CHAPTER IV

CONTINUOUS SOLUTION OF $g(xoy^{-1}) = g(x)g(y)+f(x)f(y)$ ON ABELIAN TOPOLOGICAL GROUP

Let G be an abelian topological group, F be a topological field of characteristic different from 2. In this chapter we shall determine all continuous solutions of

$$\text{(A)} \qquad g(xoy^{-1}) \quad = \quad g(x)g(y) + f(x)f(y)$$

on G to F. This result is also applied to the case where G is any abelian 2 - divisible topological group.

Definition 4.1 Let G be a topological group, F be a topological field. By a continuous solution of the functional equation

$$\text{(A)} \qquad g(xoy^{-1}) \quad = \quad g(x)g(y) + f(x)f(y)$$

on G to F, we mean any solution (f,g) for which f and g are continuous.

Lemma 4.2 Let $S_1, \ldots, S_n$ be disjoint subsets of a topological space X such that $\bigsqcup_{i=1}^{n} S_i = X$, f be any function from the topological space X into a T_1 - space Y such that for each $i = 1, \ldots, n$, there exist c_i such that $f(x) = c_i$ for all x in S_i, i.e. f is constant on each S_i, $i = 1, \ldots, n$. Then f is continuous if and only if each S_i is open, for $i = 1, \ldots, n$.

Proof Assume that $f : X \longrightarrow Y$ is a continuous function such that $f(x) = c_i$ on S_i, $i = 1, \ldots, n$. Since Y is a T_1 - space, hence we can choose open neighborhoods N_i's of c_1 in V such that $c_i \notin N_i$, $i = 2, \ldots, n$. Let $N = \bigcap_{i=1}^{n} N_i$. We see that N is a non-empty open neighborhood of c_1 such that $c_2, \ldots, c_n \notin N$. Since f is continuous, hence $f^{-1}(N)$ is open. Observe that $f^{-1}(N) = S_1$. Hence S_1 is open. Similarly we can show that $S_2, \ldots, S_n$ are open. Hence S_i is open for $i = 1, \ldots, n$.

Conversely, assume that $f(x) = c_i$ on each open set S_i, $i = 1, \ldots, n$. Let O be any open set in Y, and $I = \{ i : c_i \in O \}$ Observe that $f^{-1}(O) = \bigcup_{i \in I} f^{-1}(c_i) = \bigcup_{i \in I} S_i$. Hence $f^{-1}(O)$, being a union of open sets, is open. Therefore f is continuous.

Corollary 4.3 Let f be any function from a topological group G into a T_1 - topological field F such that f is constant on each coset of a subgroup H of finite index in G.
Then f is continuous if and only if H is open.

Proof Since H is of finite index in G, hence G is the finite union of distinct cosets of H. According to Lemma 4.2 we see that f is continuous if and only if each coset of H is open. But each coset of H is open if and only if H is open. Hence f is continuous if and only if H is open.

Theorem 4.4 Let G be an abelian topological group, F be a topological field of characteristic different from 2. Then a solution (f,g) of

$$(A) \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F of the form

$$(4.4.1) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2},$$

where h is a homomorphism from G into M(F) , is continuous if and only if h is continuous.

Proof Assume that (f,g) given by (4.4.1) is a continuous solution of (A). It follows that

$$g(x) + i\, f(x) = \frac{h(x) + h(x^{-1})}{2} + i\left[\frac{h(x) - h(x^{-1})}{2i}\right]$$

$$= \frac{2\, h(x)}{2}$$

$$= h(x)\,.$$

Since f and g are continuous, hence h is continuous. Conversely, if h is a continuous homomorphism from G into M(F), then it is clear that

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

are continuous.

Theorem 4.5 Let G be an abelian topological group, F be a T_1 - topological field of characteristic different from 2. Then the continuous solution of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F are those and only those (f,g) of the forms :

(4.5.1) $f(x) = b$, $g(x) = a$ for all x in G, where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$\text{(4.5.2)} \qquad f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is an open subgroup of index 2 in G and a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$\text{(4.5.3)} \qquad f(x) = \begin{cases} 0, & x \in H \\ d, & x \notin H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ c, & x \notin H \end{cases}$$

where H is an open subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$\text{(4.5.4)} \qquad f(x) = \begin{cases} 0, & x \in H \text{ or } x_1H \\ d, & x \in x_2H \\ -d, & x \in x_3H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is an open subgroup of index 4 in G such that G/H is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$(4.5.5) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is continuous homomorphism from G into M(F).

