CHAPTER 5

THE DEVIATIONS OF THE IMPULSE RESPONSE

5.1 Introduction

For a perfect binary m-sequence signal, its autocorrelation function is of triangular shape (see figure 2.1) with the base width $2\Delta t$ and the bandwidth is of the order of $\frac{1}{\Delta t}$. This autocorrelation function may be approximated to those of white noise depending on the value of Δt. If the value of Δt is very small, then the autocorrelation function of a perfect binary m-sequence signal is closely to a delta function. When the input signal is an imperfect m-sequence, the autocorrelation function can not be approximated to those of white noise. The effects of these input signals are derived in the following sections.

5.2 The Effect of Perfect M-sequence Input Signal

When the perfect binary m-sequence is used as an the input signal of the linear system shown in the figure 5.1. The crosscorrelation between the input signal x(t) and the output signal y(t) in eqn. (1.5) can be rewritten as

$$\phi_{xy}(\tau) = \int_0^T \left\{ h(\tau)+(u-\tau)h'(\tau)+\frac{(u-\tau)^2}{2!}h''(\tau)+\dots \right\} \phi_{xx}(\tau-u)du \tag{5.1}$$

Let $V=\tau-u$, eqn. (5.1) becomes

$$\phi_{xy}(\tau) = \int_{\tau-T}^{\tau} \left\{ h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots \right\} \phi_{xx}(V)dV \tag{5.2}$$

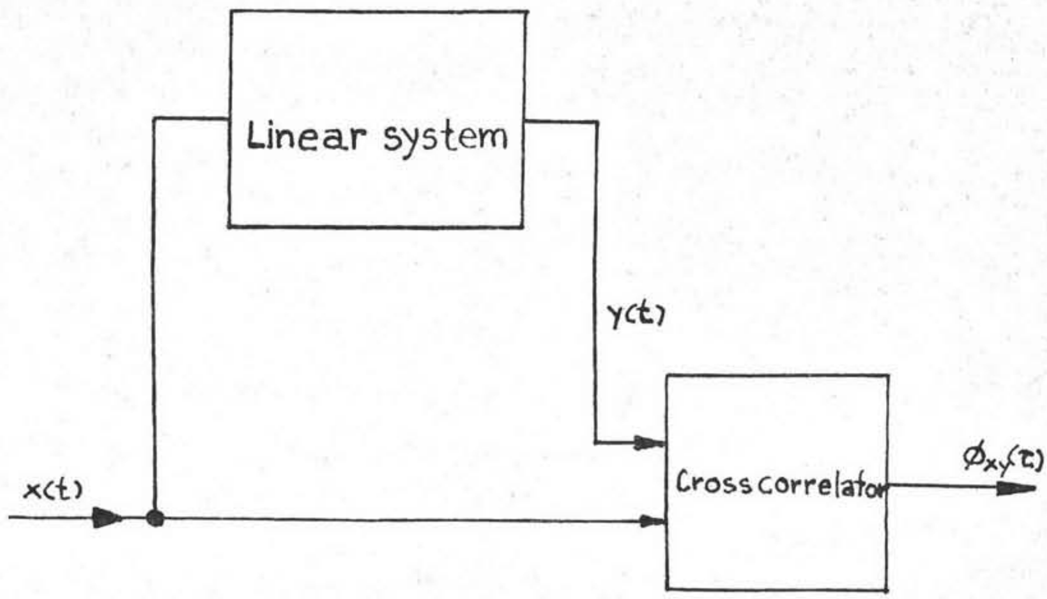


Figure 5.1

The cross correlation between the input binary m-sequence signal, x(t) and the output signal, y(t)

Since the values of the autocorrelation function of the input signal x(t) obtained from the property (h) in chapter 2 are

$$\phi_{xx}(V) = a^2\left(1-\frac{V}{\Delta t}\frac{2^n}{2^n-1}\right) \quad , \text{ for } \quad 0 \leq V \leq \Delta t \qquad (5.3)$$

$$= a^2\left(1+\frac{V}{\Delta t}\frac{2^n}{2^n-1}\right) \quad , \text{ for } \quad 0 \geq V \geq -\Delta t \qquad (5.4)$$

and

$$= \frac{-a^2}{2^n-1} \quad , \text{ for } \Delta t \leq V \leq (2^n-2)\Delta t \quad (5.5)$$

From eqns. (5.2), (5.3), (5.4), and (5.5), we have

$$\begin{aligned}
\phi_{xy}(\tau) &= \int_0^{\Delta t}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}a^2\left(1-\frac{V}{\Delta t}\frac{2^n}{2^n-1}\right)dV \\
&+\int_{-\Delta t}^{0}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}a^2\left(1+\frac{V}{\Delta t}\frac{2^n}{2^n-1}\right)dV \\
&+\int_{\Delta t}^{(2^n-2)\Delta t}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}\left(\frac{-a^2}{2^n-1}\right)dV \\
&= \int_0^{\Delta t}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}\frac{a^2 2^n}{2^n-1}\left(1-\frac{V}{\Delta t}\right)dV \\
&+\int_{-\Delta t}^{0}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}\frac{a^2 2^n}{2^n-1}\left(1+\frac{V}{\Delta t}\right)dV \\
&-\int_0^{T} h(u)\frac{a^2}{2^n-1}\,du
\end{aligned} \qquad (5.6)$$

It is not necessary to expand the impulse response h(u) in the third integral term in the right-hand side of eqn. (5.4), because the value of this third integral term is constant and can be measured easily at the time shift $\tau = T$. It is assumed that the period T of the m-sequence is greater than the settling time of the linear system, and $h(u) = 0$ for $u < 0$. Thus, eqn. (5.6) becomes

$$\phi_{xy}(\tau) = \frac{a^2 2^n}{2^n-1}\int_0^{\Delta t}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1-\frac{V}{\Delta t})dV$$
$$+\frac{a^2 2^n}{2^n-1}\int_{-\Delta t}^{0}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1+\frac{V}{\Delta t})dV$$
$$-A \qquad (5.7)$$

where $A = \frac{a^2}{2^n-1}\int_0^T h(u)du$. It is convenient to considered three separate cases: $\tau=0$, $0 \leqslant \tau \leqslant \Delta t$, $\tau \geqslant \Delta t$.

(a) For the case $\tau = 0$,

The first term in the right-hand side of eqn. (5.7) is zero because $h(u) = 0$ in the range $0 \leqslant v \leqslant \Delta t$, then eqn. (5.7) becomes

$$\phi_{xy}(0) = \frac{a^2 2^n}{2^n-1}\int_{-\Delta t}^{0}\left\{h(0)-Vh(0)+\frac{V^2}{2!}h(0)+\dots\right\}(1+\frac{V}{t})dV - A$$
$$= \frac{a^2 2^n}{2^n-1}\left\{h(0)\frac{\Delta t}{2}+h'(0)\frac{\Delta t^2}{6}+h''(0)\frac{\Delta t^3}{24}+h'''(0)\frac{\Delta t^4}{120}+\dots\right\} -A \qquad (5.8)$$

(b) For the case $0 \leqslant \tau \leqslant \Delta t$,

The value of $h(u) = 0$ in the range $\tau \leqslant V \leqslant \Delta t$. Thus, we have

$$\phi_{xy}(\tau)+A = \frac{a^2 2^n}{2^n-1}\int_0^{\tau}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1-\frac{V}{\Delta t})dV$$
$$+\frac{a^2 2^n}{2^n-1}\int_{-\Delta t}^{0}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1+\frac{V}{\Delta t})dV$$
$$= \frac{a^2 2^n}{2^n-1}\left\{(1+\frac{2\tau}{\Delta t}-\frac{\tau^2}{\Delta t^2})(\frac{\Delta t}{2})h(\tau)+(1-\frac{3\tau^2}{\Delta t^2}+\frac{2\tau^3}{\Delta t^3})(\frac{\Delta t^2}{6})h'(\tau)\right.$$
$$\left.+(1+\frac{4\tau^3}{\Delta t^3}-\frac{3\tau^4}{\Delta t^4})(\frac{\Delta t^3}{24})h''(\tau)+(1-\frac{5\tau^4}{\Delta t^4}+\frac{4\tau^5}{\Delta t^5})(\frac{\Delta t^4}{120})h'''(\tau)+\dots\right\} \qquad (5.9)$$

(c) For the case $\tau \geqslant \Delta t$,

All of the integral terms in eqn. (5.7) is used, we have

$$\phi_{xy}(\tau)+A = \frac{a^2 2^n}{2^n-1}\int_0^{\Delta t}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1-\frac{V}{\Delta t})dV$$
$$+\frac{a^2 2^n}{2^n-1}\int_{-\Delta t}^{0}\left\{h(\tau)-Vh'(\tau)+\frac{V^2}{2!}h''(\tau)+\dots\right\}(1+\frac{V}{\Delta t})dV$$

$$\phi_{xy}(\tau)+A = \frac{a^2 2^n}{2^n-1}\left\{h(\tau)\Delta t+h''(\tau)\frac{\Delta t^3}{12}+h^{iv}(\tau)\frac{\Delta t^5}{360}+\dots\right\} \qquad (5.10)$$

