CHAPTER III

GENERAL SOLUTION OF $g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$ ON ABELIAN GROUP.

Let G be an abelian group, F be a field of characteristic different from 2. In this chapter we shall determine all functions $f, g : G \longrightarrow F$ satisfying the functional equation

$$(A) \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y) ,$$

for all x, y in G. The main result of this chapter is Theorem 3.30.

<u>Definition 3.1</u> Let G be any group, F be any field. By a <u>solution</u> of the functional equation

$$(A) \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F, we mean an ordered pair (f,g) where f, g are functions from G into F such that (A) holds for all x, y in G.

<u>Definition 3.2</u> Let f, g be any functions from an arbitrary group G into an arbitrary field F such that (f, g) is a solution of

$$(A) \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y) .$$

The subset $H = \{x : g(x) = g(e)\}$ will be called the <u>period</u> of g.

Lemma 3.3 Let f, g be any functions from an arbitrary group G into an arbitrary field F such that (f, g) is a solution of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y) .$$

Then we have

$$\text{(3.3.1)} \qquad g(x)^2 + f(x)^2 = g(e) ,$$

$$\text{(3.3.2)} \qquad f(e)^2 = g(e) - g(e)^2,$$

$$\text{(3.3.3)} \qquad f(x)f(e) = g(x) - g(x)g(e) ,$$

$$\text{(3.3.4)} \qquad g(x^{-1}) = g(x) ,$$

$$\text{(3.3.5)} \qquad f(x^{-1}) = f(x) \text{ or } f(x^{-1}) = - f(x) ,$$

for all x in G .

Proof Setting $y = x$ in (A) we get

$$g(x)^2 + f(x)^2 = g(xox^{-1}) = g(e) .$$

Therefore

$$g(x)^2 + f(x)^2 = g(e) ,$$

for all x in G , i.e. (3.3.1) holds.

In particular when $x = e$ we have

$$g(e)^2 + f(e)^2 = g(e) .$$

Thus (3.3.2) holds.

For an arbitrary x , we have

$$g(x) = g(xoe) = g(x)g(e) + f(x)f(e) .$$

Therefore

$$f(x)f(e) = g(x) - g(x)g(e) ,$$

for all x in G, i.e. (3.3.3) holds.

It follows from (A) that

$$g(x^{-1}) = g(eox^{-1}) = g(e)g(x) + f(e)f(x) .$$

Hence

$$g(x^{-1}) = g(x) ,$$

for all x in G, i.e. (3.3.4) holds.

Using (3.3.1) , we have

$$f(x)^2 = g(e) - g(x)^2 ,$$

and

$$f(x^{-1})^2 = g(e) - g(x^{-1})^2 ,$$

hence, it follows from (3.3.4) that $f(x^{-1})^2 = f(x)^2$. Therefore

$$f(x^{-1}) = f(x) \quad \text{or} \quad f(x^{-1}) = -f(x)$$

for all x in G, i.e. (3.3.5) holds.

Lemma 3.4 Let f, g be any functions from an arbitrary group G into an arbitrary field F such that (f, g) is a solution of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y).$$

Then the followings hold for all x in G ;

(3.4.1) if $g(x) = g(e)$ then $f(x) = f(e)$,

(3.4.2) if $g(x) = -g(e)$ then $f(x) = -f(e)$,

(3.4.3) $\left[1 - g(e)\right]\left[g(x)^2 - g(e)^2\right] = 0$.

Proof To show (3.4.1) we assume that $g(x) = g(e)$. From (3.3.2) and (3.3.3) we find that

$$\begin{aligned} f(x)f(e) &= g(x) - g(x)g(e) \\ &= g(e) - g(e)^2 \\ &= f(e)^2 . \end{aligned}$$

Thus

$$f(e)\left[f(x) - f(e)\right] = 0 .$$

Case I . Assume that $f(e) \neq 0$. It follows that $f(x) = f(e)$.

Case II . Assume that $f(e) = 0$. By (3.3.1) and (3.3.2) we see that

$$\begin{aligned} f(x)^2 &= g(e) - g(x)^2, \\ &= g(e) - g(e)^2 , \end{aligned}$$

$$= f(e)^2,$$

$$= 0 .$$

Hence

$$f(x) = 0 = f(e) .$$

That is (3.4.1) holds.

To show (3.4.2) we assume that $g(x) = -g(e)$. From (3.3.2) and (3.3.3) we find that

$$f(x)f(e) = g(x) - g(x)g(e)$$

$$= -g(e) + g(e)^2$$

$$= -f(e)^2.$$

Thus

$$f(e)\left[f(x) + f(e)\right] = 0.$$

If $f(e) \neq 0$, then $f(x) = - f(e)$. Suppose that $f(e) = 0$. It can be verified in the same way as Case II that $f(x)^2 = 0$.

Hence

$$f(x) = 0 = -f(e).$$

That is (3.4.2) holds.

From (3.3.3) we find that

$$f(x)f(e) = g(x) - g(x)g(e),$$

$$= g(x)\left[1 - g(e)\right].$$

Thus

$$f(x)^2f(e)^2 = g(x)^2\left[1 - g(e)\right]^2,$$

$$0 = g(x)^2\left[1 - g(e)\right]^2 - f(x)^2f(e)^2.$$

Using (3.3.1) and (3.3.2) we obtain

$$0 = g(x)^2\left[1 - g(e)\right]^2 - \left[g(e)-g(x)^2\right]\left[g(e)-g(e)^2\right]$$

$$= \left[1 - g(e)\right]\left[g(x)^2 - g(x)^2g(e) - g(e)^2 + g(x)^2g(e)\right]$$

$$= \left[1 - g(e)\right]\left[g(x)^2 - g(e)\right]$$

That is (3.4.3) holds.

<u>Theorem 3.5</u> Let G be an arbitrary group, F be an arbitrary field. The only solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that

(Z) $$g(e) = 0,$$

is (f,g) such that f and g are identically zero.

<u>Proof</u> It is clear that if f, g are identically zero, then f, g satisfy (A) and (Z).

Suppose that (f, g) is a solution of (A) on G to F such that g satisfies (Z). By (3.3.2) we see that

$$f(e)^2 = g(e) - g(e)^2 = 0.$$

Hence $f(e) = 0.$

Using (3.3.3) we have

$$g(x) = g(x)g(e) + f(x)f(e) = 0.$$

Thus $g(x) = 0$ for all x in G .

It follows from (3.3.1) that

$$f(x)^2 = g(e) - g(x)^2 = 0.$$

Thus $f(x) = 0$ for all x in G.

Hence f and g are identically zero.

<u>Lemma 3.6</u> Let f, g be any functions from an abelian group G into an arbitrary field F such that (f, g) is a solution of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y).$$

Then the period H of g is a subgroup of G. If $g(e) \neq 1$, then H is either all of G or is of index 2 in G. In the case where H is of index 2 we have $g(x) = -g(e)$ and $f(x) = -f(e)$ for any $x \notin H$.

<u>Proof</u> Since e belongs to H, hence H is not empty. Let x, y be any elements of H. Therefore $g(x) = g(e) = g(y)$. Hence, by (3.4.1), we have

$f(x) = f(e) = f(y)$. Thus,

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y),$$

$$= g(e)^2 + f(e)^2$$

$$= g(e).$$

The last equality follows from (3.3.2). Therefore xoy^{-1} belongs to H.

Hence H is a subgroup of G.

Suppose that $g(e) \neq 1$. By (3.4.3) we have $g(x)^2 - g(e)^2 = 0$. This implies that $g(x)^2 = g(e)^2$. Hence $g(x) = \pm\ g(e)$. Thus,

(3.6.1) $\{x : g(x) = \pm g(e)\}$ is the group G.