Proof By Theorem 3.30, the solution of (A) are those and only those (f,g) of the forms :

(4.5.1) $f(x) = b$, $g(x) = a$ for all x in G, where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$(4.5.2)' \qquad f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$(4.5.3)' \qquad f(x) = \begin{cases} 0, & x \in H \\ d, & x \notin H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ c, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$(4.5.4)' \qquad f(x) = \begin{cases} 0, & x \in H \text{ or } x_1H \\ d, & x \in x_2H \\ -d, & x \in x_3H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that G/H is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$(4.5.5)' \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).

By Corollary 4.3, f,g in (4.5.2)', (4.5.3)' and (4.5.4)' are continuous iff the subgroup H's are open, and by Theorem 4.4, f,g in (4.5.5)' are continuous iff h is continuous. Note that f,g in (4.5.1) are constant on G, thus they are continuous. Hence the continuous solutions of (A) are those and only those (f,g) given by (4.5.1) or (4.5.2) or (4.5.3) or (4.5.4) or (4.5.5).

Remark 4.6 Note that if G has no open subgroup of index 2, then it can not have any open subgroup of index 4. Hence, for such G the continuous solutions of (A) must be those and only those (f,g) of the forms :

(4.6.1) $f(x) = b$, $g(x) = a$ for all x in G where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$(4.6.2) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a continuous homomorphism from G into M(F).

Note that our proof of Theorem 4.5 makes uses of the property T_1 of the topological field F only when G has open subgroup H of index 2 or 4. Hence in the case where G has no open subgroup of index 2, we need not assume that F is T_1, i.e. we may omit the assumption that F is T_1 from the hypothesis of the Theorem.

Lemma 4.7 Any abelian 2 - divisible group G has no subgroup of index 2.

Proof Suppose that there exists a subgroup H of index 2 in G. Hence $G - H \neq \emptyset$. Choose $x \in G-H$. Since G is 2 - divisible, hence there exists y in G such that $x = yoy$. If $y \notin H$, then $yH \neq H$. However H is of index 2, hence H and yH are the only two elements of the quotient group $G/_H$. Since $G/_H$ has order 2, therefore $H = yHoyH = (yoy)H$. Thus $x = yoy \in H$. If $y \in H$, then $x = yoy \in H$. In any case we have $x \in H$, which is a contradiction. Hence G has no subgroup of index 2.

Theorem 4.8 Let G be an abelian 2-divisible topological group, F be a topological field of characteristic different from 2. Then the continuous solutions of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F are those and only those (f,g) of the forms:

(4.8.1) $f(x) = b$, $g(x) = a$ for all x in G where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$\text{(4.8.2)} \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i} \; , \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a continuous homomorphism from G into $M(F)$.

Proof Since G is an abelian 2 - divisible topological group, by Lemma 4.7, G has no subgroup of index 2, hence G has no open subgroup of index 2. It follows from Remark 4.6 that the continuous solutions of (A) must be those and only those (f,g) of the forms (4.8.1) or (4.8.2).

Corollary 4.9 Let V be a topological vector space over F, where F is $\mathbb{R}$ or $\mathbb{C}$. Then the continuous solutions of

(A) $$g(x - y) = g(x)g(y) + f(x)f(y)$$

on V to F are those and only those (f,g) of the forms:

(4.9.1) $f(x) = b$, $g(x) = a$ for all x in V where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

(4.9.2) $$f(x) = \frac{h(x) - h(-x)}{2i}, \quad g(x) = \frac{h(x) + h(-x)}{2}$$

where h is a continuous homomorphism from V into $M(F)$.

Proof Observe that V is an abelian 2 - divisible topological group. By Theorem 4.8, continuous solutions of (A) on V to F must be those and only those (f,g) of the forms (4.9.1) or (4.9.2).