CHAPTER I

FUNDAMENTAL INVERSE SEMIGROUPS

Munn has characterized a fundamental inverse semigroup as a certain semigroup of mappings in [5]. In this chapter, we introduce his significant result . We study further about necessary and sufficient conditions of some kinds of inverse semigroups to be fundamental. An example to show that an inverse subsemigroup and a homomorphic image of a fundamental inverse semigroup need not be fundamental is given. Moreover, it is shown that a homomorphism from a fundamental inverse semigroup which is one-to-one on the set of all idempotents is an isomorphism.

A semigroup S is said to be <u>fundamental</u> if and only if the only congruence on S contained in the Green's relation $\mathcal{H}$ is the identity congruence on S.

A congruence ρ on a semigroup S is an <u>idempotent-separating congruence</u> if every ρ-class contains at most one idempotent of S.

Howie has shown in [4] that any inverse semigroup S has the maximum idempotent-separating congruence, $\mu(S)$ or μ, and

$$\mu = \{ (a,b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S) \};$$

equivalently,

$$\mu = \{ (a,b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S) \},$$

moreover, $\mu \subseteq \mathcal{H}$.

Let S be a semigroup. Any $\mathcal{H}$-class of S contains at most one idempotent [[2], Lemma 2.15]. Then any congruence on S contained in $\mathcal{H}$ is an idempotent-separating congruence.

Hence, an inverse semigroup S is fundamental if and only if the maximum idempotent-separating congruence μ of S is the identity congruence.

Let X be a set. A <u>one-to-one partial transformation</u> of X is a one-to-one map from a subset of X onto a subset of X. For a one-to-one partial transformation α of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α; respectively. Let I_X denote the set of all one-to-one partial transformations of X. If $\alpha \in I_X$ with $\Delta\alpha = \nabla\alpha = \phi$, then α is called the <u>empty transformation</u> and denoted by 0. The product on I_X is defined as follows : For $\alpha, \beta \in I_X$, let $\alpha\beta = 0$ if $\nabla\alpha \cap \Delta\beta = \phi$, otherwise, let $\alpha\beta : (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \rightarrow (\nabla\alpha \cap \Delta\beta)\beta$ be the composite map; it is clear that $\nabla(\alpha\beta) = (\nabla\alpha \cap \Delta\beta)\beta$. Then I_X is an inverse semigroup with zero and identity,

$$E(I_X) = \{\alpha \in I_X \mid \alpha \text{ is the identity map on } \Delta\alpha\},$$

and for each $\alpha \in I_X$, the inverse map of α, α^{-1}, is the inverse element of α in I_X and $\Delta(\alpha^{-1}) = \nabla(\alpha)$, $\nabla(\alpha^{-1}) = \Delta(\alpha)$ [[2]]. The inverse semigroup I_X is called <u>the symmetric inverse semigroup on the set</u> X.

Let S be a semigroup. An ideal A of S is called a <u>principal ideal</u> if and only if $A = S^1aS^1$ for some $a \in S$. Then, if E is a semilattice, then a principal ideal of E is of the form Ee for some $e \in E$.

Let E be a semilattice. The notation T_E denotes the following :

$T_E = \{\alpha \in I_E \mid \alpha$ is an isomorphism, $\Delta\alpha$ and $\nabla\alpha$ are principal ideals of $E\}$.

Then

$T_E = \{\alpha \in I_E \mid \alpha$ is an homomorphism, $\Delta\alpha = Ee$ and $\nabla\alpha = Ef$ for some $e, f \in E\}$.

Recall that the relation $\leq$ definedon a semilattice E by

$$e \leq f \quad \text{if and only if} \quad e = ef \ (= fe)$$

is the natural partial order on E.

Then for a semilattice E, for $e \in E$, Ee is a principal ideal of E having e as the maximum element, and hence for any $f \in E$, $f \in Ee$ if and only if $f \leq e$.

We give a following remark : Let $\alpha \in I_E$, $\Delta\alpha$ and $\nabla\alpha$ be ideals of E. If α is an isomorphism and $\Delta\alpha$ is a principal ideal, then $\nabla\alpha$ is also principal. A proof is given as follow : Let $e \in E$ such that $\Delta\alpha = Ee$. Then $e\alpha \in \nabla\alpha$. Let $x \in \nabla\alpha$. Then $x\alpha^{-1} \in \Delta\alpha = Ee$, so $x\alpha^{-1} \leq e$. Thus $(x\alpha^{-1})e = x\alpha^{-1}$. Since α is a homomorphism,

$$x(e\alpha) = ((x\alpha^{-1})e)\alpha = (x\alpha^{-1})\alpha = x$$

which implies $x \leq e\alpha$ and $x \in \nabla\alpha$. Therefore $\nabla\alpha = E(e\alpha)$ and so $(Ee)\alpha = E(e\alpha)$.