Suppose that H is not all of G. Let xH, yH be any cosets of H in G such that $xH \neq H \neq yH$. Hence x, y do not belong to H. Therefore $g(x) = -g(e) = g(y)$. Hence, by (3.4.2) it follows that $f(x) = -f(e) = f(y)$. Thus

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y),$$

$$= (-g(e))^2 + (-f(e))^2$$

$$= g(e).$$

Again, the last equality follows from (3.3.2). Therefore xoy^{-1}

belongs to H. Hence x, y belong to the same coset. That is $xH = yH$, which implies that H is of index 2 in G.

To prove the last argument, assume that H is of index 2 in G. By (3.6.1) we have $g(x) = -g(e)$, for any $x \notin H$. It follows from (3.4.2) that $f(x) = -f(e)$, for any $x \notin H$.

Lemma 3.7 Let f, g be functions from an abelian group G into a field F of characteristic different from 2. If (f, g) is a solution of

$$\text{(A)} \qquad g(x \circ y^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F, then f and g are constant on each coset of the period of g.

Proof Let H be the period of g. By (3.4.1) we have $f(x) = f(e)$, for any x in H. By Definition 3.2, we have $g(x) = g(e)$, for any $x \in H$. Hence, f and g are constant on H.

Let xH be any left coset of H, and y be any element of xH. Therefore, $y = x \circ h$ for some h in H. Thus $x = y \circ h^{-1}$ and

$$\begin{aligned} g(x) &= g(y \circ h^{-1}) \\ &= g(y)g(h) + f(y)f(h) \\ &= g(y)g(e) + f(y)f(e) \end{aligned}$$

From (3.3.3) we have $g(y) = g(y)g(e) + f(y)f(e)$. Hence

$$g(x) = g(y). \tag{3.7.1}$$

That is, g is constant on each coset of H. It follows from (3.3.3) that

$$\begin{aligned} f(x)\,f(e) &= g(x) - g(x)g(e) \\ &= g(y) - g(y)g(e) \\ &= f(y)f(e). \end{aligned}$$

Case I. Assume that $f(e) \neq 0$. Hence, it follows that $f(x) = f(y)$.

Case II. Assume that $f(e) = 0$. Using (3.3.1) and (3.7.1) we have

$$\begin{aligned} f(x)^2 &= g(e) - g(x)^2 \\ &= g(e) - g(y)^2 \\ &= f(y)^2. \end{aligned}$$

Therefore $f(x) = f(y)$ or $f(x) = -f(y)$.

Suppose that $f(x) \neq f(y)$. Hence $f(x) = -f(y)$. Observe that if $f(y) = 0$, we have $f(x) = 0 = f(y)$. Hence $f(y) \neq 0$. By (A) and (3.7.1) we have

$$\begin{aligned} g(xoy^{-1}) &= g(x)g(y) + f(x)f(y), \\ &= g(y)^2 - f(y)^2. \end{aligned}$$

But,

$$g(xoy^{-1}) = g(xo(xoh)^{-1})$$
$$= g(h)$$
$$= g(e)$$
$$= g(y)^2 + f(y)^2.$$

The last equality follows from (3.3.1). Thus

$$g(y)^2 - f(y)^2 = g(y)^2 + f(y)^2$$
$$2f(y)^2 = 0 .$$

Hence $f(y) = 0$, we have a contradiction. Therefore

$$f(x) = f(y).$$

Hence f is constant on each coset of H.

Theorem 3.8 Let G be an abelian group, F be a field of characteristic different from 2. Then the solutions of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F are those and only those (f, g) of the form

$$f(x) = f_o(xH) , \quad g(x) = g_o(xH)$$

for all x in G where H is a subgroup of G and (f_o, g_o) is a solution of

$$(A_o) \qquad g_o(XoY^{-1}) = g_o(X)g_o(Y) + f_o(X)f_o(Y)$$

on G/H to F such that g_o satisfies

$$(C_o) \qquad g_o(X) \neq g_o(H)$$

for any X in G/H such that $X \neq H$.

<u>Proof</u> Assume that (f, g) is a solution of (A) on G to F. Let H be the period of g. By Lemma 3.6, H is a normal subgroup of G.

Defined $f_o, g_o : G/H \longrightarrow F$ by

$$g_o(xH) = g(x), \quad f_o(xH) = f(x).$$

By Lemma 3.7 we see that f_o, g_o are well defined.

Note that, for any xH, yH in G/H, we have

$$\begin{aligned} g_o(xHo(yH)^{-1}) &= g_o(xHoy^{-1}H), \\ &= g_o(xoy^{-1}H), \\ &= g(xoy^{-1}), \end{aligned}$$

and

$$g_o(xH)g_o(yH) + f_o(xH)f_o(yH) = g(x)g(y) + f(x)f(y) = g(xoy^{-1}).$$

Thus

$$g_o(xHo(yH)^{-1}) = g_o(xH)g_o(yH) + f_o(xH)f_o(yH).$$

Hence (f_o, g_o) is a solution of (A_o) on G/H to F.

Let x be any element of G such that $x \notin H$. Therefore $g(x) \neq g(e)$. Since $g(x) = g_o(xH)$ and $g(e) = g_o(H)$, hence

$$g_o(xH) \neq g_o(H),$$

i.e. g_o satisfies (C_o).

Conversely, suppose that (f_o, g_o) is a solution of (A_o) on G/H to F such that g_o satisfies (C_o) and $f_o(xH) = f(x)$, $g_o(xH) = g(x)$ where H is a subgroup of G. Hence

$$\begin{aligned} g(xoy^{-1}) &= g_o((xoy^{-1})H) \\ &= g_o(xHoy^{-1}H) \\ &= g_o(xH)g_o(yH) + f_o(xH)f_o(yH) \\ &= g(x)g(y) + f(x)f(y). \end{aligned}$$

Therefore (f, g) is a solution of (A).

<u>Remark 3.9</u> By Theorem 3.8, we see that to determine all solutions of (A) on G to F, we need to determine the various subgroups H of G and for each H we need only to determine all solutions (f_o, g_o) of (A_o) on the quotient group G/H to F such that g_o satisfies (C_o). Hence it is sufficient to look for any solution of

$$\text{(A)} \qquad g_o(xoy^{-1}) = g_o(x)g_o(y) + f_o(x)f_o(y)$$

on any group $\bar{G}$ such that g_o satisfies

(C) $$g_o(x) \neq g_o(e),$$

for any x in $\bar{G}$ such that $x \neq e$.

<u>Lemma 3.10</u> Let f, g be functions from an abelian group G into a field F of characteristic different from 2. If (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F then f satisfies either

(D) $$f(x^{-1}) = - f(x)$$

for all x in G ; or

(D*) $$f(x^{-1}) = f(x)$$

for all x in G

<u>Proof</u> Suppose that the lemma is not true . Hence there exist x_1, y_1 in G such that $f(x_1^{-1}) \neq -f(x_1)$ and $f(y_1^{-1}) \neq f(y_1)$. By (3.3.5) we have $f(x_1^{-1}) = f(x_1)$ and $f(y_1^{-1}) = -f(y_1)$. Note that if $f(x_1) = 0$, then we must have $f(x_1^{-1}) = 0$, which implies that $f(x_1^{-1}) = -f(x_1)$. Hence $f(x_1) \neq 0$. By a similar argument we have $f(y_1) \neq 0$.