By the first approximation, the eqns. (5.8), (5.9), and (5.10) reduce to

$$\phi_{xy}(0)+A = \frac{a^2 2^n}{2^n-1}h(0)\left(\frac{\Delta t}{2}\right) \quad , \text{ for } \tau = 0 \qquad (5.11)$$

$$\phi_{xy}(\tau)+A = \frac{a^2 2^n}{2^n-1}h(\tau)\left(1+\frac{2\tau}{\Delta t}-\frac{\tau^2}{\Delta t^2}\right)\left(\frac{\Delta t}{2}\right) \quad , \text{ for } 0 \leq \tau \leq \Delta t \qquad (5.12)$$

and $$\phi_{xy}(\tau)+A = \frac{a^2 2^n}{2^n-1}h(\tau)(\Delta t) \quad , \text{ for } \tau \geq \Delta t \qquad (5.13)$$

The error terms which give an indication of the accuracy for the impulse response obtained are those of the derivatives of $h(\tau)$ in eqns. (5.8), (5.9), (5.10). These errors are approximately to

$$h'(0)\frac{\Delta t}{3} \quad , \text{ for } \tau = 0$$

$$\frac{h'(\tau)}{3}\left\{\frac{\Delta t^3-3\tau^2\Delta t+2\tau^3}{\Delta t^2+2\tau\Delta t-\tau^2}\right\}+\frac{h''(\tau)}{12}\left\{\frac{\Delta t^4+4\tau^3\Delta t-3\tau^4}{\Delta t^2+2\tau\Delta t-\tau^2}\right\}, \text{ for } 0 \leq \tau \leq \Delta t$$

and $$h''(\tau)\frac{\Delta t^2}{12} \quad , \text{ for } \tau \geq \Delta t$$

respectively.

For examples of the linear system has its impulse response $h(\tau) = Be^{-\tau/T_1}$ where B and T_1 are constants. It can be shown that the error terms are of magnitude;

$$-\frac{\Delta t}{3T_1}h(0) \quad , \text{ for } \tau = 0$$

$$\left\{\frac{-\Delta t^3+3\tau^2\Delta t-2\tau^3}{3T_1(\Delta t^2+2\tau\Delta t-\tau^2)}+\frac{\Delta t^4+4\tau^3\Delta t-3\tau^4}{12T_1^2(\Delta t^2+2\tau\Delta t-\tau^2)}\right\}h(\tau) \quad , \text{ for } 0 \leq \tau \leq \Delta t$$

$$\frac{\Delta t^2}{12T_1^2}h(\tau) \quad , \text{ for } \tau \geq \Delta t$$

The percentage errors may be expressed as the fractions of the error and the estimate value of $h(\tau)$ obtained from eqns. (5.11), (5.12), and (5.13). They are

(i) For $\Delta t = \frac{T_1}{10}$

$$-3.33\% \qquad , \text{ for } \tau = 0$$

$$-\left\{\frac{39T_1^4 - 1200\tau^2T_1^2 + 76000\tau^3T_1 + 30000\tau^4}{12T_1^4 + 240\tau T_1^3 - 1200\tau^2T_1^2}\right\}\% \qquad , \text{ for } 0 \leq \tau \leq \Delta t$$

and $$0.083\% \qquad , \text{ for } \tau \geq \Delta t$$

(ii) For $\Delta t = T_1$

$$-33.33\% \qquad , \text{ for } \tau = 0$$

$$\frac{100}{12T_1^2}\left\{\frac{-3T_1^4 + 12\tau^2T_1^2 - 4\tau^3T_1 - 3\tau^4}{T_1^4 + 2\tau T_1^3 - \tau^2T_1^2}\right\}\% \qquad , \text{ for } 0 \leq \tau \leq \Delta t$$

$$8.33\% \qquad , \text{ for } \tau \geq \Delta t$$

It can be seen that the error due to the small value of Δt will be less than the error due to the large value of Δt.

5.3 The Effect of Transition Error Signal

The error is increased in evaluation of the impulse response, when the transition error signal is considered. The crosscorrelation function $\phi_{xy_e}(\tau)$ is given by eqn. (3.5), on the right-hand side of which is the second convolution integral term, and let

$$I(\tau) = \int_0^T h(u)\phi_{xe}(\tau - u)du \qquad (5.14)$$

The error terms in eqn. (5.14) are the additional error terms of the errors discussed in the section 5.2. These errors may be determined by the Taylor series expansion of $h(u)$ about $h(\tau)$. Thus, we have

$$I(\tau) = C_0 h(\tau) + C_1 h'(\tau) + C_2 h''(\tau) + \dots \tag{5.15}$$

where
$$C_r = \frac{(-1)^{r+1}}{r!} \int_{\tau}^{\tau - T} \phi_{xe}(V) V^r \, dV \quad , \; r = 0,1,2,\dots \tag{5.16}$$

and
$$V = \tau - u$$

Substituting for $\phi_{xe}(V)$ obtained from the table 3.2, eqn. (5.16) becomes

$$\begin{aligned}
C_r = \frac{(-1)^{r+1}}{r!} \cdot \Bigg\{ & \int_{k_s\Delta t+\theta_+}^{k_s\Delta t} \frac{a 2^{n-2}}{(2^n-1)\Delta t} \left[(\gamma - \epsilon(v)) - (\lambda - \mu(v))\right] v^r dv \\
& + \int_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+} \frac{a 2^{n-2}}{(2^n-1)\Delta t} \left[\gamma - \lambda\right] v^r dv + \int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t} \frac{a 2^{n-2}}{(2^n-1)\Delta t} \left[\epsilon(v) - \mu(v)\right] v^r dv \\
& + \int_{\Delta t+\theta_+}^{\Delta t} \frac{a 2^{n-2}}{(2^n-1)\Delta t} \left[(\gamma - \epsilon(v)) + (\lambda - \mu(v))\right] v^r dv + \int_{\Delta t}^{\theta_+} \frac{a 2^{n-2}}{(2^n-1)\Delta t} \left[\gamma + \lambda\right] v^r dv \\
& + \int_{\theta_+}^{0} \frac{-a 2^{n-2}}{(2^n-1)\Delta t} \left[(\gamma - 2\epsilon(v)) + (\lambda - 2\mu(v))\right] v^r dv + \int_{0}^{\theta_+-\Delta t} \frac{-a 2^{n-2}}{(2^n-1)\Delta t} \left[\gamma + \lambda\right] v^r dv \\
& + \int_{\theta_+-\Delta t}^{-\Delta t} \frac{-a 2^{n-2}}{(2^n-1)\Delta t} \left[\epsilon(v) + \mu(v)\right] v^r dv \Bigg\}
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
= K_r \Bigg\{ & \int_{k_s\Delta t+\theta_+}^{k_s\Delta t} (\bar{D} - \bar{B}) v^r dv + \int_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+} \bar{D} v^r dv + \int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t} \bar{B} v^r dv + \int_{\Delta t+\theta_+}^{\Delta t} (D - B) v^r dv \\
& + \int_{\Delta t}^{\theta_+} D v^r dv - \int_{\theta_+}^{0} D v^r dv + \int_{\theta_+}^{0} 2B v^r dv - \int_{0}^{\theta_+-\Delta t} D v^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} B v^r dv \Bigg\}
\end{aligned} \tag{5.18}$$

where
$$K_r = \frac{(-1)^{r+1}}{r!} \left(\frac{a 2^{n-2}}{(2^n-1)\Delta t} \right)$$
$$D = \gamma + \lambda$$
$$\bar{D} = \gamma - \lambda$$
$$B = \epsilon(v) + \mu(v)$$

and $\quad \overline{B} \quad = \epsilon(\nu) - \mu(\nu)$

It is convenient to evaluate the coefficient C_r in different cases of the time shift τ as follows:

(a) for the case $\tau \geq k_s\Delta t + \theta_+$

There is no value of the time V to give the value of time $u < 0$. Thus, every term in eqn. (5.18) can be determined directly. Since the values of γ and λ are constants with respect to time, and the values $\epsilon(\nu)$ and $\mu(\nu)$ depend on overlapping time ν. Therefore, eqn. (5.18) becomes

$$
\begin{aligned}
C_r \quad = K_r\Big\{ & \overline{D}\int_{k_s\Delta t+\theta_+}^{k_s\Delta t} V^r dV - \int_{k_s\Delta t+\theta_+}^{k_s\Delta t} \overline{B} V^r dV + \overline{D}\int_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+} V^r dV + \int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t} \overline{B} V^r dV \\
& + D\int_{\Delta t+\theta_+}^{\Delta t} V^r dV - \int_{\Delta t+\theta_+}^{\Delta t} B V^r dV + D\int_{\Delta t}^{\theta_+} V^r dV - D\int_{\theta_+}^{0} V^r dV \\
& + 2\int_{\theta_+}^{0} B V^r dV - D\int_{0}^{\theta_+-\Delta t} V^r dV - \int_{\theta_+-\Delta t}^{-\Delta t} B V^r dV \Big\}
\end{aligned}
\qquad (5.19)
$$