From the above proof, we also have the following : If $\alpha \in T_E$, $e \in \Delta\alpha$, then $(Ee)\alpha = E(e\alpha)$. Hence we have

$T_E = \{\alpha \in I_E \mid \alpha$ is a homomorphism, $\Delta\alpha = Ee$ and $\nabla\alpha = E(e\alpha)$ for some $e \in E\}$.

For any $e \in E$, let ϵ_e denote the identity map on Ee. Then

$\epsilon_e \in T_E$ for all $e \in E$.

The first proposition shows that T_E is an inverse subsemigroup of I_X. The following lemma is required first :

1.1 Lemma. Let E be a semilattice. Then the following hold :

(i) For $e,f \in E$, $Ee \cap Ef = Eef$.

(ii) For $e,f \in E$, if $e \in Ef$ then $Ee \subseteq Ef$ and $e \leq f$.

(iii) For $e,f \in E$, $Ee = Ef$ if and only if $e = f$.

Proof : To show $Ee \cap Ef = Eef$ for all $e,f \in E$, let $a \in Ee \cap Ef$. Then there exist $x,y \in E$ such that $a = xe = yf$. Thus $a = xee = ae$ and $a = yff = af$. Hence $a = aef$. Therefore $a \in Eef$, and so $Ee \cap Ef \subseteq Eef$. Now, since $Ee \subseteq E$, $(Ee)f \subseteq Ef$. Because $Ef \subseteq E$, $(Ef)e \subseteq Ee$. Therefore $Eef = Efe \subseteq Ee \cap Ef$. Hence $Ee \cap Ef = Eef$.

Next, we show that $e \in Ef$ implies $Ee \subseteq Ef$ and $e \leq f$. Assume $e \in Ef$. Let $a \in Ee$, then $a = xe$ for some $x \in E$. Since $e \in Ef$, $e = yf$ for some $y \in E$. Therefore $a = xyf \in Ef$. Hence $Ee \subseteq Ef$. Because $e \in Ef$, $e \leq f$.

Finally, we show that $Ee = Ef$ if and only if $e = f$ for all $e,f \in E$. Assume $e,f \in E$ such that $Ee = Ef$. Then $e \in Ee = Ef$, so $e \leq f$. Because $f \in Ef = Ee$, $f \leq e$. Therefore $e = f$. The converse of (iii) is trivial. #

1.2 Proposition [5]. Let E be a semilattice. Then T_E is an inverse subsemigroup of I_E and

$$E(T_E) \equiv E^* = \{ \epsilon_e \mid e \in E \} .$$

Moreover, the mapping $\psi : E \to E^*$ defined by

$$e\psi = \epsilon_e \qquad (e \in E)$$

is an onto isomorphism.

Proof : It is clear from the definition of T_E that $\alpha \in I_E$, $\alpha \in T_E$ implies $\alpha^{-1} \in T_E$.

Let $\alpha_1, \alpha_2 \in T_E$. Then $\nabla\alpha_1 = Ee_1$ and $\Delta\alpha_2 = Ee_2$ for some $e_1, e_2 \in E$. By Lemma 1.1 (i), $\nabla\alpha_1 \cap \Delta\alpha_2 \neq \phi$, $\Delta(\alpha_1\alpha_2) = (\nabla\alpha_1 \cap \Delta\alpha_2)\alpha_1^{-1}$ $= (Ee_1 \cap Ee_2)\alpha_1^{-1} = (Ee_1e_2)\alpha_1^{-1}$. Since $e_1e_2 = e_2e_1 \in \Delta\alpha_1^{-1}$, we have $\Delta(\alpha_1\alpha_2) = E(e_1e_2)\alpha_1^{-1}$, that is, $\Delta(\alpha_1\alpha_2)$ is a principal ideal generated by $(e_1e_2)\alpha_1^{-1}$. Since $e_1e_2 \in \Delta\alpha_2$, we have $\nabla(\alpha_1\alpha_2) = (\nabla\alpha_1 \cap \Delta\alpha_2)\alpha_2$ $= (Ee_1e_2)\alpha_2 = E(e_1e_2)\alpha_2$. Therefore $\alpha_1\alpha_2 \in T_E$. Hence T_E is an inverse subsemigroup of I_E.

Now, we show that the semilattice of T_E is $E^* = \{\epsilon_e \mid e \in E\}$. It is clear that $E^* \subseteq E(T_E)$. To show $E(T_E) \subseteq E^*$, let $\alpha \in E(T_E)$. Then $\alpha \in E(I_E)$. Therefore α is the identity map on a subset A of E. Since $\alpha \in T_E$, $A = Ee$, for some $e \in E$, that is, $\alpha = id_{Ee} = \epsilon_e \in E^*$. Hence $E(T_E) = E^*$.