Therefore

$$g(x_1oy_1^{-1}) = g(x_1)g(y_1) + f(x_1)f(y_1),$$

and

$$g(y_1^{-1} o x_1) = g(y_1^{-1} o (x_1^{-1})^{-1})$$

$$= g(y_1^{-1})g(x_1^{-1}) + f(y_1^{-1})f(x_1^{-1})$$

$$= g(x_1)g(y_1) - f(x_1)f(y_1).$$

The last equality follows from (3.3.4). Since G is abelian, hence

$$g(x_1)g(y_1) + f(x_1)f(y_1) = g(x_1)g(y_1) - f(x_1)f(y_1),$$

$$2f(x_1)f(y_1) = 0$$

Therefore $f(x_1) = 0$ or $f(y_1) = 0$, which is a contradiction.

Hence the lemma is true.

<u>Lemma 3.11</u> Let f, g be functions from an arbitrary group G into an arbitrary field F. If (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that g satisfies

(B) $$g(e) \neq 1,$$

then f satisfies

(D^*) $$f(x^{-1}) = f(x),$$

for all x in G.

Proof   Note that if $g(e) = 0$, then by Theorem 3.5 we have $f(x) = 0$ for all x in G. Hence $f(x) = 0 = f(x^{-1})$ for all x in G. Suppose that $g(e) \neq 0$. From (3.3.2) and (B) we see that

$$f(e) \neq 0.$$

By (3.3.3) we conclude that

$$f(x) = g(x)\left[\frac{1 - g(e)}{f(e)}\right],$$

and

$$f(x^{-1}) = g(x^{-1})\left[\frac{1 - g(e)}{f(e)}\right].$$

By (3.3.4) we have $g(x^{-1}) = g(x)$ for all x in G. Hence, it follows that $f(x^{-1}) = f(x)$ for all x in G, i.e. f satisfies (D^*).

Lemma 3.12   Let f, g be functions from an abelian group G into a field F of characteristic different from 2. If (f, g) is a solution of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F then f, g must satisfies one of the followings :

(3.12.1) $\quad g(e) = 1$ and $f(x^{-1}) = -f(x)$ for all x in G ;

(3.12.2) $\quad g(e) = 1$ and $f(x^{-1}) = f(x)$ for all x in G ;

(3.12.3) $\quad g(e) \neq 1$.

Proof   Assume that (3.12.3) does not hold. Therefore

$g(e) = 1$, hence by Lemma 3.10 we have $f(x^{-1}) = - f(x)$ for all x in G or $f(x^{-1}) = f(x)$ for all x in G, i.e.(3.12.1)or (3.12.2) hold.

Clearly if (3.12.3) hold, then (3.12.1) and (3.12.2) cannot both hold. In this case, by Lemma 3.11, we have $f(x^{-1}) = f(x)$ for all x in G.

It is convenient to classify the solutions of (A) according to the conditions of Lemma 3.12. This is done in the following definition.

Definition 3.13 Let f, g be functions from an abelian group G into a field F of characteristic different from 2 such that (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G. to F.

We say that (f,g) is of class GF(I) if f, g satisfy the following additional conditions :

(B^*) $$g(e) = 1, \text{ and}$$

(D) $$f(x^{-1}) = -f(x) \quad \text{for all } x \text{ in } G.$$

We say that (f, g) is of Class GF(II) if f, g satisfy the following additional conditions :

(B^*) $$g(e) = 1, \text{ and}$$

(D^*) $\qquad f(x^{-1}) = f(x) \qquad$ for all x in G.

We say that (f, g) is of Class GF(III) if f, g satisfy the following additional condition :

(B) $\qquad g(e) \neq 1.$

Remark 3.14 It follows from Lemma 3.12 that any solution of (A) on G to F must be of Class GF(I) or Class GF(II) or Class GF(III). Hence to determine all solutions (f, g) of (A), we need only to determine (f, g) of Class GF(I), Class GF(II) and Class GF(III) . Note that if $g(e) = 0$, then (f, g) is of Class GF(III). It follows from Theorem 3.5 that f and g are identically zero. This solution will be called the trivial solution. If f or g is not identically zero, then (f, g) will be called a non-trivial solution. The functional equation (A) always has a solution on any group G, namely, the trivial solution.

Theorem 3.15 Let G be an arbitrary group, F be an arbitrary field. Then the solutions (f, g) of

(A) $\qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$

on G to F such that g satisfies

(B) $\qquad g(e) \neq 1,$

are those and only those (f, g) of the forms :

(3.15.1) $f(x) = b$, $g(x) = a$ for all x in G where a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$(3.15.2) \quad f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$.

Proof Assume that (f,g) is a solution of (A) on G to F such that g satisfies (B). Let $g(e) = a$, $f(e) = b$, and H be the period of g. It follows from (B) that $a = g(e) \neq 1$, and follows from (3.3.2) that

$$\begin{aligned} a - a^2 &= g(e) - g(e)^2 \\ &= f(e)^2 \\ &= b^2. \end{aligned}$$

Hence $a - a^2 = b^2$. By Lemma 3.6, H is either all of G or is of index 2 in G.

Case I. Assume that H is the group G. Thus $g(x) = g(e) = a$ for all x in G. By (3.4.1) we have $f(x) = f(e) = b$ for all x in G. That is, f, g are of the form(3.15.1).

Case II. Assume that H is of index 2 in G. Let x be any element of G. Suppose that $x \in H$. We have $g(x) = g(e) = a$ for all x in H. By (3.4.1) we get $f(x) = f(e) = b$ for all x in H.

Suppose that $x \notin H$. By Lemma 3.6 we have $g(x) = -g(e) = -a$. From (3.4.2) we see that $f(x) = -f(e) = -b$. Hence

$$f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

i.e. f, g are of the form (3.15.2).

Conversely, assume that f, g are of the forms (3.15.1) or (3.15.2)

<u>Case I</u>. Suppose that f, g are of the form (3.15.1), i.e. $f(x) = b$, $g(x) = a$ for all x in G where a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$.

Let x, y be any elements of G. We have $g(xoy^{-1}) = a$, and

$$g(x)g(y) + f(x)f(y) = a^2 + b^2$$
$$= a.$$

Hence

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

for all x, y in G, i.e. (f, g) is a solution of (A).

<u>Case II</u>. Assume that f, g are of the form (3.15.2), that is

$$f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$.

First, we assume that $x, y \in H$. Hence we have $xoy^{-1} \in H$. Therefore $f(x) = b = f(y)$, $g(x) = a = g(y)$, and $g(xoy^{-1}) = a$. But,

$$g(x)g(y) + f(x)f(y) = a^2 + b^2 = a.$$

Hence,

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y).$$

Next, we assume that $x, y \notin H$. We have $g(x) = -a = g(y)$, and $f(x) = -b = f(y)$. Therefore,

$$\begin{aligned} g(x)g(y) + f(x)f(y) &= (-a)(-a) + (-b)(-b) \\ &= a^2 + b^2 \\ &= a. \end{aligned}$$

Since H is of index 2 in G, hence $xoy^{-1} \in H$, therefore

$$g(xoy^{-1}) = a.$$

Hence

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y).$$

Note that if $x \notin H$ and $y \in H$, then $yox^{-1} \notin H$. Thus $xoy^{-1} = (yox^{-1})^{-1} \notin H$. Hence $g(yox^{-1}) = -a = g(xoy^{-1})$. Therefore, it remains to be considered only the case $x \in H$ and $y \notin H$. In this case we have $xoy^{-1} \notin H$, hence $g(x) = a$, $g(y) = -a$, $f(x) = b$, $f(y) = -b$ and $g(xoy^{-1}) = -a$.

But $$g(x)g(y) + f(x)f(y) = a(-a) + b(-b)$$
$$= -a^2 - b^2$$
$$= -a.$$

Hence

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y).$$

Therefore

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y),$$

for all x,y in G, i.e. (f, g) is a solution of (A).