Because each value of B and $\overline{B}$ varies only in the range of time interval θ_+. Thus, the values of integral terms which conclude B and $\overline{B}$ are zero for time variable V less than the lower limit of the integral. Hence, let dV = dt where t is a new time variable, $t \propto \nu$ that varies from the lower limit to the upper limit of each integral term, eqn. (5.19) becomes

$$
\begin{aligned}
C_r \quad = K_r\Big\{ & \overline{D}\,\frac{V^{r+1}}{r+1}\Big|_{k_s\Delta t+\theta_+}^{k_s\Delta t} + \overline{D}\,\frac{V^{r+1}}{r+1}\Big|_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+} + D\,\frac{V^{r+1}}{r+1}\Big|_{\Delta t+\theta_+}^{\Delta t} + D\,\frac{V^{r+1}}{r+1}\Big|_{\Delta t}^{\theta_+} - D\,\frac{V^{r+1}}{r+1}\Big|_{\theta_+}^{0} \\
& - D\,\frac{V^{r+1}}{r+1}\Big|_{0}^{\theta_+-\Delta t} + \int_0^{\theta_+} \overline{B}\,(k_s\Delta t+t)^r dt - \int_0^{\theta_+} \overline{B}\,((k_s-1)\Delta t+t)^r dt \\
& + \int_0^{\theta_+} B(\Delta t+t)^r dt - 2\int_0^{\theta_+} Bt^r dt + \int_0^{\theta_+} B(t-\Delta t)^r dt \Big\}
\end{aligned}
\qquad (5.20)
$$

$$C_r = K_r\Big\{\frac{\bar{D}}{r+1}\Big[\langle (k_s-1)\Delta t+\theta_+\rangle^{r+1} - \langle k_s\Delta t+\theta_+\rangle^{r+1}\Big] + \frac{D}{r+1}\Big[2\theta_+^{r+1} - \langle \theta_+-\Delta t\rangle^{r+1} - \langle \theta_+ + \Delta t\rangle^{r+1}\Big] + \int_0^{\theta_+}\bar{B}\Big[\langle k_s\Delta t+t\rangle^r - \langle (k_s-1)\Delta t+t\rangle^r\Big]dt + \int_0^{\theta_+} B\Big[\langle \Delta t+t\rangle^r - 2t^r + \langle t-\Delta t\rangle^r\Big]dt\Big\} \qquad (5.21)$$

(b) for the case $k_s\Delta t+\theta_+ > \tau > k_s\Delta t$

It can be seen that $h(u) = 0$ in the range $k_s\Delta t+\theta_+ > V > \tau$, thus the upper limit of the first integral in eqn. (5.17) will be τ. Therefore, we have

$$C_r = K_r\Big\{\int_\tau^{k_s\Delta t}(\bar{D}-\bar{B})v^r dv + \int_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+}\bar{D}v^r dv + \int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t}\bar{B}v^r dv + \int_{\Delta t+\theta_+}^{\Delta t}(D-B)v^r dv - \int_{\Delta t}^{\theta_+}Dv^r dv - \int_{\theta_+}^{0}(D-2B)v^r dv - \int_0^{\theta_+-\Delta t}Dv^r dv - \int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv\Big\} \qquad (5.22)$$

$$= K_r\Big\{\frac{\bar{D}}{r+1}\Big[\langle (k_s-1)\Delta t+\theta_+\rangle^{r+1} - \tau^{r+1}\Big] + \frac{D}{r+1}\Big[2\theta_+^{r+1} - \langle \theta_+-\Delta t\rangle^{r+1} - \langle \theta_++\Delta t\rangle^{r+1}\Big] + \int_0^{\theta_+}\bar{B}\langle (k_s-1)\Delta t+t\rangle^r dt + \int_0^{\tau-k_s\Delta t}\bar{B}\langle k_s\Delta t+t\rangle^r dt + \int_0^{\theta_+}B\Big[\langle \Delta t+t\rangle^r - 2t^r + \langle t-\Delta t\rangle^r\Big]dt \qquad (5.23)$$

where $\tau-k_s\Delta t \leq \theta_+$

(c) for the case $\tau = k_s\Delta t$

In the range $k_s\Delta t \leq V \leq k_s\Delta t+\theta_+$, the value of u is less than zero. Thus the first integral term in eqn. (5.17) will be omitted and replaced by the same integral term when the limit of the integral will be in the new range of $k_s\Delta t-T \leq V \leq k_s\Delta t+\theta_+-T$. If the settling time of the impulse response is assumed to be greater than $T-k_s\Delta t-\theta_+$, we have

$$C_r = K_r\left\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}\langle\bar{D}-\bar{B}\rangle v^r dv+\int_{k_s\Delta t}^{(k_s-1)\Delta t+\theta_+}\bar{D}v^r dv+\int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t}\bar{B}v^r dv+\int_{\Delta t+\theta_+}^{\Delta t}\langle D-B\rangle v^r dv\right.$$

$$+\int_{\Delta t}^{\theta_+}Dv^r dv-\int_{\theta_+}^{0}\langle D-2B\rangle v^r dv-\int_{0}^{\theta_+-\Delta t}Dv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv \quad (5.24)$$

$$= K_r\left\{\frac{\bar{D}}{r+1}\left[\langle(k_s-1)\Delta t+\theta_+\rangle^{r+1}-\langle k_s\Delta t\rangle^{r+1}+\langle k_s\Delta t-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right]\right.$$

$$+\frac{D}{r+1}\left[2\theta_+^{r+1}-\langle\theta_++\Delta t\rangle^{r+1}-\langle\theta_+-\Delta t\rangle^{r+1}\right]$$

$$+\int_0^{\theta_+}\bar{B}\left[\langle k_s\Delta t+t-T\rangle^r-\langle(k_s-1)\Delta t+t\rangle^r\right]dt$$

$$+\int_0^{\theta_+}B\left[\langle t+\Delta t\rangle^r-2t^r+\langle t-\Delta t\rangle^r\right]dt \quad (5.25)$$

(d) for the case $k_s\Delta t > \tau > (k_s-1)\Delta t+\theta_+$

Applying the similar techniques described in the sectios 5.3(a) and 5.3(b) to simplify the second integral term and the first integral term in eqn. (5.17) respectively. Thus, we have

$$C_r = K_r\left\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}\langle\bar{D}-\bar{B}\rangle v^r dv+\int_{\tau}^{(k_s-1)\Delta t+\theta_+}\bar{D}v^r dv+\int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t}\bar{B}v^r dv+\int_{\Delta t+\theta_+}^{\Delta t}\langle D-B\rangle v^r dv\right.$$

$$+\int_{\Delta t}^{\theta_+}Dv^r dv-\int_{\theta_+}^{0}\langle D-2B\rangle v^r dv-\int_{0}^{\theta_+-\Delta t}Dv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv \quad (5.26)$$

$$= K_r\left\{\frac{\bar{D}}{r+1}\left[\langle k_s\Delta t-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}+\langle(k_s-1)\Delta t+\theta_+\rangle^{r+1}-\tau^{r+1}\right]\right.$$

$$+\frac{D}{r+1}\left[2\theta_+^{r+1}-\langle\theta+\Delta t\rangle^{r+1}-\langle\theta_+-\Delta t\rangle^{r+1}\right]$$

$$+\int_0^{\theta_+}\bar{B}\left[\langle k_s\Delta t+t-T\rangle^r-\langle(k_s-1)\Delta t+t\rangle^r\right]dt$$

$$\left.+\int_0^{\theta_+}B\left[\langle t+\Delta t\rangle^r-2t^r+\langle t-\Delta t\rangle^r\right]dt\right\} \quad (5.27)$$

(e) for the case $\tau = (k_s-1)\Delta t+\theta_+$

Applying the similar technique described in the section 5.3(b) to simplify the first two integral terms in eqn. (5.17). Therefore, we obtain

$$C_r = K_r\left\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}(\bar{D}-\bar{B})v^r dv+\int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T}\bar{D}v^r dv+\int_{(k_s-1)\Delta t+\theta_+}^{(k_s-1)\Delta t}\bar{B}v^r dv+\int_{\Delta t+\theta_+}^{\Delta t}(D-B)v^r dv\right.$$
$$\left.+\int_{\Delta t}^{\theta_+}Dv^r dv-\int_{\theta_+}^{0}(D-2B)v^r dv-\int_{0}^{\theta_+-\Delta t}Dv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv\right\} \tag{5.28}$$

$$= K_r\left\{\frac{\bar{D}}{r+1}\left[\langle(k_s-1)\Delta t+\theta_+-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right]\right.$$
$$+\frac{D}{r+1}\left[2\theta_+^{r+1}-\langle\theta_++\Delta t\rangle^{r+1}-\langle\theta_+-\Delta t\rangle^{r+1}\right]$$
$$+\int_0^{\theta_+}\bar{B}\left[\langle k_s\Delta t+t-T\rangle^r-\langle(k_s-1)\Delta t+t\rangle^r\right]dt$$
$$\left.+\int_0^{\theta_+}B\left[\langle t+\Delta t\rangle^r-2t^r+\langle t-\Delta t\rangle^r\right]dt\right\} \tag{5.29}$$