Next, we show the mapping $\psi : E \to E^*$ defined by $e\psi = \epsilon_e$ $(e \in E)$ is an onto isomorphism. Obviously, ψ is onto. To show ψ is 1-1, let $e_1, e_2 \in E$ such that $e_1\psi = e_2\psi$. Then $\epsilon_{e_1} = \epsilon_{e_2}$. Then $\Delta\epsilon_{e_1} = \Delta\epsilon_{e_2}$ which implies $Ee_1 = Ee_2$. By Lemma 1.1 (iii), $e_1 = e_2$. We now show that ψ is a homomorphism. Let $e, f \in E$. Then $e\psi = \epsilon_e$, $f\psi = \epsilon_f$, $(ef)\psi = \epsilon_{ef}$ and $(e\psi)(f\psi) = \epsilon_e\epsilon_f$. Since ϵ_e is an identity map, $\Delta\epsilon_e\epsilon_f = (E_e \cap E_f)\epsilon_e^{-1} = E_e \cap E_f$. By Lemma 1.1 (i), $\Delta\epsilon_e\epsilon_f = E_{ef}$. Again, since ϵ_e and ϵ_f are identity maps,

$\epsilon_e \epsilon_f$ is the identity map on Eef. Therefore $\epsilon_{ef} = \epsilon_e \epsilon_f$. Thus, $(e\psi)(f\psi) = (ef)\psi$. Hence ψ is a homomorphism. The proof is completed. #

A subsemigroup T of a semigroup S is said to be <u>full</u> if and only if $E(S) \subseteq T$.

The next theorem has been shown by Munn in [5] that any fundamental inverse semigroup S is isomorphic to a full inverse subsemigroup of $T_{E(S)}$. The following two lemmas are required first :

1.3 <u>Lemma</u> [5] . Let S be an inverse semigroup and let $E = E(S)$. Then the following hold :

(i) For each $a \in S$, the map $\theta_a : Eaa^{-1} \longrightarrow Ea^{-1}a$ defined by

$$x\theta_a = a^{-1}xa \qquad (x \in Eaa^{-1})$$

is an onto isomorphism, and hence $\theta_a \in T_E$.

(ii) The map $\theta : S \to T_E$ defined by

$$a\theta = \theta_a \qquad (a \in S)$$

is a homomorphism. Moreover, the congruence on S induced by the homomorphism θ is the maximum idempotent-separating congruence μ of S, that is,

$$\mu = \{ (a,b) \in S \times S \mid \theta_a = \theta_b \} .$$

(iii) $S\theta$ is a full inverse subsemigroup of T_E, and hence S/μ is isomorphic to $S\theta$.

<u>Proof</u> : (i) Let $x,y \in Eaa^{-1}$. Then $x = eaa^{-1}$ and $y = faa^{-1}$ for some $e,f \in E$, and

$$\begin{aligned}(xy)\theta_a &= a^{-1}xya = a^{-1}(eaa^{-1})(faa^{-1})a \\ &= a^{-1}eaa^{-1}aa^{-1}faa^{-1}a = a^{-1}xaa^{-1}ya \\ &= (x\theta_a)(y\theta_a) \quad .\end{aligned}$$

Thus θ_a is a homomorphism. To show that θ_a is one-to-one, let $x\theta_a = y\theta_a$ where $x = eaa^{-1} \in Eaa^{-1}$ and $y = faa^{-1} \in Eaa^{-1}$. Then $a^{-1}xa = a^{-1}ya$. Thus $a^{-1}eaa^{-1}a = a^{-1}faa^{-1}a$, so $a^{-1}ea = a^{-1}fa$ which implies $a(a^{-1}ea)a^{-1} = a(a^{-1}fa)a^{-1}$. Therefore $eaa^{-1} = faa^{-1}$, that is, $x = y$. Hence θ_a is one-to-one. Next, to show θ_a is onto, let $m \in Ea^{-1}a$. Then $m = ea^{-1}a$ for some $e \in E$. Let $x = aea^{-1}aa^{-1}$. Then $x \in Eaa^{-1}$ since $aea^{-1} \in E$, and

$$\begin{aligned}x\theta_a &= a^{-1}xa = a^{-1}(aea^{-1}aa^{-1})a \\ &= a^{-1}aea^{-1}a = ea^{-1}aa^{-1}a \qquad \text{(because } a^{-1}a \in E\text{)} \\ &= ea^{-1}a = m \quad .\end{aligned}$$

Thus θ_a is onto. Therefore θ_a is an onto isomorphism. This proves $\theta_a \in T_E$, as required.