Note that $g(e) = a \neq 1$, hence (B) holds.

Remark 3.16 Let G be an abelian group, F be a field of characteristic different from 2. It follows from Theorem 3.15 that the solutions of (A) of Class GF(III) are those and only those (f,g) of the forms :

(3.16.1) $f(x) = b$, $g(x) = a$ for all x in G where a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

(3.16.2) $$f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a, b are elements of F such that $a \neq 1$, $a - a^2 = b^2$.

Lemma 3.17 Let f, g be any functions from an arbitrary group G into an arbitrary field F. If (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that g satisfies

(B^*) $$g(e) = 1,$$

then the followings hold ;

(3.17.1) if $f(x) = 0$ then $f(x^{-1}) = 0$,

(3.17.2) if $f(x) = 0 = f(y)$, then $f(xoy^{-1}) = 0$

Proof Using (3.3.1), (3.3.4) and (B^*) we have

$$f(x)^2 = 1 - g(x)^2$$

and

$$f(x^{-1})^2 = 1 - g(x^{-1})^2,$$

$$= 1 - g(x)^2,$$

hence,

(3.17.3) $$f(x^{-1})^2 = f(x)^2.$$

It follows from (3.17.3) that if $f(x) = 0$ then $f(x^{-1}) = 0$, i.e. (3.17.1) holds.

To prove (3.17.2), suppose that $f(x) = 0 = f(y)$. By (3.3.1) and (B^*) we have $g(x)^2 = 1$ and $g(y)^2 = 1$. Thus $\left[g(x)g(y)\right]^2 = 1$. Note that

$$g(xoy^{-1}) = g(x)g(y) + f(x)f(y) = g(x)g(y).$$

Using (3.3.1), we have

$$\begin{aligned} f(xoy^{-1})^2 &= 1 - g(xoy^{-1})^2 \\ &= 1 - \left[g(x)g(y)\right]^2 \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Hence $f(xoy^{-1}) = 0$, i.e. (3.17.2) holds.

Lemma 3.18 Let f, g be functions from an abelian group G into an arbitrary field F. If (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that f, g satisfy

(B^*) $$g(e) = 1,$$

(D) $$f(x^{-1}) = -f(x),$$

for all x in G, then (f, g) is a solution of

(E) $$f(xoy) = f(x)g(y) + g(x)f(y),$$

i.e. (E) holds for all x, y in G.

Proof Replacing xoy for y in (A) and using (3.3.4) and (D) we find that

$$g(xo(xoy)^{-1}) = g(x)g(xoy) + f(x)f(xoy),$$

$$= g(x)\left[g(x)g(y^{-1}) + f(x)f(y^{-1})\right] + f(x)f(xoy).$$

Since G is abelian, we see that

$$g(xo(xoy)^{-1}) = g(y^{-1}) = g(y) .$$

Therefore,

$$g(y) = g(x)\left[g(x)g(y) - f(x)f(y)\right] + f(x)f(xoy).$$

Hence,

$$f(x)f(xoy) = g(y) - g(x)^2g(y) + g(x)f(x)f(y),$$

$$= \left[1 - g(x)^2\right]g(y) + g(x)f(x)f(y).$$

Consequently, using (3.3.1) and (B^*) we obtain

$$f(x)f(xoy) = f(x)^2g(y) + g(x)f(x)f(y),$$

$$= f(x)\left[f(x)g(y) + g(x)f(y)\right].$$

Hence, if $f(x) \neq 0$ we have

$$f(xoy) = f(x)g(y) + g(x)f(y) .$$

In the case where $f(x) = 0$ and $f(y) \neq 0$, replacing xoy for x in (A), we can verify in the same way as above that

$$f(xoy) = f(x)g(y) + g(x)f(y) .$$

If $f(x) = 0 = f(y)$, by Lemma 3.17, we have

$$f(xoy) = f(xo(y^{-1})^{-1}) = 0 = f(x)g(y) + g(x)f(y).$$

Hence (f, g) is a solution of (E).

<u>Lemma 3.19</u> Let G be an abelian group, F be a field containing an element i with the property that $i^2 = -1$ and characteristic of F is different from 2. Then the solutions (f, g) of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that f, g satisfy

(B^*) $$g(e) = 1,$$

and

(D) $$f(x^{-1}) = -f(x),$$

for all x in G, are those and only those (f, g) of the form

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into $F^* = F - \{0\}$

<u>Proof</u> Assume that (f, g) is a solution of (A) on G to F such that f, g satisfy (B^*) and (D).

Defined $h : G \longrightarrow F$ by

$$(3.19.1) \qquad h(x) = g(x) + if(x),$$

for all x in G. h is clearly well defined. Using (3.3.1), (B^*), (D) and (3.19.1) we find that

$$h(x) - g(x) = if(x),$$

$$\left[h(x) - g(x)\right]^2 = -f(x)^2,$$

$$= g(x)^2 - 1 .$$

Therefore we obtain

$$(3.19.2) \qquad h(x)^2 - 2h(x)g(x) + 1 = 0 .$$

From (3.19.2) we see that $h(x) \neq 0$ for any x in G. Hence h is a function from G into F^*. Moreover

$$g(x) = \frac{h(x)^2 + 1}{2h(x)}$$

$$= \frac{h(x) + h(x)^{-1}}{2}$$

Similarly, using (3.3.1), (B^*), (D) and (3.19.1) we find that

$$h(x) - if(x) = g(x),$$

$$\left[h(x) - if(x)\right]^2 = g(x)^2,$$

$$= 1 - f(x)^2.$$

Therefore we obtain

$$h(x)^2 - 2i\, f(x)h(x) - 1 = 0,$$

$$f(x) = \frac{h(x)^2 - 1}{2i\, h(x)}$$

$$= \frac{h(x) - h(x)^{-1}}{2i}$$

It remains only to be proved that h defined by (3.19.1) is a homomorphism, that is $h(xoy) = h(x)h(y)$ for all x, y in G. observe that for any x, y in G we have

$$h(x)h(y) = \left[g(x) + i\, f(x)\right]\left[g(y) + i\, f(y)\right]$$

$$= \left[g(x)g(y) - f(x)f(y)\right] + i\left[f(x)g(y) + g(x)f(y)\right]$$

$$= \left[g(x)g(y^{-1}) + f(x)f(y^{-1})\right] + i\, f(xoy),$$

$$= g(xoy) + i\, f(xoy)$$

$$= h(xoy).$$

Here the third equality follows from (3.3.4), (D) and Lemma 3.18; the fourth and fifth equalities follow from (A) and (3.19.1) respectively. Hence h is a homomorphism. Thus,

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

for all x in G.

Conversely, assume that $g(x) = \dfrac{h(x) + h(x^{-1})}{2}$, $f(x) = \dfrac{h(x) - h(x^{-1})}{2i}$

for all x in G where h is a homomorphism from G into F^*. Hence

$$g(x)g(y) = \frac{h(x) + h(x^{-1})}{2} \cdot \frac{h(y) + h(y^{-1})}{2}$$

$$= \frac{h(x)h(y) + h(x)\,h(y^{-1}) + h(x^{-1})h(y)+h(x^{-1})h(y^{-1})}{4}$$

and

$$f(x)f(y) = \frac{h(x) - h(x^{-1})}{2i} \cdot \frac{h(y) - h(y^{-1})}{2i}$$

$$= \frac{h(x)h(y) - h(x)h(y^{-1}) - h(x^{-1})h(y) + h(x^{-1})h(y^{-1})}{-4}$$

Thus

$$g(x)g(y)+f(x)f(y) = \frac{2\,h(x)h(y^{-1}) + 2\,h(x^{-1})h(y)}{4}$$

$$= \frac{h(x)h(y^{-1}) + h(x^{-1})h(y)}{2}$$

$$= h(xoy^{-1}) + h((xoy^{-1})^{-1})$$

$$= g(xoy^{-1}).$$

That is (A) holds. Also we have

$$g(e) = \frac{h(e) + h(e^{-1})}{2}$$

$$= h(e)$$

$$= 1 ,$$

and

$$f(x^{-1}) = \frac{h(x^{-1}) - h(x)}{2i}$$

$$= - \frac{h(x) - h(x^{-1})}{2i}$$

$$= - f(x).$$

That is, (B^*) and (D) hold .

