.$$

Since T is a strictly pseudo-contractive mapping, there exists a number k, $0 \leq k < 1$ such that if $x, y \in E$, then

$$|Tx-Ty|^2 \leq |x-y|^2 + k|(I-T)x - (I-T)y|^2 .$$

Thus, if $y = q$ and $x = v_n$, then

$$(6) \qquad |Tv_n-q|^2 \leq |v_n-q|^2 + k|v_n-Tv_n|^2 .$$

Substituting (6) into (5), we obtain

$$(7) \qquad |v_{n+1}-q|^2 \leq |v_n-q|^2 - c_n(1-c_n-k)|Tv_n-v_n|^2.$$

Therefore adding these inequalities with m,m+1,...,n for n, we derive the following inequality

$$(8) \qquad \sum_{i=m}^{n} c_i(1-c_i-k)|Tv_i-v_i|^2 \leq |v_m-q|^2 - |v_{n+1}-q|^2 .$$

Since $k < 1$, there exists $\varepsilon > 0$ such that $0 < \varepsilon < 1-k$. For sufficiently large n, n = N say, we have $c_N = \frac{1}{N+1} < \varepsilon < 1-k$.

Since $\{c_n\}$ is decreasing, therefore $c_n < \varepsilon < 1-k$ for all $n \geq N$.

Then if $n > m \geq N$, (8) becomes

$$(9) \qquad \sum_{i=m}^{n} c_i(1-\varepsilon-k)|Tv_i-v_i|^2 \leq |v_m-q|^2 - |v_{n+1}-q|^2$$

or

$$(1-\varepsilon-k)\sum_{i=m}^{n} c_i|Tv_i-v_i|^2 \leq |v_m-q|^2 - |v_{n+1}-q|^2 .$$

The last member is bounded, therefore the series on the left hand side is bounded. But since $\sum_{i=1}^{\infty} c_i$ diverges, this should implies that $\liminf_{n\to\infty} |Tv_n - v_n| = 0$, which in turn implies from the compactness of E that there is a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that converges to a certain point p of F(T).

Since p is a fixed point of T, from (7), we see that if $n \geq N$, then

$$|v_{n+1} - p| \leq |v_n - p| .$$

The conditions $\lim_{k\to\infty} v_{n_k} = p$ and $\{|v_n - p|\}\downarrow$ in n for all n sufficiently large yield $\lim_{n\to\infty} v_n = p$, by Lemma 3.14 .

Q.E.D.