(ii) To show that $\theta_{ab} = \theta_a\theta_b$, for all $a,b \in S$, let $a,b \in S$. First, we claim that the map $\psi : Ea^{-1}a \rightarrow Eaa^{-1}$ defined by

$$y\psi = aya^{-1} \qquad (y \in Ea^{-1}a)$$

is an onto isomorphism and ψ is θ_a^{-1}. By the same proof as in (i), ψ is also an onto isomorphism. Next, we show that ψ is θ_a^{-1} . Let $x = ea^{-1}a \in Ea^{-1}a$ $(e \in E)$. Then we have

$$\begin{aligned}x\psi\theta_a &= (ea^{-1}a)\psi\theta_a = (aea^{-1}aa^{-1})\theta_a \\ &= a^{-1}(aea^{-1})a = ea^{-1}aa^{-1}a \\ &= ea^{-1}a = x \quad .\end{aligned}$$

Thus $\psi\theta_a$ is the identity map on $Ea^{-1}a$. For $y = eaa^{-1} \in Eaa^{-1} (e \in E)$, we have

$$
\begin{aligned}
y\theta_a\psi &= (eaa^{-1})\theta_a\psi &&= (a^{-1}eaa^{-1}a)\psi \\
&= a(a^{-1}ea)a^{-1} &&= eaa^{-1}aa^{-1} \\
&= eaa^{-1} &&= y \ .
\end{aligned}
$$

Therefore $\theta_a\psi$ is the identity map on Eaa^{-1}. These imply that ψ is θ_a^{-1} . Next, consider the following :

$$\Delta\theta_{ab} = E(ab)(ab)^{-1} = Eabb^{-1}a^{-1}$$

and

$$
\begin{aligned}
\Delta\theta_a\theta_b &= (Ea^{-1}a \cap Ebb^{-1})\theta_a^{-1} \\
&= (Ea^{-1}abb^{-1})\theta_a^{-1} \qquad \text{(by Lemma 1.1(i))} \\
&= a(Ea^{-1}abb^{-1})a^{-1} \\
&= aEbb^{-1}a^{-1} \ .
\end{aligned}
$$

Claim that $Eabb^{-1}a^{-1} = aEbb^{-1}a^{-1}$. To show this, let $x \in Eabb^{-1}a^{-1}$. Then $x = eabb^{-1}a^{-1}$ for some $e \in E$, and so $x = eaa^{-1}abb^{-1}a^{-1} = a(a^{-1}ea)bb^{-1}a^{-1}$. Since $a^{-1}ea \in E$, $x \in aEbb^{-1}a^{-1}$. Hence $Eabb^{-1}a^{-1} \subseteq aEbb^{-1}a^{-1}$. To show $aEbb^{-1}a^{-1} \subseteq Eabb^{-1}a^{-1}$, let $y \in aEbb^{-1}a^{-1}$. Then $y = afbb^{-1}a^{-1}$ for some $f \in E$, and hence $y = (afa^{-1})abb^{-1}a^{-1}$. Because $afa^{-1} \in E$, $y \in Eabb^{-1}a^{-1}$. Therefore $Eabb^{-1}a^{-1} = aEbb^{-1}a^{-1}$. This shows that $\Delta\theta_{ab} = \Delta\theta_a\theta_b$. To show $y\theta_{ab} = y\theta_a\theta_b$ for all $y \in \Delta\theta_{ab} = \Delta\theta_a\theta_b$, let $y \in \Delta\theta_{ab}$. Then

$$
\begin{aligned}
y\theta_{ab} &= (ab)^{-1}y(ab) &&= b^{-1}a^{-1}yab \\
&= (a^{-1}ya)\theta_b &&= y\theta_a\theta_b \ .
\end{aligned}
$$

Therefore $\theta_{ab} = \theta_a\theta_b$. Hence θ is a homomorphism.

The next proof is to show that the congruence on S induced by the homomorphism θ is the maximum idempotent-separating congruence μ of S, that is, to show $\mu = \{(a,b) \in S \times S \mid \theta_a = \theta_b\}$. Let