Theorem 3.20 Let G be an abelian group, F be a field of characteristic different from 2. Then a solution of

(A) $$g(xoy^{-1}) = g(x)\, g(y) + f(x)f(y)$$

on G to F is of Class GF(I) if and only if f, g are of the form

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).

Proof Assume that (f, g) is of Class GF(I), i.e.(f, g) is a solution of (A) on G to F such that f and g satisfy

(B^*) $$g(e) = 1,$$

and

(D) $$f(x^{-1}) = -f(x).$$

for all x in G.

Case I Suppose that F contains an element i. Therefore $M(F) = F - \{0\}$. By Lemma 3.19, it follows that

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).

Case II Suppose that F contains no element i. Therefore $M(F) = \Delta(F)$. Since f(x), g(x) belong to F, hence they also belong to C(F) for all x in G. C(F) is now a field of characteristic different from 2 which contain an element i, hence by Lemma 3.19 we have

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into $C(F) - \{0\}$.

To show that h(x) belongs to M(F) for all x in G, let us recall that f(x), g(x) belong to F for all x in G, hence by Lemma 2.1.1 h(x) belongs to $\Delta(F) = M(F)$ for all x in G. Conversely, suppose that $g(x) = \frac{h(x) + h(x^{-1})}{2}$, $f(x) = \frac{h(x)-h(x^{-1})}{2i}$ where h is a homomorphism from G into M(F). It can be verified as in Lemma 3.19 that f, g satisfy (A), (B^*) and (D). In the case where $M(F) = \Delta(F)$, it follows from Lemma 2.1.1 that f(x), g(x) belong to F for all x in G. Hence (f, g) is a solution of (A) on G to F of Class GF(I).

Lemma 3.21 Let G be an abelian non-boolean group, F be a field of characteristic different from 2. Then there does not exist any (f, g) of Class GF(II) such that g satisfies

(C) $g(x) \neq g(e)$,

for any $x \neq e$.

Proof Suppose that there exists (f, g) of Class GF(II) such that g satisfies

(C) , i.e. (f, g) is a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that f, g satisfy

(B*) $$g(e) = 1,$$

(D*) $$f(x^{-1}) = f(x) \quad \text{for all } x \text{ in } G, \text{ and}$$

(C) $$g(x) \neq g(e) \quad \text{for any } x \neq e,$$

Let x be any element of G such that $x^2 \neq e$. Using (3.3.4) and (D*) we obtain

$$\begin{aligned} g(x^2) &= g(xox) \\ &= g(x)\, g(x^{-1}) + f(x)\, f(x^{-1}) \\ &= g(x)^2 + f(x)^2 \\ &= g(e). \end{aligned}$$

Thus $g(x^2) = g(e)$ where $x^2 \neq e$, which is contrary to (C). Hence there does not exist any (f,g) of Class GF(II) .

Lemma 3.22 Let G be a boolean group of order 2, F be a field of characteristic different from 2. Then a solution of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that g satisfies

(C) $$g(x) \neq g(e),$$

for any $x \neq e$, is of Class GF(II) if, and only if there exist elements c, d of F where $c \neq 1$, $c^2 + d^2 = 1$, such that

$$f(x) = \begin{cases} 0, & x = e \\ d, & x \neq e \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ c, & x \neq e . \end{cases}$$

Proof Let $f, g : G \longrightarrow F$ be such that

$$f(x) = \begin{cases} 0, & x = e \\ d, & x \neq e \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ c, & x \neq e \end{cases}$$

where c, d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$. Then it can be verified that f, g satisfy

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

for all x, y in G,

(B^*) $g(e) = 1$

(C) $g(x) \neq g(e)$ for any $x \neq e$,

and

(D^*) $f(x^{-1}) = f(x)$ for all x in G.

Hence (f, g) is of Class GF(II) such that g satisfies (C). Conversely, assume that (f, g) is of Class GF(II) such that g satisfies (C), i.e f, g satisfy (A) , (B^*), (C) and (D^*) . Let $c = g(x_o)$ and $d = f(x_o)$ where x_o is the element of G such that $x_o \neq e$. It follows from (B^*) and (C) that $g(x_o) \neq 1$, hence $c = g(x_o) \neq 1$. By (3.3.1) and (B^*) we have $g(x_o)^2 + f(x_o)^2 = 1$. Thus $c^2 + d^2 = 1$. Hence, we have

$$f(x) = \begin{cases} 0, & x = e \\ d, & x \neq e \end{cases} , \quad g(x) = \begin{cases} 1, & x = e \\ c, & x \neq e . \end{cases}$$

<u>Remark 3.23</u> Note that when G is the trivial group, i.e. G contains e alone, we see that (f, g) where $f(e) = 0$ and $g(e) = 1$, is the only solution of Class GF(II) such that g satisfies (C).

<u>Lemma 3.24</u> Let f, g be any functions from an arbitrary group G into an arbitrary field F such that (f, g) is a solution of

(A) $g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$

and g satisfies

(C) $g(x) \neq g(e)$ for any $x \neq e$.

For any x, y in G, if $f(x) = f(y)$ and $g(x) = g(y)$ then $x = y$.

<u>Proof</u> Assume that x,y are any elements of G such that $f(x) = f(y)$, and $g(x) = g(y)$. Hence

$$\begin{aligned} g(xoy^{-1}) &= g(x)g(y) + f(x)f(y) \\ &= g(x)^2 + f(x)^2 \\ &= g(e). \end{aligned}$$

It follows from (C) that $xoy^{-1} = e$. Hence $x = y$.

<u>Lemma 3.25</u> Let f, g be functions from a boolean group G of order 4 into a field F of characteristic different from 2 such that (f, g) is of Class GF(II) and g satisfies (C), then there exists an element $x \neq e$ in G such that $g(x) = -1$.

<u>Proof</u> Let G be a boolean group of order 4, say $G = \{ e, x_1, x_2, x_3 \}$ and the multiplication table of G be as follows :

•	e	x_1	x_2	x_3
e	e	x_1	x_2	x_3
x_1	x_1	e	x_3	x_2
x_2	x_2	x_3	e	x_1
x_3	x_3	x_2	x_1	e

Assume that (f, g) is of Class GF(II) such that g satisfies (C), i.e. f, g : $G \longrightarrow F$ satisfy

(A) $g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$

for all x,y in G,

(B^*) $g(e) = 1$,

(C) $g(x) \neq g(e)$ for any $x \neq e$, and

(D^*) $f(x^{-1}) = f(x)$ for all x in G.

Let $f(x_i) = b_i$, $g(x_i) = a_i$ for i = 1, 2, 3 .

It follows from (A), (B^*),(3.3.1)and the multiplication table of G that the followings hold :

(3.25.1) $a_1^2 + b_1^2 = 1$,

(3.25.2) $a_2^2 + b_2^2 = 1$,

(3.25.3) $a_3^2 + b_3^2 = 1$,

(3.25.4) $a_1a_2 + b_1b_2 = a_3$,

(3.25.5) $a_1a_3 + b_1b_3 = a_2$,

(3.25.6) $a_2a_3 + b_2b_3 = a_1$.