(f) for the case $(k_s-1)\Delta t+\theta_+ > \tau > (k_s-1)\Delta t$

Applying the similar techniques described in the sections 5.3(a) and 5.3(b) to simplify the third integral term and the first two integral terms in eqn. (5.17) respectively. Thus, we have

$$C_r = K_r\left\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}\langle\bar{D}-\bar{B}\rangle v^r dv+\int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T}\bar{D}v^r dv+\int_{\tau}^{(k_s-1)\Delta t}\bar{B}v^r dv+\int_{\Delta t+\theta_+}^{\Delta t}\langle D-B\rangle v^r dv\right.$$
$$\left.+\int_{\Delta t}^{\theta_+}Dv^r dv-\int_{\theta_+}^{0}\langle D-2B\rangle v^r dv-\int_{0}^{\theta_+-\Delta t}Dv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv\right\} \tag{5.30}$$

$$= K_r\left\{\frac{\bar{D}}{r+1}\left[\langle(k_s-1)\Delta t+\theta_+-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right]\right.$$
$$+\frac{D}{r+1}\left[2\theta_+^{r+1}-\langle\theta+\Delta t\rangle^{r+1}-\langle\theta_+-\Delta t\rangle^{r+1}\right]$$
$$+\int_0^{\theta_+}\bar{B}(k_s\Delta t+t-T)^r dt-\int_0^{\tau-(k_s-1)\Delta t}\bar{B}\langle(k_s-1)\Delta t+t\rangle^r dt$$
$$\left.+\int_0^{\theta_+}B\left[\langle t+\Delta t\rangle^r-2t^r+\langle t-\Delta t\rangle^r\right]dt\right\} \tag{5.31}$$

where $\tau-(k_s-1)\Delta t \geq \theta_+$

(g) for the case $(k_s-1)\Delta t \geq \tau \geq \Delta t+\theta_+$

Applying the similar technique described in the section

5.3(b) to simplify the first three integral terms in eqn. (5.17). Therefore, we obtain

$$C_r = K_r\left\{ \int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T} (\bar{D}-\bar{B})v^r dv + \int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T} \bar{D}v^r dv + \int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T} \bar{B}v^r dv + \int_{\Delta t+\theta_+}^{\Delta t} (D-B)v^r dv \right.$$

$$\left. + \int_{\Delta t}^{\theta_+} Dv^r dv - \int_{\theta_+}^{0} (D-2B)v^r dv - \int_{0}^{\theta_+-\Delta t} Dv^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} Bv^r dv \right. \qquad (5.32)$$

$$= K_r\left\{ \frac{\bar{D}}{r+1}\left[\langle (k_s-1)\Delta t+\theta_+-T\rangle^{r+1} - \langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right] \right.$$

$$+ \frac{D}{r+1}\left[2\theta_+^{r+1} - \langle\theta_+ + \Delta t\rangle^{r+1} - \langle\theta_+ - \Delta t\rangle^{r+1}\right]$$

$$+ \int_0^{\theta_+} \bar{B}\left[\langle k_s\Delta t+t-T\rangle^r - \langle (k_s-1)\Delta t+t-T\rangle^r\right]dt$$

$$\left. + \int_0^{\theta_+} B\left[\langle t+\Delta t\rangle^r - 2t^r + \langle t-\Delta t\rangle^r\right]dt \right\} \qquad (5.33)$$

(h) for the case $\Delta t+\theta_+ > \tau > \Delta t$

Applying the similar techniques described in the sections 5.3(a) and 5.3(b) to simplify the fourth integral term and the first three integral terms in eqn. (5.17) respectively. Thus, we have

$$C_r = K_r\left\{ \int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T} (\bar{D}-\bar{B})v^r dv + \int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T} \bar{D}v^r dv + \int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T} \bar{B}v^r dv + \int_{\tau}^{\Delta t} (D-B)v^r dv \right.$$

$$\left. + \int_{\Delta t}^{\theta_+} Dv^r dv - \int_{\theta_+}^{0} (D-2B)v^r dv - \int_{0}^{\theta_+-\Delta t} Dv^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} Bv^r dv \right\} \qquad (5.34)$$

$$= K_r\left\{ \frac{\bar{D}}{r+1}\left[\langle (k_s-1)\Delta t+\theta_+-T\rangle^{r+1} - \langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right] \right.$$

$$+ \frac{D}{r+1}\left[2\theta_+^{r+1} - \tau^{r+1} - \langle\theta_+ - \Delta t\rangle^{r+1}\right]$$

$$+ \int_0^{\theta_+} \bar{B}\left[\langle k_s\Delta t+t-T\rangle^r - \langle (k_s-1)\Delta t+t-T\rangle^r\right]dt$$

$$\left. + \int_0^{\tau-\Delta t} B\langle\Delta t+t\rangle^r dt + \int_0^{\theta_+} B\left[\langle t-\Delta t\rangle^r - 2t^r\right]dt \right\} \qquad (5.35)$$

where $\tau - \Delta t \geq \theta_+$

(i) for the case $\tau = \Delta t$

Applying the similar technique described in the section 5.3(b) to simplify the first four integral terms in eqn. (5.17). Therefore, we obtain

$$C_r = K_r\Big\{ \int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T} \langle \bar{D}-\bar{B}\rangle v^r dv + \int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T} \bar{D} v^r dv + \int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T} \bar{B} v^r dv + \int_{\Delta t+\theta_+-T}^{\Delta t-T} \langle D-B\rangle v^r dv$$
$$+ \int_{\Delta t}^{\theta_+} D v^r dv - \int_{\theta_+}^{0} \langle D-2B\rangle v^r dv - \int_{0}^{\theta_+-\Delta t} D v^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} B v^r dv \Big\} \quad (5.36)$$

$$= K_r\Big\{ \frac{\bar{D}}{r+1}\Big[\langle (k_s-1)\Delta t+\theta_+-T\rangle^{r+1} - \langle k_s\Delta t+\theta_+-T\rangle^{r+1}\Big]$$
$$+ \frac{D}{r+1}\Big[2\theta_+^{r+1} - \langle \theta_+-\Delta t\rangle^{r+1} - \Delta t^{r+1} + \langle \Delta t-T\rangle^{r+1} - \langle \Delta t+\theta_+-T\rangle^{r+1}\Big]$$
$$+ \int_0^{\theta_+} \bar{B}\Big[\langle k_s\Delta t+t-T\rangle^r - \langle (k_s-1)\Delta t+t-T\rangle^r\Big] dt$$
$$+ \int_0^{\theta_+} B\Big[\langle t-\Delta t\rangle^r - 2t^r + \langle \Delta t+t-T\rangle^r\Big] dt \Big\} \quad (5.37)$$

(j) for the case $\Delta t > \tau > \theta_+$

Applying the similar techniques described in the sections 5.3(a) and 5.3(b) to simplify the fifth integral term and the first four integral terms in eqn. (5.17) respectively. Thus, we have

$$C_r = K_r\Big\{ \int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T} \langle \bar{D}-\bar{B}\rangle v^r dv + \int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T} \bar{D} v^r dv + \int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T} \bar{B} v^r dv + \int_{\Delta t+\theta_+-T}^{\Delta t-T} \langle D-B\rangle v^r dv$$
$$+ \int_{\tau}^{\theta_+} D v^r dv - \int_{\theta_+}^{0} \langle D-2B\rangle v^r dv - \int_{0}^{\theta_+-\Delta t} D v^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} B v^r dv \quad (5.38)$$

$$= K_r\Big\{ \frac{\bar{D}}{r+1}\Big[\langle (k_s-1)\Delta t+\theta_+-T\rangle^{r+1} - \langle k_s\Delta t+\theta_+-T\rangle^{r+1}\Big]$$
$$+ \frac{D}{r+1}\Big[2\theta_+^{r+1} - \langle \theta_+-\Delta t\rangle^{r+1} - \tau^{r+1} + \langle \Delta t-T\rangle^{r+1} - \langle \Delta t+\theta_+-T\rangle^{r+1}\Big]$$
$$+ \int_0^{\theta_+} \bar{B}\Big[\langle k_s\Delta t+t-T\rangle^r - \langle (k_s-1)\Delta t+t-T\rangle^r\Big] dt$$
$$+ \int_0^{\theta_+} B\Big[\langle t-\Delta t\rangle^r - 2t^r + \langle \Delta t+t-T\rangle^r\Big] dt \quad (5.39)$$