$\rho = \{(a,b) \in S \times S \mid \theta_a = \theta_b\}$. Let $(a,b) \in \rho$. Then $\theta_a = a\theta = b\theta = \theta_b$, so $\Delta\theta_a = \Delta\theta_b$. Hence $Eaa^{-1} = Ebb^{-1}$. Let $e \in E$. Then $(eaa^{-1})\theta_a = (eaa^{-1})\theta_b$ and $(ebb^{-1})\theta_a = (ebb^{-1})\theta_b$, so $b^{-1}eaa^{-1}b = a^{-1}eaa^{-1}a = a^{-1}ea$, $a^{-1}ebb^{-1}a = b^{-1}ebb^{-1}b = b^{-1}eb$. Since $eaa^{-1} \in E$, $eaa^{-1}bb^{-1} \in Ebb^{-1}$. Therefore $(eaa^{-1}bb^{-1})\theta_a = (eaa^{-1}bb^{-1})\theta_b$ and hence $a^{-1}eaa^{-1}bb^{-1}a = b^{-1}eaa^{-1}bb^{-1}b$ which implies $a^{-1}ebb^{-1}a = b^{-1}eaa^{-1}b$. Hence $aea^{-1} = beb^{-1}$. This shows $(a,b) \in \mu$. Therefore $\rho \subseteq \mu$. Next, let $(a,b) \in \mu$. Then $a^{-1}ea = b^{-1}eb$ for all $e \in E$; equivalently, $aea^{-1} = beb^{-1}$ for all $e \in E$. We want $\theta_a = \theta_b$, let $x \in \Delta\theta_a = Eaa^{-1}$. Then $x = eaa^{-1}$ for some $e \in E$, so $b^{-1}eb \in E(S)$ and

$$x = aa^{-1}eaa^{-1} = ab^{-1}eba^{-1} = bb^{-1}ebb^{-1} = ebb^{-1}$$

which belongs to Ebb^{-1}. Thus $Eaa^{-1} \subseteq Ebb^{-1}$. Similarly, we also have that $Ebb^{-1} \subseteq Eaa^{-1}$. Therefore $\Delta\theta_a = \Delta\theta_b$. Let $x \in Eaa^{-1} = Ebb^{-1}$. Then $x = eaa^{-1}$, $x = fbb^{-1}$ for some $e,f \in E$. Hence $x = eeaa^{-1} = ex = efbb^{-1}$ and $x = ffbb^{-1} = fx = feaa^{-1} = efaa^{-1}$. It then follows that

$$a^{-1}xa = a^{-1}efaa^{-1}a = a^{-1}efa$$

and

$$b^{-1}xb = b^{-1}efbb^{-1}b = b^{-1}efb .$$

Since $ef \in E$ and $(a,b) \in \mu$, $a^{-1}efa = b^{-1}efb$. Thus $a^{-1}xa = b^{-1}xb$. Hence $x\theta_a = x\theta_b$. This shows that $\theta_a = \theta_b$ which implies $(a,b) \in \rho$. Therefore $\mu \subseteq \rho$. Hence $\mu = \rho$ as desired.

(iii) We now show that $S\theta$ is a full inverse subsemigroup of T_E and $S/\mu \cong S\theta$. Since S is an inverse semigroup and $S\theta$ is a homomorphic image of S, $S\theta$ is an inverse subsemigroup of T_E [Introduction, page 4]. To show $E(T_E) \subseteq S\theta$, let $\epsilon_e \in E(T_E) (e \in E)$. It is clearly seen that $\epsilon_e = \theta_e$. Then $\epsilon_e = e\theta \in S\theta$. Hence $E(T_E) \subseteq S\theta$. Therefore $S\theta$ is a full inverse subsemigroup of T_E. Since μ is the congruence induced by the homomorphism θ, we have $S/\mu \cong S\theta$. #

1.4 <u>Lemma</u>. Let $\alpha, \beta \in I_X$, X be a nonempty set. If $\alpha \mathcal{H} \beta$, then $\Delta\alpha = \Delta\beta$ and $\nabla\alpha = \nabla\beta$.

<u>Proof</u> : Suppose $\alpha \mathcal{H} \beta$. Then $\alpha \mathcal{L} \beta$ and $\alpha \mathcal{R} \beta$, so $I_X\alpha = I_X\beta$ and $\alpha I_X = \beta I_X$. Since $\alpha \in I_X\beta$ and $\beta \in I_X\alpha$, $\alpha = \gamma\beta$ and $\beta = \gamma'\alpha$ for some $\gamma, \gamma' \in I_X$. From $\alpha = \gamma\beta$, we have $\nabla\alpha \subseteq \nabla\beta$, and from $\beta = \gamma'\alpha$, we have $\nabla\beta \subseteq \nabla\alpha$. Hence $\nabla\alpha = \nabla\beta$. Since $\alpha \in \beta I_X$ and $\beta \in \alpha I_X$, $\alpha = \beta\lambda$ and $\beta = \alpha\lambda'$ for some $\lambda, \lambda' \in I_X$. Since $\alpha = \beta\lambda$, $\Delta\alpha \subseteq \Delta\beta$. Since $\beta = \alpha\lambda'$, we have $\Delta\beta \subseteq \Delta\alpha$. Therefore $\Delta\alpha = \Delta\beta$. #

1.5 <u>Theorem</u> [5] . Let S be an inverse semigroup and $E = E(S)$. Then S is fundamental if and only if S is isomorphic to a full inverse subsemigroup of T_E .