Suppose that there does not exist any element x of G such that $g(x) = -1$. Hence $g(x_i) = a_i \neq -1$, for $i = 1,2,3$. It follows from (B^*) and (C) that $a_i \neq 1$ for $i = 1,2,3$. Hence by (3.25.1), (3.25.2) and (3.25.3) we have $b_i \neq 0$ for $i = 1,2,3$. Using (3.25.1), (3.25.4), (3.25.5) and (3.25.6) we obtain

$$\begin{aligned} a_1 &= \left[a_1a_3 + b_1b_3\right]\left[a_1a_2 + b_1b_2\right] + b_2b_3 , \\ &= a_1^2a_2a_3 + a_1a_3b_1b_2 + a_1a_2b_1b_3 + b_1^2b_2b_3 + b_2b_3 , \\ &= (1 - b_1^2)a_2a_3 + a_1a_3b_1b_2 + a_1a_2b_1b_3 + b_1^2b_2b_3 + b_2b_3 , \\ &= a_2a_3 - b_1^2a_2a_3 + a_1a_3b_1b_2 + a_1a_2b_1b_3 + b_1^2b_2b_3 + b_2b_3 , \\ &= a_1 - b_1^2a_2a_3 + a_1a_3b_1b_2 + a_1a_2b_1b_3 + b_1^2b_2b_3 . \end{aligned}$$

Hence

$$\begin{aligned} 0 &= b_1\left[- b_1a_2a_3 + a_1a_3b_2 + a_1a_2b_3 + b_1b_2b_3\right] \\ &= b_1\left[- a_2a_3 + b_2b_3\right] + a_1\left[a_3b_2 + a_2b_3\right] \\ &= b_1\left[b_2b_3 - a_1 + b_2b_3\right] + a_1\left[a_3b_2 + a_2b_3\right] \\ &= 2\, b_1b_2b_3 - a_1b_1 + a_1\left[a_3b_2 + a_2b_3\right] . \end{aligned}$$

Therefore

$$(3.25.7) \qquad 2\, b_1b_2b_3 + a_1\left[-b_1 + a_3b_2 + a_2b_3\right] = 0 .$$

Similarly using (3.25.2), (3.25.4), (3.25.5) and (3.25.6) we have

(3.25.8) $$2b_1b_2b_3 + a_2\left[-b_2 + a_3b_1 + a_1b_3\right] = 0,$$

and using (3.25.3), (3.25.4) , (3.25.5) and (3.25.6) we have

(3.25.9) $$2b_1b_2b_3 + a_3\left[-b_3 + a_2b_1 + a_1b_2\right] = 0.$$

It follows from (3.25.7) and (3.25.8) that

$$0 = a_1\left[-b_1 + a_3b_2 + a_2b_3\right] - a_2\left[-b_2 + a_3b_1 + a_1b_3\right]$$

$$= -a_1b_1 + a_1a_3b_2 + a_1a_2b_3 + a_2b_2 - a_2a_3b_1 - a_1a_2b_3$$

$$= b_2\left[a_1a_3 + a_2\right] - b_1\left[a_1 + a_2a_3\right]$$

$$= b_2\left[a_2 - b_1b_3 + a_2\right] - b_1\left[a_1 + a_1 - b_2b_3\right]$$

$$= 2a_2b_2 - b_1b_2b_3 - 2a_1b_1 + b_1b_2b_3$$

Thus

$$a_1b_1 = a_2b_2 .$$

Similarly, it follows from (3.25.7) and (3.25.9) that

$$a_3b_3 = a_1b_1 .$$

Hence

(3.25.10) $$a_2 = \frac{a_1b_1}{b_2}, \quad a_3 = \frac{a_1b_1}{b_3} .$$

Substitute(3.25.10) in (3.25.5) we have

$$a_1 \cdot \frac{a_1b_1}{b_3} + b_1b_3 = \frac{a_1b_1}{b_2}$$

$$a_1^2b_1b_2 + b_1b_2b_3^2 = a_1b_1b_3 .$$

Therefore

$$a_1b_3 = b_2\left[a_1^2 + b_3^2\right]. \tag{3.25.11}$$

Similarly, substitute (3.25.10) in (3.25.6) we get

$$\frac{a_1b_1}{b_2} \cdot \frac{a_1b_1}{b_3} + b_2b_3 = a_1.$$

Hence,

$$a_1^2b_1^2 + b_2^2b_3^2 = a_1b_2b_3. \tag{3.25.12}$$

It follows from (3.25.11) and (3.25.12) that

$$a_1^2b_1^2 + b_2^2b_3^2 = \left[a_1^2 + b_3^2\right]b_2^2,$$

Therefore

$$a_1^2\left[b_1^2 - b_2^2\right] = 0. \tag{3.25.13}$$

From (3.25.13) we conclude that

$$a_1 = 0 \quad \text{or} \quad b_1^2 - b_2^2 = 0$$

<u>Case I</u>. Suppose that $a_1 = 0$. It follows from (3.25.1) that $b_1 = 1$ or $b_1 = -1$.

<u>Case 1</u>. Suppose that $b_1 = 1$. It follows from (3.25.4) and (3.25.5) respectively that $b_2 = a_3$ and $b_3 = a_2$. Substitute these in (3.25.6) we have

$$a_2a_3 + a_2a_3 = a_1 = 0,$$

$$2a_2a_3 = 0,$$

hence $a_2 = 0$ or $a_3 = 0$.

Case 1.1 Assume that $a_2 = 0$. It follows from (3.25.2) that $b_2 = 1$ or $b_2 = -1$.

Suppose that $b_2 = 1$. Hence $g(x_1) = a_1 = 0 = a_2 = g(x_2)$ and $f(x_1) = b_1 = 1 = b_2 = f(x_2)$. It follows from Lemma 3.24 that $x_1 = x_2$, which is a contradiction.

Suppose that $b_2 = -1$. By (3.25.4) we have

$$\begin{aligned} a_3 &= a_1a_2 + b_1b_2 \\ &= 0 + 1(-1) \\ &= -1. \end{aligned}$$

It follows from (3.25.3) that $b_3 = 0$, which is a contradiction.

Case 1.2 Assume that $a_3 = 0$. It can be verified in the same way as in Case 1.1 that $x_1 = x_3$, $b_2 = 0$, which are contradictions.

Case 2 Suppose that $b_1 = -1$. It follows from (3.25.4) and (3.25.5) respectively that $-b_2 = a_3$ and $-b_3 = a_2$. Substitute these in (3.25.6) we have

$$\begin{aligned} a_2a_3 + (-a_2)(-a_3) &= 0, \\ 2a_2a_3 &= 0. \end{aligned}$$

Hence

$$a_2 = 0 \text{ or } a_3 = 0.$$

It can be verified in the same way as in Case 1 that these lead to, $x_1 = x_2$, $x_1 = x_3$, $b_3 = 0$, $b_2 = 0$, which are contradictions.

<u>Case II</u>. Suppose that $b_1^2 - b_2^2 = 0$. That is $b_1 = b_2$ or $b_1 = -b_2$. In each case we have $a_1 = a_2$ or $a_1 = -a_2$.

<u>Case 1</u>. Assume that $b_1 = b_2$, $a_1 = a_2$. Hence $f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$. It follows from Lemma 3.24 that $x_1 = x_2$, which is a contradiction.