(k) for the case $\tau = \theta_+$

Applying the similar technique described in the section 5.3(b) to simplify the first five integral terms in eqn. (5.17). Therefore, we obtain

$$C_r = K_r\Big\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}\langle\bar{D}-\bar{B}\rangle v^r dv+\int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T}\bar{D}v^r dv+\int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T}\bar{B}v^r dv+\int_{\Delta t+\theta_+-T}^{\Delta t-T}\langle D-B\rangle v^r dv$$
$$+\int_{\Delta t-T}^{\theta_+-T}Dv^r dv-\int_{\theta_+}^{0}\langle D-2B\rangle v^r dv-\int_{0}^{\theta_+-\Delta t}Dv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv\Big\} \quad (5.40)$$

$$= K_r\Big\{\frac{\bar{D}}{r+1}\left[\langle(k_s-1)\Delta t+\theta_+-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right]$$
$$+\frac{D}{r+1}\left[\langle\theta_+-T\rangle^{r+1}+\theta_+^{r+1}-\langle\Delta t+\theta_+-T\rangle^{r+1}-\langle\theta_+-\Delta t\rangle^{r+1}\right]$$
$$+\int_0^{\theta_+}\bar{B}\left[\langle k_s\Delta t+t-T\rangle^r-\langle(k_s-1)\Delta t+t-T\rangle^r\right]dt$$
$$+\int_0^{\theta_+}B\left[\langle\Delta t+t-T\rangle^r+\langle t-\Delta t\rangle^r-2t^r\right]dt\Big\} \quad (5.41)$$

(l) for the case $\theta_+ > \tau > 0$

Applying the similar techniques described in the sections 5.3(a) and 5.3(b) to simplify the sixth integral term and the first five integral terms in eqn. (5.17) respectively. Thus, we have

$$C_r = K_r\Big\{\int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T}\langle\bar{D}-\bar{B}\rangle v^r dv+\int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T}\bar{D}v^r dv+\int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T}\bar{B}v^r dv+\int_{\Delta t+\theta_+-T}^{\Delta t-T}\langle D-B\rangle v^r dv$$
$$+\int_{\Delta t-T}^{\theta_+-T}Dv^r dv-\int_{\tau}^{0}\langle D-2B\rangle v^r dv-\int_{0}^{\theta_+-\Delta t}Bv^r dv-\int_{\theta_+-\Delta t}^{-\Delta t}Bv^r dv\Big\} \quad (5.42)$$

$$= K_r\Big\{\frac{\bar{D}}{r+1}\left[\langle(k_s-1)\Delta t+\theta_+-T\rangle^{r+1}-\langle k_s\Delta t+\theta_+-T\rangle^{r+1}\right]$$
$$+\frac{D}{r+1}\left[\langle\theta_+-T\rangle^{r+1}-\langle\theta_++\Delta t-T\rangle^{r+1}+\tau^{r+1}-\langle\theta_+-\Delta t\rangle^{r-1}\right]$$
$$+\int_0^{\theta_+}\bar{B}\left[\langle k_s\Delta t+t-T\rangle^r-\langle(k_s-1)\Delta t+t-T\rangle^r\right]dt$$
$$+\int_0^{\theta_+}B\left[\langle\Delta t+t-T\rangle^r+\langle t-\Delta t\rangle^r\right]dt-2\int_0^{\tau}Bt^r dt\Big\} \quad (5.43)$$

(m) for the case $\tau = 0$

Applying the similar technique described in the section 5.3(b) to simplify the first six integral terms in eqn. (5.17). Therefore, we obtain

$$
\begin{aligned}
C_r = K_r \Big\{ & \int_{k_s\Delta t+\theta_+-T}^{k_s\Delta t-T} (\bar{D}-\bar{B}) v^r dv + \int_{k_s\Delta t-T}^{(k_s-1)\Delta t+\theta_+-T} \bar{D} v^r dv + \int_{(k_s-1)\Delta t+\theta_+-T}^{(k_s-1)\Delta t-T} \bar{B} v^r dv + \int_{\Delta t+\theta_+-T}^{\Delta t-T} (D-B) v^r dv \\
& + \int_{\Delta t-T}^{\theta_+-T} D v^r dv - \int_{\theta_+-T}^{-T} (D-2B) v^r dv - \int_{0}^{\theta_+-\Delta t} D v^r dv - \int_{\theta_+-\Delta t}^{-\Delta t} B v^r dv \Big\}
\end{aligned} \tag{5.44}
$$

$$
\begin{aligned}
= K_r \Big\{ & \frac{\bar{D}}{r+1} \left[\langle (k_s-1)\Delta t+\theta_+-T \rangle^{r+1} - \langle k_s\Delta t+\theta_+-T \rangle^{r+1} \right] \\
& + \frac{D}{r+1} \left[2\langle \theta_+-T \rangle^{r+1} - \langle \Delta t+\theta_+-T \rangle^{r+1} - \langle -T \rangle^{r+1} - \langle \theta_+-\Delta t \rangle^{r+1} \right] \\
& + \int_0^{\theta_+} \bar{B} \left[\langle k_s\Delta t+t-T \rangle^r - \langle (k_s-1)\Delta t+t-T \rangle^r \right] dt \\
& + \int_0^{\theta_+} B \left[\langle \Delta t+t-T \rangle^r + \langle t-\Delta t \rangle^r - 2\langle t-T \rangle^r \right] dt
\end{aligned} \tag{5.45}
$$

It is very complicated to evaluate the value of $I(\tau)$ when the transition error is non-reversible. It can be seen that the value of $I(\tau)$ obtained is directly proportional to the " degree of non-reversibility ", $(\gamma - \lambda)$ and $(\epsilon(t)-\mu(t))^8$. The value of $I(\tau)$ for non-reversible transition error depends on the value of k_s in each m-sequence used as an input signal.

When the reversible transition error is considered in evaluation for $I(\tau)$, the values of D, $\bar{D}$, B, and $\bar{B}$ are replaced by

$$
\left.
\begin{aligned}
D &= 2\alpha \\
\bar{D} &= 0 \\
B &= 2\beta(t) \\
\bar{B} &= 0 \\
\theta_+ = \theta_- &= \theta
\end{aligned}
\right\} \tag{5.46}
$$

If the settling time of the impulse response is than T-2Δt. From eqns. (5.21) to (5.33), all the values of C_r are the same as

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[2\theta^{r+1} - \langle\theta-\Delta t\rangle^{r+1} - \langle\theta+\Delta t\rangle^{r+1}\right] + 2\int_0^{\theta} \beta(t)\left[\langle\Delta t+t\rangle^{r} - 2t^{r} + \langle t-\Delta t\rangle^{r}\right]dt \right\} \quad ,\text{for } \tau \geq \Delta t+\theta \quad (5.47)$$

The value of C_r in eqn. (5.35) becomes

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[2\theta^{r+1} - \tau^{r+1} - \langle\theta-\Delta t\rangle^{r+1}\right] + 2\int_0^{\tau-\Delta t} \beta(t)\langle\Delta t+t\rangle^{r} dt + 2\int_0^{\theta} \beta(t)\left[\langle t-\Delta t\rangle^{r} - 2t^{r}\right]dt \right\} \quad ,\text{for } \Delta t+\theta > \tau > \Delta t \quad (5.48)$$

The value of C_r in eqn. (5.37) becomes

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[2\theta^{r+1} - \langle\theta-\Delta t\rangle^{r+1} - \Delta t^{r+1}\right] + 2\int_0^{\theta} \beta(t)\left[\langle t-\Delta t\rangle^{r} - 2t^{r}\right]dt \right\} \quad ,\text{for } \tau = \Delta t \quad (5.49)$$

The value of C_r in eqn. (5.39) becomes

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[2\theta^{r+1} - \langle\theta-\Delta t\rangle^{r+1} - \tau^{r+1}\right] + 2\int_0^{\theta} \beta(t)\left[\langle t-\Delta t\rangle^{r} - 2t^{r}\right]dt \right\} \quad ,\text{for } \Delta t > \tau > \theta \quad (5.50)$$

The value of C_r in eqn. (5.41) becomes

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[\theta^{r+1} - \langle\theta-\Delta t\rangle^{r+1}\right] + 2\int_0^{\theta} \beta(t)\left[\langle t-\Delta t\rangle^{r} - 2t^{r}\right]dt \right\} \quad ,\text{for } \tau = \theta \quad (5.51)$$

The value of C_r in eqn. (5.43) becomes

$$C_r = K_r\left\{ \frac{2\alpha}{r+1}\left[\tau^{r+1} - \langle\theta-\Delta t\rangle^{r+1}\right]\right.$$

$$+2\int_0^{\theta}\beta(t)\langle t-\Delta t\rangle^{r}dt-4\int_0^{\tau}\beta(t)t^{r}dt\Big\}, \text{for } \theta>\tau>0 \quad (5.52)$$

