CHAPTER VI

RELATION BETWEEN W_c AND W

In this chapter we will consider a certain behaviour of the transformation of the Wiener integral, under the scalar multiplication $y \longmapsto y/\sqrt{c}$, which leads to the relation between W_c and W.

<u>Definition 6.1</u>. Let A be a subset of C. We define

$$A/\sqrt{c} = \{x \in C : x = y/\sqrt{c}, \quad y \in A\}.$$

<u>Theorem 6.2</u>. If $A \in \mathcal{B}(C)$ then $A/\sqrt{c} \in \mathcal{B}(C)$.

<u>Proof</u>. We divide the proof into 3 steps :

<u>Step 1</u>. We show that $(A/\sqrt{c})' = A'/\sqrt{c}$, where $(A/\sqrt{c})' = C - A/\sqrt{c}$ and $A' = C - A$.

If $x \in (A/\sqrt{c})'$, then $x \notin A/\sqrt{c}$, so that $\sqrt{c}\,x \notin A$. Therefore $\sqrt{c}\,x \in A'$ and hence $x \in A'/\sqrt{c}$. Conversely, if $x \in A'/\sqrt{c}$, then $\sqrt{c}\,x \in A'$, so that $\sqrt{c}\,x \notin A$. Therefore $x \notin A/\sqrt{c}$ and hence $x \in (A/\sqrt{c})'$.

<u>Step 2</u>. We show that $\bigcup_{i=1}^{\infty} (A_i/\sqrt{c}) = \bigcup_{i=1}^{\infty} A_i/\sqrt{c}$.

If $x \in \bigcup_{i=1}^{\infty} (A_i/\sqrt{c})$, then $x \in A_i/\sqrt{c}$ for some i, so that $\sqrt{c}\,x \in \bigcup_{i=1}^{\infty} A_i$. Therefore $x \in \bigcup_{i=1}^{\infty} A_i/\sqrt{c}$. Conversely, if $x \in \bigcup_{i=1}^{\infty} A_i/\sqrt{c}$, then $\sqrt{c}\,x \in \bigcup_{i=1}^{\infty} A_i$, so that $x \in A_i/\sqrt{c}$ for some i. Therefore $x \in \bigcup_{i=1}^{\infty} (A_i/\sqrt{c})$.

Step 3. Let $\mathcal{A} = \{A \in \mathcal{B}(C) : A/\sqrt{c} \in \mathcal{B}(C)\}$. Then

(i). Since $y \longmapsto y/\sqrt{c}$ is a homeomorphism on C, it follows that if $A \subseteq C$ is open then $A/\sqrt{c}$ is open and hence $A/\sqrt{c} \in \mathcal{B}(C)$. Therefore $\mathcal{A}$ contains all open sets in C.

(ii). Let $A \in \mathcal{A}$, then A and $A/\sqrt{c} \in \mathcal{B}(C)$. Since $\mathcal{B}(C)$ is a σ-algebra, A' and $(A/\sqrt{c})' \in \mathcal{B}(C)$. It follows from step 1 that $A' \in \mathcal{A}$.

(iii). Let $A_i \in \mathcal{A}$ for $i = 1,2,\ldots$, then A_i and $A_i/\sqrt{c} \in \mathcal{B}(C)$ for all i. Since $\mathcal{B}(C)$ is a σ-algebra, $\bigcup_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty}(A_i/\sqrt{c}) \in \mathcal{B}(C)$. It follows from step 2 that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

From (i), (ii) and (iii) we have that $\mathcal{A}$ is a σ-algebra containing all open sets in C and must therefore contain the collection $\mathcal{B}(C)$ of all Borel sets, since $\mathcal{B}(C)$ is the smallest σ-algebra containing open sets. Hence $\mathcal{A} = \mathcal{B}(C)$.

Q.E.D.