<u>Proof</u> : First, let S be isomorphic to a full inverse subsemigroup S' of T_E . Let $(\alpha,\beta) \in \mu(S')$, the maximum idempotent-separating congruence of S'. Since $\mu(S') \subseteq \mathcal{H}'$, the Green's relation $\mathcal{H}$ on S', it follows from Lemma 1.4 that $\Delta\alpha = \Delta\beta$ and $\nabla\alpha = \nabla\beta$. Then there exists $e \in E$ such that $\Delta\alpha = Ee = \Delta\beta$. Let $g \in Ee$. Then $Eg \subseteq Ee$ and $g \leq e$ by Lemma 1.1 (ii). Thus $g = ge = eg$. Therefore

$$\Delta(\epsilon_g\alpha) = (\nabla\epsilon_g \cap \Delta\alpha)\epsilon_g^{-1} = Eg \cap Ee = Ege = Eg.$$

Since $\Delta\alpha = Ee$, we have $\nabla\alpha^{-1} = Ee$; and hence

$$\Delta(\alpha^{-1}\epsilon_g\alpha) = (Ee \cap Eg)\alpha$$
$$= (Eeg)\alpha \qquad \text{(Lemma 1.1 (i))}$$
$$= (Eg)\alpha$$
$$= E(g\alpha) = \Delta(\epsilon_{g\alpha}) .$$

But $\alpha^{-1}\epsilon_g\alpha$ is an idempotent in T_E , hence $\alpha^{-1}\epsilon_g\alpha$ is the identity map on $\Delta(\alpha^{-1}\epsilon_g\alpha) = \Delta(\epsilon_{g\alpha})$ which implies $\alpha^{-1}\epsilon_g\alpha = \epsilon_{g\alpha}$. Similarly, $\beta^{-1}\epsilon_g\beta = \epsilon_{g\beta}$. Because S' is full and $\epsilon_g \in E(T_E)$, $\epsilon_g \in E(S')$. But $(\alpha,\beta) \in \mu(S')$, so $\alpha^{-1}\epsilon_g\alpha = \beta^{-1}\epsilon_g\beta$. Thus $\epsilon_{g\alpha} = \epsilon_{g\beta}$ and hence $Eg\alpha = \Delta(\epsilon_{g\alpha}) = \Delta(\epsilon_{g\beta}) = Eg\beta$. By Lemma 1.1 (iii) , $g\alpha = g\beta$. Since this holds for all $g \in Ee$, it follows that $\alpha = \beta$. Thus $\mu(S')$ is the identity congruence on S'. Hence S' is fundamental, and then S is fundamental.

Conversely, assume S is fundamental. Let θ be the homomorphism from S into T_E defined as in Lemma 1.3 (ii). By Lemma 1.3 (iii), $S\theta$ is a full inverse subsemigroup of T_E and $S/\mu \cong S\theta$. Because S is fundamental, $\mu = \iota$, the identity congruence on S, and so $S/\mu \cong S$. Hence $S \cong S\theta$ which is a full inverse subsemigroup of T_E . #

From Theorem 1.5, the following corollary follows easily:

1.6 <u>Corollary</u>. If E is a semilattice, then T_E is a fundamental inverse subsemigroup of I_E .

Let S be a semigroup and T be a subset of S. The <u>centralizer of</u> T <u>in</u> S is the set $\{ x \in S \mid xt = tx \text{ for all } t \in T \}$ which is denoted by C(T). Because any two idempotents of an inverse semigroup commute, it follows that for any inverse semigroup S, $E(S) \subseteq C(E(S))$.

It has been shown in [4] that an inverse semigroup S is fundamental if and only if $E(S) = C(E(S))$, the centralizer of E(S) in S.

A symmetric inverse semigroup on any set is fundamental.