The value of C_r in eqn. (5.45) becomes

$$C_r = K_r\Big\{\frac{2\alpha}{r+1}\Big[-\langle\theta-\Delta t\rangle^{r+1}\Big] + 2\int_0^{\theta}\beta(t)\langle t-\Delta t\rangle^{r}dt\Big\}, \text{for } \tau=0 \quad (5.53)$$

It seen that the value of the coefficient C_r does not depend on the value of k_s for all m-sequences. The results of $I(\tau)$ in eqn. (5.15) are rewritten as the function of τ in four different forms.

$$I(\tau)=\frac{a2^n}{2^n-1}\Big\{-\frac{1}{2}\alpha\Delta t h'(\tau)+\frac{1}{2}\langle\alpha\theta\Delta t-\Delta t\int_0^{\theta}\beta(t)dt\rangle h''(\tau) + \dots\Big\}, \text{for } \tau\geq\Delta t+\theta \quad (5.54)$$

$$I(\Delta t)=\frac{a2^n}{2^n-1}\Big\{\frac{-\alpha\theta+\int_0^{\theta}\beta(t)dt}{2\Delta t}h(\Delta t)+\frac{\alpha\langle\frac{\theta^2}{2}+\theta\Delta t-\Delta t^2\rangle-\int_0^{\theta}\beta(t)\langle t+\Delta t\rangle dt}{2\Delta t}h'(\Delta t)$$
$$+\frac{\alpha\langle-\frac{\theta^3}{3}-\theta^2\Delta t+\theta\Delta t^2\rangle+\int_0^{\theta}\beta(t)\langle t^2+2t\Delta t-\Delta t^2\rangle dt}{4\Delta t}h''(\Delta t) + \dots\Big\}, \text{for } \tau=\Delta t \quad (5.55)$$

$$I(\theta)=\frac{a2^n}{2^n-1}\Big\{\frac{-\alpha\Delta t+\int_0^{\theta}\beta(t)dt}{2\Delta t}h(\theta)+\frac{\alpha\langle\theta\Delta t-\frac{\Delta t^2}{2}\rangle-\int_0^{\theta}\beta(t)\langle t+\Delta t\rangle dt}{2\Delta t}h'(\theta)$$
$$+\frac{\alpha\langle-\theta^2\Delta t+\theta\Delta t^2-\frac{\Delta t^3}{3}\rangle-\int_0^{\theta}\beta(t)\langle\Delta t^2-2t\Delta t-t^2\rangle dt}{4\Delta t}h''(\theta) + \dots\Big\}, \text{for } \tau=\theta \quad (5.56)$$

$$I(0)=\frac{a2^n}{2^n-1}\Big\{\frac{-\alpha\langle\Delta t-\theta\rangle-\int_0^{\theta}\beta(t)dt}{2\Delta t}h(0)+\frac{-\frac{\alpha}{2}\langle\Delta t+\theta\rangle^2+\int_0^{\theta}\beta(t)\langle t-\Delta t\rangle dt}{2\Delta t}h'(0)$$
$$+\frac{-\frac{\alpha}{3}\langle\Delta t-\theta\rangle^3-\int_0^{\theta}\beta(t)\langle t-\Delta t\rangle^2dt}{4\Delta t}h''(0)+\dots\Big\}, \text{for } \tau=0 \quad (5.57)$$

It is seen that eqn. (5.47) to eqn. (5.53) depend on the waveform and the area of the non zero parts of the transition error. These will be discussed in three particular types as follows.

5.3.1 Reversible transition error of the exponential form

This type of transition error is defined as a transducer having the time constant T_e for both positive- and negative- going transitions[8]. Following each transition of the input signal x(t) from the +a to the -a state, the error signal e(t) becomes

$$e(t) = 2ae^{-t/T_e} \tag{5.58}$$

It may be shown that

$$\alpha = \int_0^{\theta} e(t)dt = 2aT_e(1-e^{-\theta/T_e}) \tag{5.59}$$

and

$$\beta(t) = \int_0^{t} e(t)dt = 2aT_e(1-e^{-t/T_e}) \tag{5.60}$$

Theoretically, this type of error signal e(t) will return to zero after an infinite time, but, provided that e(t) is close to zero when $t=\theta \leq \Delta t$. Choosing this small value of the error e(t) to be 1% of e(0), from eqn. (5.58) we have

$$0.02a = 2ae^{-\theta/T_e} \tag{5.61}$$

$$T_e \doteq \frac{\theta}{4.6} \tag{5.62}$$

and

$$\alpha \doteq 2aT_e \tag{5.63}$$

Substituting for the values of α and $\beta(t)$ into eqn. (5.54) to eqn. (5.57), we obtain

$$I(\tau) \doteq \frac{a2^n}{2^n-1}\left\{-aT_e\Delta t\,h'(\tau)+a\Delta t\,T_e^2h''(\tau)+\dots\right\} \qquad ,\text{for } \tau \geq \Delta t+\theta \quad (5.64)$$

$$I(\Delta t) \doteq \frac{a2^n}{2^n-1}\left\{-\frac{aT_e^2}{\Delta t}h(\Delta t)+\left(aT_e^2-aT_e\Delta t+\frac{aT_e^3}{\Delta t}\right)h'(\Delta t)\right.$$

$$\left.+\left(\frac{aT_e^2\Delta t}{2}-aT_e^3-\frac{aT_e^4}{\Delta t}\right)h''(\Delta t)+\dots\right\} \qquad ,\text{for } \tau=\Delta t \quad (5.65)$$

$$I(\theta) \doteq \frac{a2^n}{2^n-1}\left\{\left(\frac{aT_e\theta}{\Delta t}-\frac{aT_e^2}{\Delta t}-aT_e\right)h(\theta)+\left(aT_e^2+\frac{aT_e^3}{\Delta t}-\frac{aT_e\theta^2}{2\Delta t}-\frac{aT_e\Delta t}{2}\right)h'(\theta)\right.$$

$$\left.+\left(\frac{aT_e\theta^3}{6\Delta t}-\frac{aT_e\Delta t^2}{6}-\frac{aT_e^4}{\Delta t}-aT_e^3\right)h''(\theta)+\dots\right\} ,\text{for } \tau=\theta \quad (5.66)$$

$$I(0) \doteq \frac{a2^n}{2^n-1}\left\{\left(\frac{aT_e^2}{\Delta t}-aT_e\right)h(0)+\left(aT_e^2-\frac{aT_e^3}{\Delta t}-\frac{aT_e\Delta t}{2}\right)h'(0)\right.$$

$$\left.+\left(\frac{aT_e^2\Delta t}{2}-\frac{aT_e\Delta t^2}{6}-aT_e^3+\frac{aT_e^4}{\Delta t}\right)h''(0)+\dots\right\} ,\text{for } \tau=0 \quad (5.67)$$

Since we assumed that $T_e \doteq \frac{\theta}{5}$, thus we obtain

$$I(\tau) \doteq \frac{a2^n}{2^n-1}\left\{-\frac{a\theta\Delta t}{5}h'(\tau)+\frac{a\theta^2\Delta t}{25}h''(\tau)+\dots\right\} ,\text{for } \tau \geq \Delta t+\theta \quad (5.68)$$

$$I(\Delta t) \doteq \frac{a2^n}{2^n-1}\left\{-\frac{a\theta^2}{25\Delta t}h(\Delta t)+\left(\frac{a\theta^2}{25}-\frac{a\theta\Delta t}{5}+\frac{a\theta^3}{125\Delta t}\right)h'(\Delta t)\right.$$

$$\left.+\left(\frac{a\theta^2\Delta t}{50}-\frac{a\theta^3}{125}-\frac{a\theta^4}{625\Delta t}\right)h''(\Delta t)+\dots\right\} ,\text{for } \tau=\Delta t \quad (5.69)$$

$$I(\theta) \doteq \frac{a2^n}{2^n-1}\left\{\left(\frac{4a\theta^2}{25\Delta t}-\frac{a\theta}{5}\right)h(\theta)+\left(\frac{a\theta^2}{25}-\frac{23a\theta^3}{250\Delta t}-\frac{a\theta\Delta t}{10}\right)h'(\theta)\right.$$

$$\left.+\left(\frac{119a\theta^4}{3750\Delta t}-\frac{a\theta\Delta t^2}{30}-\frac{a\theta^3}{125}\right)h''(\theta)+\dots\right\} ,\text{for } \tau=\theta \quad (5.70)$$

$$I(0) \doteq \frac{a2^n}{2^n-1}\left\{\left(\frac{a\theta^2}{25\Delta t}-\frac{a\theta}{5}\right)h(0)+\left(\frac{a\theta^2}{25}-\frac{a\theta^3}{125\Delta t}-\frac{a\theta\Delta t}{10}\right)h'(0)\right.$$

$$\left.+\left(\frac{a\theta^2\Delta t}{50}-\frac{a\theta\Delta t^2}{30}-\frac{a\theta^3}{125}+\frac{a\theta^4}{625\Delta t}\right)h''(0)+\dots\right\} ,\text{for } \tau=0 \quad (5.71)$$