Theorem 6.3. Let Γ be a Wiener measurable subset of C. Then $W_c(\Gamma) = W(\Gamma/\sqrt{c})$. Moreover if F is any measurable functional defined on Γ

$$\int_{\Gamma} F(y)\, dW_c(y) = \int_{\Gamma/\sqrt{c}} F(\sqrt{c}\, x)\, dW(x).$$

in the sense that the existence of one side implies that of the other and the validity of the equality.

Proof. We first let F be a functional satisfying the conditions in Theorem 5.5 and let n, M, $C_{M,n}$ and C_M be as in the proof of Theorem 5.5. Then according to (5.8), we have

$$\int_{C_{M,n}} F(L_n(y))dW_c(y) = \gamma_n \int_{-M}^{M} \cdots \int_{-M}^{M} H(\xi_1,\ldots,\xi_n)\exp\left\{-\sum_{j=1}^{n}\frac{(\xi_j-\xi_{j-1})^2}{ct_j-ct_j}\right\}d\xi_1,\ldots d\xi_n. \quad \ldots\ldots\ldots\ldots(1)$$

where $\gamma_n = \left\{\pi^n c^n t_1(t_2-t_1)\ldots(t_n-t_{n-1})\right\}^{-\frac{1}{2}}$.

If x is the image of y under the transformation $y \longmapsto y/\sqrt{c}$ and if $L_n(x)$, $L_n(y)$ are the polygonalized functions corresponding to x and y respectively, then according to Definition 5.1

$$L_n y(t) = \sqrt{c}\, x(t_j) + \frac{\sqrt{c}\, x(t_{j+1}) - \sqrt{c}\, x(t_j)}{t_{j+1} - t_j}\cdot(t - t_j)$$

$$= \sqrt{c}\, L_n x(t). \quad \ldots\ldots\ldots\ldots(2)$$

If we write

$$\xi_j = y(t_j)\ ,\quad \eta_j = x(t_j)\ ,\quad j = 0,\ldots,n,$$

then under (2) we have

$$\xi_j = \sqrt{c}\ \eta_j\ ,\qquad j = 0,\ldots, n. \quad \ldots\ldots\ldots\ldots(3)$$

Since $\dfrac{\partial(\xi_1,\ldots,\xi_n)}{\partial(\eta_1,\ldots,\eta_n)} = \begin{vmatrix} \sqrt{c} & 0 & 0\ldots\ldots 0 \\ 0 & \sqrt{c} & 0\ldots\ldots 0 \\ 0 & 0 & 0\ldots\ldots \sqrt{c} \end{vmatrix} = c^{n/2}$,

on applying the transformation (3) to the Lebesgue integral in (1) we find that

$$\int_{C_{M,n}} F(L_n(y))dW_c(y) = c^{n/2}\gamma_n \int_{-M/\sqrt{c}}^{M/\sqrt{c}} \cdots\cdots \int_{-M/\sqrt{c}}^{M/\sqrt{c}} H(\sqrt{c}\,\eta_1, \ldots, \sqrt{c}\,\eta_n)$$

$$\cdot \exp\left\{-\sum_{j=1}^{n} \frac{(\eta_j - \eta_{j-1})^2}{t_j - t_{j-1}}\right\} d\eta_1 \cdots\cdots d\eta_n$$

$$= \int_{C_{M,n/\sqrt{c}}} H(\sqrt{c}\,x(t_1), \ldots, \sqrt{c}\, x(t_n))dW(x).$$

$$= \int_{C_{M,n/\sqrt{c}}} F(L_n(\sqrt{c}\, x))dW(x).$$

This gives us a transformation formula over $C_{M,n}$ for the polygonalized functions under the transformation $y \longmapsto y/\sqrt{c}$. If we let $n \longrightarrow \infty$ (over the sequence $\{2^0, 2^1, \ldots\}$) and then M, according to the properties of F we obtain

$$\int_C F(y)dW_c(y) = \int_C F(\sqrt{c}\, x)dW(x).$$

Thus, if Γ is any Wiener measurable set and F is a measurable functional defined on Γ, then as the same proof as in Theorem 5.10 we obtain the theorem.

<u>Q.E.D.</u>