1.7 Theorem [1] . For any set X, I_X is fundamental.

Proof : To show that I_X is fundamental, it suffices to show $C(E(I_X)) = E(I_X)$. Because I_X is an inverse semigroup, $E(I_X) \subseteq C(E(I_X))$. Suppose $C(E(I_X)) \neq E(I_X)$. Then there exists $\alpha \in C(E(I_X))$ such that $\alpha \notin E(I_X)$. Since

$$E(I_X) = \{ \beta \in I_X \mid \beta \text{ is the identity map on } \Delta\beta \},$$

α is not the identity map on $\Delta\alpha$. Then there exists $x \in \Delta\alpha$ such that $x\alpha \neq x$. Let δ be the identity map on the set $\{x\}$. Then $\delta \in E(I_X)$, Since $\Delta\delta = \nabla\delta = \{x\}$, we have

$$\Delta(\delta\alpha) = (\nabla\delta \cap \Delta\alpha)\delta^{-1} = (\{x\} \cap \Delta\alpha)\delta = \{x\}.$$

If $x \in \nabla\alpha$, then

$$\Delta(\alpha\delta) = (\nabla\alpha \cap \Delta\delta)\alpha^{-1} = (\nabla\alpha \cap \{x\})\alpha^{-1} = \{x\alpha^{-1}\}.$$

If $x \notin \nabla\alpha$, then $\alpha\delta = 0$. Since $x\alpha \neq x$ and α is one-to-one, $x \neq x\alpha^{-1}$. Hence $\Delta(\delta\alpha) \neq \Delta(\alpha\delta)$, so $\delta\alpha \neq \alpha\delta$. It follows that $\alpha \notin C(E(I_X))$, which is a contradiction. Thus $C(E(I_X)) = E(I_X)$. This proves I_X is fundamental as required. #

Let S be a semilattice. Then $E(S) = S = C(E(S))$. Hence S is fundamental. An another way to prove this, let $a,b \in S$ such that $a\mu b$. Then $aea^{-1} = beb^{-1}$ for all $e \in E(S) = S$. Since S is a semilattice,

$a^2 = a = a^{-1}$ and $b^2 = b = b^{-1}$, and so $ae = be$ for all $e \in E(S) = S$. Then

$$a = a^2 = ba$$

and

$$b = b^2 = ab = ba,$$

so $a = b$. This shows that μ is the identity congruence on S, and therefore S is fundamental.

Let G be a group. Then $E(G) = \{1\}$ where 1 is the identity of G. Then $C(E(G)) = G$. Hence the group G is fundamental if and only if G is a trivial group.

Let S be a semigroup. For each $a \in S$, let H_a denote the $\mathcal{H}$-class of S containing a. If e is an idempotent of S, then H_e is the maximum subgroup of S having e as its identity [Introduction, page 7].

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For each $\alpha \in Y$, let e_α denote the identity of the group G_α. Then

$$E(S) = \{ e_\alpha \mid \alpha \in Y \}.$$

Since S is the disjoint union of the subgroups G_α, it follows that for each $\alpha \in Y$, G_α is the maximum subgroup of S having e_α as its identity. Therefore, G_α is an $\mathcal{H}$-class of S for all $\alpha \in Y$. Moreover, $\mathcal{H}$ is a congruence. To show this, let $a,b,c \in S$ and $a \mathcal{H} b$. Then $a,b \in G_\beta$ for some $\beta \in Y$. Let $\lambda \in Y$ and $c \in G_\lambda$. Then $ac, bc, cb \in G_{\beta\lambda}$ and so $ac \mathcal{H} bc$ and $ca \mathcal{H} cb$. Hence $\mathcal{H}$ is an idempotent-separating congruence. But the maximum idempotent-separating congruence μ is contained in $\mathcal{H}$. Therefore $\mu = \mathcal{H}$. Thus, if

$S = \bigcup_{\alpha \in Y} G_\alpha$ is fundamental, then $S = E(S) = \{e_\alpha \mid \alpha \in Y\}$.

Hence any semilattice Y of groups is fundamental if and only if it is a semilattice which is isomorphic to Y .

We further study an inverse subsemigroup and a homomorphic image of a fundamental inverse semigroup. We can show that an inverse subsemigroup and a homomorphic image of a fundamental inverse semigroup are not necessarily fundamental. An example is given as follows :

Let $X = \{a,b\}$. Then the symmetric inverse semigroup on X, I_X, is fundamental [Theorem 1.7]. Let G_X be the permutation group on X. Then G_X is a group of order 2 and it is an inverse subsemigroup of I_X. Because G_X is not a nontrivial group, G_X is not fundamental.

Let 0 and 1 be the zero and the identity of I_X and α_1, α_2, α_3, α_4, α_5 be one-to-one partial transformations on X defined as follow :

$$\Delta\alpha_1 = \{a\} = \nabla\alpha_1 ,$$
$$\Delta\alpha_2 = \{b\} = \nabla\alpha_2 ,$$
$$\Delta\alpha_3 = \{a\} , \quad \nabla\alpha_3 = \{b\} ,$$
$$\Delta\alpha_4 = \{b\} , \quad \nabla\alpha_4 = \{a\} ,$$

$\Delta\alpha_5 = \{a,b\} = \nabla\alpha_5$ such that $a\alpha_5 = b$, $b\alpha_5 = a$.