If the first order linear system ($h(\tau)=Be^{-\tau/T_1}$) is considered , the approximate error terms have the magnitudes when

(i) $\theta = \Delta t$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{\Delta t^2}{5T_1}h(\tau) \qquad ,\text{for } \tau \geq \Delta t+\theta$$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{19\ \Delta t^2}{125\,T_1}h(\Delta t) \qquad ,\text{for } \tau=\Delta t$$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{38\ \Delta t^2}{250\,T_1}h(\theta) \qquad ,\text{for } \tau=\theta$$

and
$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{17\ \Delta t^2}{250\,T_1}h(0) \qquad ,\text{for } \tau=0$$

(ii) $\theta = \frac{\Delta t}{10}$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{\Delta t^2}{50\,T_1}h(\tau) \qquad ,\text{for } \tau \geq \Delta t+\theta$$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{2449\ \Delta t^2}{125000\,T_1}h(\Delta t) \qquad ,\text{for } \tau=\Delta t$$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{2423\ \Delta t^2}{250000\,T_1}h(\theta) \qquad ,\text{for } \tau=\theta$$

$$\left(\frac{a^2z^n}{z^n-1}\right)\frac{1201\ \Delta t^2}{125000\,T_1}h(0) \qquad ,\text{for } \tau=0$$

5.3.2 Reversible transition error of the constant-acceleration form

This type of transition error $e(t)$ may be expressed by

$$e(t) = 2a\left(1-\frac{t^2}{\theta^2}\right) \tag{5.72}$$

It can be shown that

$$\alpha = \int_0^\theta e(t)\,dt = \frac{4}{3}a\theta \tag{5.73}$$

and
$$\beta(t) = \int_0^t e(t)\,dt = 2a\left(t-\frac{t^3}{3\theta^2}\right) \tag{5.74}$$

Substituting for the values of α and $\beta(t)$ into eqn. (5.54) to eqn. (5.57), we obtain

$$I(\tau) = \frac{az^n}{z^n-1}\left\{-\frac{2}{3}a\theta\Delta t\,h'(\tau)+\frac{1}{4}a\theta^2\Delta t\,h''(\tau)+\ldots\right\}, \text{for } \tau\geq\Delta t+\theta \tag{5.75}$$

$$I(\Delta t) = \frac{az^n}{z^n-1}\left\{-\frac{a\theta^2}{4\Delta t}h(\Delta t) + \left(\frac{a\theta^3}{15\Delta t}+\frac{a\theta^2}{4}-\frac{2a\theta\Delta t}{3}\right)h'(\Delta t)\right.$$

$$+\left\langle \frac{3a\theta^3}{5} + \frac{a\theta^2\Delta t}{6} - \frac{a\theta^4}{72\Delta t}\right\rangle h''(\Delta t) + \ldots \Big\} \quad \text{,for } \tau = \Delta t \qquad (5.76)$$

$$I(\theta) = \frac{a2^n}{2^n-1}\Big\{\left\langle -\tfrac{2}{3}a\theta + \frac{5a\theta^2}{12\Delta t}\right\rangle h(\theta) + \left\langle \tfrac{1}{4}a\theta^2 - \tfrac{1}{3}a\theta\Delta t - \frac{4a\theta^3}{15\Delta t}\right\rangle h'(\theta)$$

$$+ \left\langle \frac{a\theta^2\Delta t}{8} - \frac{a\theta^3}{15} - \frac{a\theta\Delta t^2}{9} + \frac{7a\theta^4}{72\Delta t}\right\rangle h''(\Delta t) + \ldots\Big\} \quad \text{,for } \tau = \theta \qquad (5.77)$$

$$I(0) = \frac{a2^n}{2^n-1}\Big\{\left\langle -\tfrac{2}{3}a\theta + \frac{a\theta^2}{4\Delta t}\right\rangle h(0) + \left\langle \frac{a\theta^2}{4} - \frac{a\theta\Delta t}{3} - \frac{a\theta^3}{15\Delta t}\right\rangle h'(0)$$

$$+ \left\langle \frac{a\theta^2\Delta t}{8} - \frac{a\theta^3}{15} - \frac{a\theta\Delta t^2}{9} + \frac{a\theta^4}{72\Delta t}\right\rangle h''(0) + \ldots\Big\} \quad \text{,for } \tau = 0 \qquad (5.78)$$

If the same first order linear system as in the section 5.3.1 is considered, the error terms have the magnitudes when

(i) $\theta = \Delta t$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{2\Delta t^2}{3T_1}h(\tau)$,for $\tau \geq \Delta t+\theta$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{21\ \Delta t^2}{60T_1}h(\Delta t)$,for $\tau = \Delta t$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{21\ \Delta t^2}{60T_1}h(\theta)$,for $\tau = \theta$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{3\ \Delta t^2}{20T_1}h(0)$,for $\tau = 0$

(ii) $\theta = \frac{\Delta t}{10}$

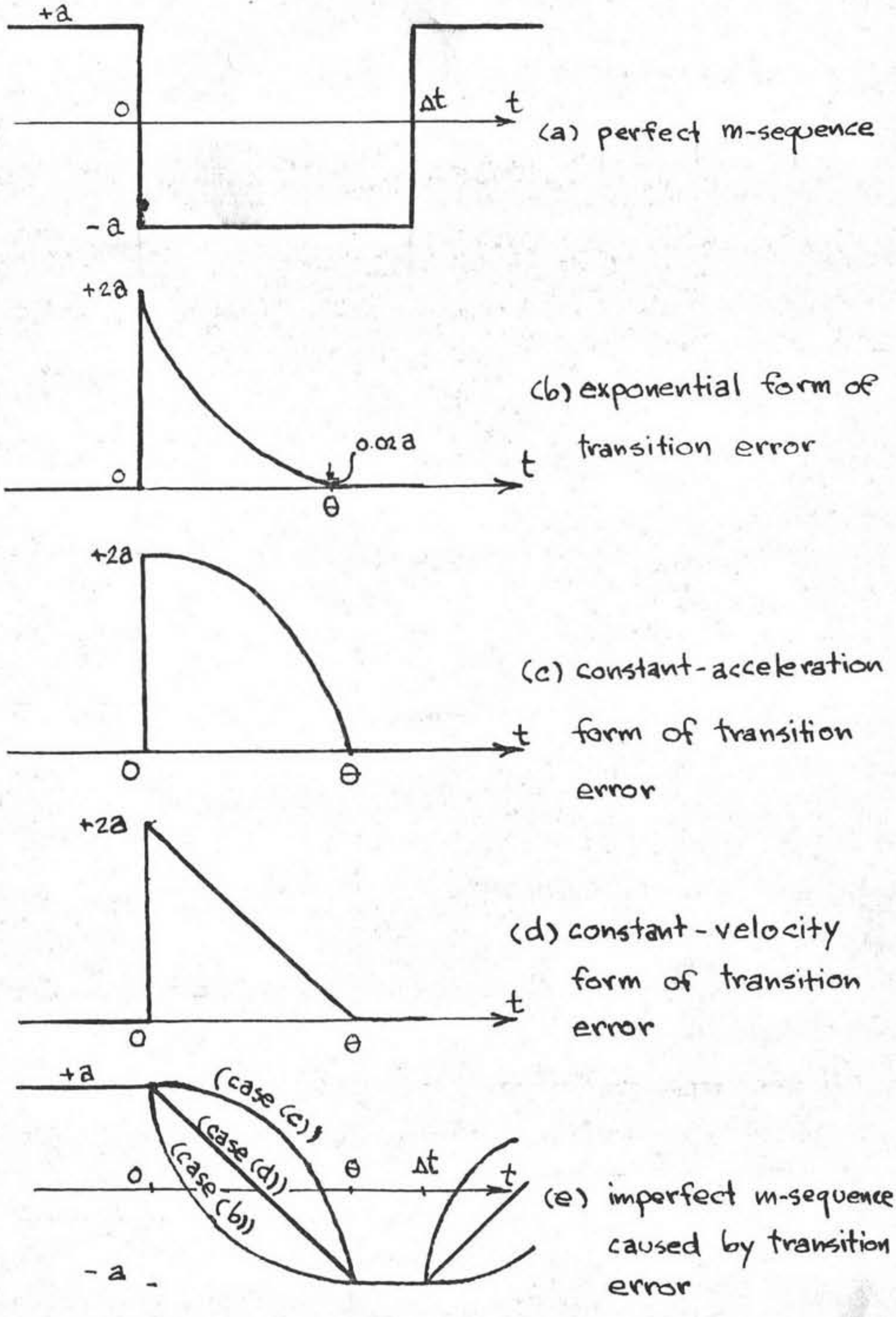
$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{2\ \Delta t^2}{30T_1}h(\tau)$,for $\tau \geq \Delta t+\theta$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{2423\ \Delta t^2}{30000T_1}h(\Delta t)$,for $\tau = \Delta t$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{311\ \Delta t^2}{10000T_1}h(\theta)$,for $\tau = \theta$