<u>Case 2</u>. Assume that $b_1 = b_2$, $a_1 = -a_2$. From (3.25.5) we have

$$-a_2a_3 + b_2b_3 = a_2.$$

It follows from this and (3.25.6) that

$$\begin{aligned} 2b_2b_3 &= a_1 + a_2 \\ &= -a_2 + a_2 \\ &= 0. \end{aligned}$$

Hence

$$b_2b_3 = 0.$$

Therefore $b_2 = 0$ or $b_3 = 0$, which is a contradiction.

<u>Case 3</u> Assume that $b_1 = -b_2$, $a_1 = a_2$. It can be verified in the same way as in Case 2 that $b_2b_3 = 0$, which implies that $b_2 = 0$ or $b_3 = 0$. We again have a contradiction.

Case 4 Assume that $b_1 = -b_2$, $a_1 = -a_2$. From (3.25.4) we have

$$
\begin{aligned}
a_3 &= a_1a_2 + b_1b_2 \\
&= -a_1^2 - b_1^2 \\
&= -(a_1^2 + b_1^2) \\
&= -1.
\end{aligned}
$$

Hence $b_3 = 0$, which is a contradiction. Therefore there exists x in G such that $g(x) = -1$. It follows from (C) that $x \neq e$.

Lemma 3.26 Let $G = \{e, x_1, x_2, x_3,\}$ be a boolean group of order 4, F be a field of characteristic different from 2. Then a solutions (f, g) of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that g satisfies

$$\text{(C)} \qquad g(x) \neq g(e),$$

for any $x \neq e$, is of Class GF(II) if, and only if there exist elements c, d of F where $c \neq \pm 1$, $c^2 + d^2 = 1$, such that

$$
f(x) = \begin{cases} 0, & x = e \text{ or } x_1 \\ d, & x = x_2 \\ -d, & x = x_3 \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ -1, & x = x_1 \\ c, & x = x_2 \\ -c, & x = x_3. \end{cases}
$$

<u>Proof</u> Assume that (f, g) is of Class GF(II) such that g satisfies (C), i.e. f,g satisfy

(A) $g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$

for all x,y in G,

(B^*) $g(e) = 1$,

(C) $g(x) \neq g(e)$ for any $x \neq e$, and

(D^*) $f(x^{-1}) = f(x)$ for all x in G.

By Lemma 3.25 , there exists $x \neq e$ in G such that $g(x) = -1$. With out loss of generality, we may assume that $g(x_1) = -1$. Hence $f(x_1) = 0$. It follows from (A) and the multiplication table of G that

$$g(x_1)g(x_2) + f(x_1)f(x_2) = g(x_1 o x_2^{-1}) = g(x_3).$$

Therefore

$$(3.26.1) \qquad -g(x_2) = g(x_3).$$

Suppose that $f(x_2) = 0$. Hence, by (3.3.1) and (B^*), we have $g(x_2) = 1$ or $g(x_2) = -1$, Since $g(e) = 1$, it follows from (C) that $g(x_2) \neq 1$. Hence $g(x_2) = -1 = g(x_1)$ and $f(x_2) = 0 = f(x_1)$. It follows from Lemma 3.24 that $x_1 = x_2$, which is a contradiction. Hence $f(x_2) \neq 0$. From (A) and the multiplication table of G we obtain

(3.26.2) $$g(x_2)g(x_3) + f(x_2)f(x_3) = g(x_1).$$

It follows from (3.26.1) and (3.26.2) that

$$-g(x_2)^2 + f(x_2)f(x_3) = -1,$$

$$f(x_2)f(x_3) = -\left[1 - g(x_2)^2\right]$$

$$= -f(x_2)^2.$$

The last equality follows from (3.3.1) and (B^*) . Hence,

$$f(x_2)f(x_3) = -f(x_2)^2.$$

Since $f(x_2) \neq 0$, we see that $f(x_3) = -f(x_2)$.

Let $c = g(x_2)$, $d = f(x_2)$. We have $d \neq 0$, hence $c \neq \pm 1$. It follows from (3.3.1) and (B^*) that $c^2 + d^2 = 1$.

Hence we have

$$f(x) = \begin{cases} 0, & x = e \text{ or } x_1 \\ d, & x = x_2 \\ -d, & x = x_3 \end{cases}, \qquad g(x) = \begin{cases} 1, & x = e \\ -1, & x = x_1 \\ c, & x = x_2 \\ -c, & x = x_3 \end{cases}$$

where $c \neq \pm 1$ and $c^2 + d^2 = 1$.

Lemma 3.27 Let G be a boolean group of order greater than 4, F be a field of characteristic different from 2. Then there does not exist any (f,g) of Class GF(II) such that g satisfies (C).

<u>Proof</u> Suppose that there exists (f, g) of Class GF(II) such that g satisfies (C), i.e. f,g satisfy

(A) $g(xoy^{-1}) \quad = \quad g(x)g(y) + f(x)f(y)$

for all x,y in G,

(B^*) $g(e) \quad = \quad 1,$

(C) $g(x) \quad \neq \quad g(e)$ for any $x \neq e$; and

(D^*) $f(x^{-1}) \quad = \quad f(x)$ for all x in G.

Let x_1, x_2 be any elements of G such that they are distinct and different from the element e. Since G is a boolean group, hence $H_1 = \{e, x_1, x_2, x_1x_2\}$ is a subgroup of G. By Lemma 3.25, there exists $x \neq e$ in H_1 such that $g(x) = -1$. We shall assume that $g(x_1) = -1$, Since $|G| > 4$, there exists x_3 in G such that $x_3 \notin H_1$. Hence $H_2 = \{e, x_2, x_3, x_2x_3\}$ is also a subgroup of G. By Lemma 3.25 there exists x in H_2 such that $g(x) = -1$. It follows from Lemma 3.26 that for all $x \neq x_1$ in H_1, $g(x) \neq -1$. Since $x_3 \notin H_1$, hence $x_2x_3 \notin H_1$. Let $z = x_3$ or x_2x_3 . Hence $g(z) = -1 = g(x_1)$ and $f(z) = 0 = f(x_1)$. It follows from Lemma 3.24 that $z = x_1$, which is a contradiction. Hence, there does not exist any (f,g) of Class GF(II) such that g satisfies (C).

Lemma 3.28 Let G be a boolean group, F be a field of characteristic different from 2. Then a solution (f,g) of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F such that g satisfies

$$\text{(C)} \qquad g(x) \neq g(e),$$

for any $x \neq e$, is of Class GF(II) if, and only if f,g are of the forms :

$$(3.28.1) \qquad f(x) = \begin{cases} 0, & x = e \\ d, & x \neq e \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ c, & x \neq e \end{cases}$$

where c, d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$, if $|G| = 1$ or 2 ; or

$$(3.28.2) \qquad f(x) = \begin{cases} 0, & x = e \text{ or } x_1 \\ d, & x = x_2 \\ -d, & x = x_3 \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ -1, & x = x_1 \\ c, & x = x_2 \\ -c, & x = x_3 \end{cases}$$

where c, d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$, if $|G| = 4$, in other word G is the Klein four group.

Proof Assume that $f,g : G \longrightarrow F$ are of the form (3.28.1) or (3.28.2). Then it can be verified that (f,g) is of Class GF (II) such that g satisfies (C).

Conversely, assume that (f,g) is a solution of (A) such that (f,g) is of Class GF(II) and g satisfies (C). We shall determine (f,g) according to the order of G.

Case I Assume that G is trivial, i.e. G contains e alone. By Remark 3.23, we see that (f,g) where $f(e) = 0$, $g(e) = 1$, is the only solution of Class GF(II) such that g satisfies (C). Hence f,g are of the form (3.28.1).