Hence $I_X = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 1\}$ and its multiplication table is as follows :

$\cdot$	0	α_1	α_2	α_3	α_4	α_5	1
0	0	0	0	0	0	0	0
α_1	0	α_1	0	α_3	0	α_3	α_1
α_2	0	0	α_2	0	α_4	α_4	α_2
α_3	0	0	α_3	0	α_1	α_1	α_3
α_4	0	α_4	0	α_2	0	α_2	α_4
α_5	0	α_4	α_3	α_2	α_1	1	α_5
1	0	α_1	α_2	α_3	α_4	α_5	1

Let $T = \{0, \alpha_5, 1\}$. From the above table, we have T as a subsemigroup of I_X. Since $\alpha_5^{-1} = \alpha_5$, T is an inverse subsemigroup of S. Moreover, $\alpha_5^2 = 1$, so $E(T) = \{0, 1\}$. It is clearly seen that T is commutative. Then the centralizer of E(T) in T is T. Hence, $C(E(T)) = T \neq E(T)$, so T is not fundamental [[4], Theorem 2.7].

Let $A = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. It follows from the table that A is an ideal of I_X and $I_X = A \cup \{\alpha_5, 1\}$.

To show that $T = \{0, \alpha_5, 1\}$ is a homomorphic image of I_X, let $\psi : I_X \longrightarrow T$ be defined by

$$\alpha\psi = \begin{cases} \alpha & \text{if } \alpha \in \{\alpha_5, 1\} , \\ 0 & \text{if } \alpha \in A . \end{cases}$$

Let $\alpha, \beta \in I_X$.

<u>Case $\alpha, \beta \in A$.</u> Since A is a subsemigroup of I_X, $\alpha\beta \in A$, so $\alpha\psi$, $\beta\psi$, $\alpha\beta\psi$ are all 0. Therefore $(\alpha\beta)\psi = (\alpha\psi)(\beta\psi)$.

<u>Case $\alpha, \beta \in \{\alpha_5, 1\}$.</u> Then $\alpha\beta \in \{\alpha_5, 1\}$ and hence

$$(\alpha\beta)\psi = \begin{cases} 1 & \text{if } \alpha = \beta = 1 , \\ 1 & \text{if } \alpha = \beta = \alpha_5 , \\ \alpha_5 & \text{if either } \alpha = \alpha_5,\ \beta = 1 \text{ or } \alpha = 1,\ \beta = \alpha_5 , \end{cases}$$

and

$$(\alpha\psi)(\beta\psi) = \begin{cases} 1.1 = 1 & \text{if } \alpha = \beta = 1, \\ \alpha_5\alpha_5 = 1 & \text{if } \alpha = \beta = \alpha_5 , \\ \alpha_5 1 = \alpha_5 & \text{if } \alpha = \alpha_5,\ \beta = 1, \\ 1\alpha_5 = \alpha_5 & \text{if } \alpha = 1,\ \beta = \alpha_5 . \end{cases}$$

Case $\alpha \in A$, $\beta \in \{\alpha_5, 1\}$. Then $\alpha\beta$, $\beta\alpha \in A$ since A is an ideal of S. Therefore

$$\alpha\psi = 0,\ (\alpha\beta)\psi = 0,\ (\beta\alpha)\psi = 0 , \text{ so}$$

$$(\alpha\psi)(\beta\psi) = 0 = (\alpha\beta)\psi$$

and

$$(\beta\psi)(\alpha\psi) = 0 = (\beta\alpha)\psi .$$

Hence ψ is an onto homomorphism. Thus T is a homomorphic image of I_X.

The following proposition shows that a homomorphic image of a fundamental inverse semigroup S by a homomorphism which is one-to-one on E(S) is isomorphic to S, and hence it is fundamental.

1.7 Proposition. Let $\psi : S \to T$ be a homomorphism from an inverse semigroup S onto an inverse semigroup T such that for $e,f \in E(S)$, $e\psi = f\psi$ implies $e = f$. If S is fundamental, then ψ is an onto isomorphism, and hence T is isomorphic to S.

Proof : Let ρ be the congruence on S induced by ψ, that is,

$$a\rho b \iff a\psi = b\psi \qquad (a,b \in S) .$$

Since ψ is one-to-one on E(S), each class of ρ contains at most one idempotent of S. Then ρ is an idempotent-separating congruence on S, and hence $\rho \subseteq \mu$, the maximum idempotent-separating congruence on S. Because S is fundamental, μ is the identity congruence, so ρ is the identity congruence on S. Then ψ is one-to-one. Therefore ψ is an onto isomorphism, and hence T is isomorphic to S. #