$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{309\ \Delta t^2}{10000T_1}h(0)$,for $\tau = 0$

Figure 5.2



+a
0
Δt
t
-a
(a) perfect m-sequence
+2a
0
0.02 a
θ
t
(b) exponential form of transition error
+2a
0
θ
t
(c) constant-acceleration form of transition error
+2a
0
θ
t
(d) constant-velocity form of transition error
+a
(case (c))
(case (d))
(case (b))
0
θ
Δt
t
-a
(e) imperfect m-sequence caused by transition error

5.3.3 Reversible transition error of the constant-velocity form

This type of transition error $e(t)$ can be expressed by

$$e(t) = 2a\left(1-\frac{t}{\theta}\right) \tag{5.79}$$

Thus we have

$$\alpha = \int_0^{\theta} e(t)\,dt = a\theta \tag{5.80}$$

and

$$\beta(t) = \int_0^{t} e(t)\,dt = 2a\left(t-\frac{t^2}{2\theta}\right) \tag{5.81}$$

Substituting for α and $\beta(t)$ into eqn. (5.54) to eqn. (5.17), gives

$$I(\tau) = \frac{a2^n}{2^n-1}\left\{-\frac{1}{2}a\theta\Delta t\,h'(\tau) + \frac{1}{6}a\theta^2\Delta t\,h''(\tau) + \ldots\right\}, \text{ for } \tau \geq \Delta t+\theta \tag{5.82}$$

$$I(\Delta t) = \frac{a2^n}{2^n-1}\left\{-\frac{a\theta^2}{6\Delta t}h(\Delta t) + \left(\frac{a\theta^3}{24\Delta t} + \frac{a\theta^2}{6} - \frac{a\theta\Delta t}{2}\right)h'(\Delta t)\right.$$
$$\left. + \left(\frac{a\theta^2\Delta t}{12} - \frac{a\theta^3}{24} - \frac{a\theta^4}{120\Delta t}\right)h''(\Delta t) + \ldots\right\}, \text{ for } \tau = \Delta t \tag{5.83}$$

$$I(\theta) = \frac{a2^n}{2^n-1}\left\{\left(-\frac{a\theta}{2} + \frac{a\theta^2}{3\Delta t}\right)h(\theta) + \left(\frac{a\theta^2}{6} - \frac{a\theta\Delta t}{4} - \frac{5a\theta^3}{24\Delta t}\right)h'(\theta)\right.$$
$$\left. + \left(\frac{a\theta^2\Delta t}{12} - \frac{a\theta^3}{24} - \frac{a\theta\Delta t^2}{12} + \frac{3a\theta^4}{40\Delta t}\right)h''(\theta) + \ldots\right\}, \text{ for } \tau = \theta \tag{5.84}$$

$$I(0) = \frac{a2^n}{2^n-1}\left\{\left(-\frac{a\theta}{2} + \frac{a\theta^2}{6\Delta t}\right)h(0) + \left(\frac{a\theta^2}{6} - \frac{a\theta\Delta t}{4} - \frac{a\theta^3}{24\Delta t}\right)h'(0)\right.$$
$$\left. + \left(\frac{a\theta^2\Delta t}{12} - \frac{a\theta^3}{24} - \frac{a\theta\Delta t^2}{12} + \frac{a\theta^4}{120\Delta t}\right)h''(0) + \ldots\right\}, \text{ for } \tau = 0 \tag{5.85}$$

When the same first order linear system as in the section 5.3.1 is considered, the error terms have the magnitudes when

(i) $\theta = \Delta t$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{\Delta t^2}{2T_1}h(\tau) \qquad , \text{for } \tau \geq \Delta t+\theta$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{7\ \Delta t^2}{24T_1}h(\Delta t) \qquad ,\text{for } \tau=\Delta t$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{7\ \Delta t^2}{24T_1}h(\theta) \qquad ,\text{for } \tau=\theta$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{\Delta t^2}{8T_1}h(0) \qquad ,\text{for } \tau=0$$

(ii) $\theta = \frac{\Delta t}{10}$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{\Delta t^2}{20T_1}h(\tau) \qquad ,\text{for } \tau \geq \Delta t+\theta$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{1159\Delta t^2}{24000T_1}h(\Delta t) \qquad ,\text{for } \tau=\Delta t$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{565\Delta t^2}{24000T_1}h(\theta) \qquad ,\text{for } \tau=\theta$$

$$\left(\frac{a^2 2^n}{2^n-1}\right)\frac{561\Delta t^2}{24000T_1}h(0) \qquad ,\text{for } \tau=0$$

The percentage errors of the error terms due to three types of the reversible transition error are illustrated in table 5.1.

5.4 The Effect of the External Noise

The errors in evaluation of the impulse response may be increased when the external noise is considered. The second convolution integral in the right-hand side of eqn. (4.5) is the effect of this noise signal to the value of crosscorrelation function $\phi_{xy_i}(\tau)$. Let the second convolution integral be

$$\rho(\tau) = \int_0^T h(u)\phi_{xn_i}(\tau - u)du \tag{5.86}$$

If the impulse response h(u) is expanded by Taylor series about $u=\tau$, we have

Table 5.1

The approximate percentage error caused by non-reversible transition error in first order linear system with $h(\tau)=Be^{-\tau/T_1}$

Types of error	Values of τ	$\theta=\Delta t$		$\theta=\Delta t/10$		Note
		$\Delta t=T_1/10$	$\Delta t=T_1$	$\Delta t=T_1/10$	$\Delta t=T_1$	
Exponential form	$\tau \geq \Delta t+\theta$	2%	20%	0.2%	2%	a
	$\tau=\Delta t$	1.54%	15.4%	0.196%	1.96%	b
	$\tau=\theta$	1.54%	15.4%	0.163%	1.63%	c
	$\tau=0$	1.36%	13.6%	0.192%	1.92%	d
Constant-acceleration form	$\tau \geq \Delta t+\theta$	6.67%	66.67%	0.67%	6.67%	a
	$\tau=\Delta t$	3.5%	35%	0.808%	8.08%	b
	$\tau=\theta$	3.5%	35%	0.523%	5.23%	c
	$\tau=0$	3%	30%	0.618%	6.18%	d
Constant-velocity form	$\tau \geq \Delta t+\theta$	5%	50%	0.5%	5%	a
	$\tau=\Delta t$	2.917%	29.17%	0.483%	4.83%	b
	$\tau=\theta$	2.917%	29.17%	0.396%	3.96%	c
	$\tau=0$	2.5%	25%	0.468%	4.68%	d

Note: The magnitude of first order derivative terms is compared to

(a) $\frac{a^2 2^n}{(2^n-1)}(\Delta t)\,h(\tau)$

(b) $\frac{a^2 2^n}{(2^n-1)}(\Delta t)\,h(\tau)$

(c) $\frac{a^2 2^n}{(2^n-1)}\left(\frac{\Delta t}{2}\right)\left(1+\frac{2\tau}{\Delta t}-\frac{a^2}{2\Delta t^2}\right)h(\theta)$

(d) $\frac{a^2 2^n}{2^n-1}\left(\frac{\Delta t}{2}\right)h(0)$

$$\rho(\tau) = K_0 h(\tau) + K_1 h'(\tau) + \ldots \tag{5.87}$$

where
$$K_r = \frac{1}{r!}\int_0^T (t-\tau)^r \phi_{xn_i}(\tau - t)\,dt \tag{5.88}$$

The value of K_r depends on the waveform of the external noise which may be taken as

(i) The time polynomial function signal

The crosscorrelation between the input signal x(t) and the time polynomial function signal $n_i(t)$ can be obtained from eqn. (4.12)

$$\phi_{xn_i}(\tau - t) = A_0 + A_1(\tau - t) + \ldots + A_m(\tau - t)^m \tag{5.89}$$

From eqn. (5.88) and eqn. (5.89), we have (see Appendix G)

$$\rho(\tau) = B_0 h(\tau) + B_1 h'(\tau) + \ldots \tag{5.90}$$

where
$$B_r = \frac{1}{r!}\sum_{i=0}^{m} \frac{A_i(-1)^i}{r+i+1}\left\{(T-\tau)^{r+i+1} - (-\tau)^{r+i+1}\right\}$$

(ii) The white noise signal

The crosscorrelation between the input signal x(t) and the white noise signal $n_i(t)$ is obtained from eqn. (4.22), thus we have

$$\phi_{xn_i}(\tau - t) = a\sqrt{K} \tag{5.91}$$

From eqn. (5.88) and eqn. (5.91), we have (see Appendix G)

$$\rho(\tau) = D_0 h(\tau) + D_1 h'(\tau) + \ldots \tag{5.92}$$

where
$$D_r = \frac{(-1)^{r+1}}{(r+1)!}\, a\sqrt{K}\left\{(\tau - T)^{r+1} - \tau^{r+1}\right\}$$

It can be seen that the values of $\rho(\tau)$ also depend on the value of Δt as the values of $I(\tau)$ do.