Case II Assume that G is of order 2. It follows from Lemma 3.22 that any (f,g) of Class GF(II) such that g satisfies (C) are of the form :

$$(3.28.1) \qquad f(x) = \begin{cases} 0, & x = e \\ d, & x \neq e \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ c, & x \neq e \end{cases}$$

where c, d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$.

Case III Assume that G is of order 4, i.e. G is the Klein four group, say $G = \{e, x_1, x_2, x_3\}$. It follows from Lemma 3.25 that (f,g) of Class GF(II) such that g satisfies (C) are of the form :

$$(3.28.2) \qquad f(x) = \begin{cases} 0, & x = e \text{ or } x_1 \\ d, & x = x_2 \\ -d, & x = x_3 \end{cases}, \quad g(x) = \begin{cases} 1, & x = e \\ -1, & x = x_1 \\ c, & x = x_2 \\ -c, & x = x_3 \end{cases}$$

where c, d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

Case IV Assume that G is of order greater than 4.

It follows from Lemma 3.27 that there does not exist any (f,g) of Class GF(II) such that g satisfies (C)

Hence, by combining the whole cases we have that f,g are of the forms (3.28.1) or (3.28.2).

Theorem 3.29 Let G be an abelian group, F be a field of characteristic different from 2. Then a solution (f,g) of

$$\text{(A)} \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F is of Class GF(II) if and only if f,g are of the forms :

$$\text{(3.29.1)} \qquad f(x) = \begin{cases} 0, & x \in H \\ d, & x \notin H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ c, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$\text{(3.29.2)} \qquad f(x) = \begin{cases} 0, & x \in H \text{ or } x_1H \\ d, & x \in x_2H \\ -d, & x \in x_3H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that $G/_H$ is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

<u>Proof</u> Assume that $f, g : G \longrightarrow F$ are of the forms (3.29.1) or (3.29.2). Since H is a subgroup of G, hence we can write $f_o, g_o : G/_H \longrightarrow F$ according to (3.29.1) or (3.29.2) as follows :

$$(3.29.3) \qquad f_o(x) = \begin{cases} 0, & x = H \\ d, & x \neq H \end{cases}, \quad g_o(x) = \begin{cases} 1, & x = H \\ c, & x \neq H \end{cases}$$

where H is a subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$(3.29.4) \qquad f_o(x) = \begin{cases} 0, & x = H \text{ or } x_1H \\ d, & x = x_2H \\ -d, & x = x_3H \end{cases}, \quad g_o(x) = \begin{cases} 1, & x = H \\ -1, & x = x_1H \\ c, & x = x_2H \\ -c, & x = x_3H \end{cases}$$

where H is a subgroups of index 4 in G such that $G/_H$ is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$. By Lemma 3.28, we see that (3.29.3) and (3.29.4) are the solutions of

$$(A_o) \qquad g_o(X \circ Y^{-1}) = g_o(X)g_o(Y) + f_o(X)f_o(Y)$$

on $G/_H$ to F such that g_o satisfies

$$(C_o) \qquad g_o(X) \neq g_o(H)$$

for any $X \neq H$. Therefore, it follows from Theorem 3.8 that (f,g) where $f(x) = f_o(xH)$, $g(x) = g_o(xH)$, is a solution of (A) . Hence (f,g) of the forms (3.29.1) or (3.29.2) is a solution of (A) of Class GF (II). Conversely, assume that (f,g) is of Class GF(II). By Theorem 3.8, we have $f(x) = f_o(xH)$, $g(x) = g_o(xH)$ where H is a subgroup of G and (f_o,g_o) is a solution of

$$(A_o) \qquad g_o(X \circ Y^{-1}) \quad = \quad g_o(X)\, g_o(Y) + f_o(X) f_o(Y)$$

on $G/_H$ to F such that g_o satisfies

$$(C_o) \qquad g_o(X) \quad \neq \quad g_o(H) \qquad \text{for any } X \neq H.$$

Observe that (f_o,g_o) is of Class GF(II) such that g_o satisfies (C_o).

Suppose that the quotient group $G/_H$ is not a boolean group, hence $G/_H$ is a non-boolean abelian group. By Lemma 3.21, there does not exist any (f_o,g_o) of Class GF(II) such that g_o satisfies (C_o), we have a contradiction. Hence $G/_H$ is the boolean group. It follows from Lemma 3.28 that f_o,g_o are of the forms (3.29.3) or (3.29.4). By Theorem 3.8 it follows that f,g are of the forms (3.29.1) or (3.29.2).

Theorem 3.30 Let G be an abelian group, F be a field of characteristic different from 2. Then the solutions of

$$\text{(A)} \qquad g(x \circ y^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F are those and only those (f,g) of the forms :

(3.30.1) $f(x) = b$, $g(x) = a$ for all x in G, where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$\text{(3.30.2)} \qquad f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases} , \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$; or

$$\text{(3.30.3)} \qquad f(x) = \begin{cases} 0, & x \in H \\ d, & x \notin H \end{cases} , \quad g(x) = \begin{cases} 1, & x \in H \\ c, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$\text{(3.30.4)} \qquad f(x) = \begin{cases} 0, & x \in H \text{ or } x_1H \\ d, & x \in x_2H \\ -d, & x \in x_3H \end{cases} , \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that $G/_H$ is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$\text{(3.30.5)} \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i} , \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).

<u>Proof</u> Assume that (f,g) is a solution of (A) on G. By Remark 3.14, (f,g) must be of Class GF(I) or Class GF(II) or Class GF(III).

By Theorem 3.20, we see that (f,g) is of Class GF(I) if and only if there exists a homomorphism h from G into $M(F)$ such that

$$(3.30.5) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i} , \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

for all x in G.

By Theorem 3.29, we see that (f,g) is of Class GF(II) if and only if f,g are of the forms:

$$(3.30.3) \qquad f(x) = \begin{cases} 0 , & x \in H \\ d , & x \notin H \end{cases} , \quad g(x) = \begin{cases} 1 , & x \in H \\ c , & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and c,d are elements of F such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$(3.30.4) \qquad f(x) = \begin{cases} 0 , & x \in H \text{ or } x_1H \\ d , & x \in x_2H \\ -d , & x \in x_3H \end{cases} , \quad g(x) = \begin{cases} 1 , & x \in H \\ -1 , & x \in x_1H \\ c , & x \in x_2H \\ -c , & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that $G/_H$ is the Klein four group and c,d are elements of F such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

By Remark 3.16, we see that (f,g) is of Class GF(III) if and only if f,g are of the forms:

(3.30.1) $f(x) = b, \quad g(x) = a$ for all x in G where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$, or

(3.30.2) $$f(x) = \begin{cases} b, & x \in H \\ -b, & x \notin H \end{cases}, \quad g(x) = \begin{cases} a, & x \in H \\ -a, & x \notin H \end{cases}$$

where H is a subgroup of index 2 in G and a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$.

Hence (f,g) is a solution of (A) if and only if f and g are of the forms (3.30.1) or (3.30.2) or (3.30.3) or (3.30.4) or (3.30.5).

<u>Remark 3.31</u> Note that if G is any abelian group which has no subgroup of index 2, then it follows that G has no subgroup of index 4 such that $G/_H$ is the Klein four group. By Theorem 3.30, there does not exist any (f,g) of the forms (3.30.2) or (3.30.3) or (3.30.4). Hence the solutions of

(A) $$g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

on G to F are those and only those (f,g) of the forms :

(3.31.1) $f(x) = b$, $g(x) = a$ for all x in G where a,b are elements of F such that $a \neq 1$, $a - a^2 = b^2$, or

(3.31.2) $$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).