CHAPTER I

THE L^p SPACES

In this chapter, we shall construct the classical L^p spaces over $\mathcal{T}$. Before we can do this, we must obtain the important inequalities of Minkowski and of Hölder which, in turn, can be easily obtained through convexity argument.

1. Convex functions and Inequalities.

1.1 Definition. A real-valued function φ defined on an open interval (a,b) where $-\infty \leq a < b \leq +\infty$, is called convex, if $\varphi((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)\varphi(x_1) + \lambda\varphi(x_2)$ holds whenever $a < x_1 < b$, $a < x_2 < b$ and $0 \leq \lambda \leq 1$.

Geometrically, the convexity of $\varphi(x)$ means that for any triple $x_1 \leq x \leq x_2$, the point $(x, \varphi(x))$ on the graph of the function $x \longmapsto \varphi(x)$ is always below or on the line segment joining the points $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$.

1.2 Theorem. φ is convex in (a,b) if and only if

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t} \quad \text{whenever } a < s < t < u < b.$$

Proof. Suppose φ is convex. Since t can be written in the form $\frac{u-t}{u-s} . s + \frac{t-s}{u-s} . u$,

$$\begin{aligned} \varphi(t) &= \varphi\left(\frac{u-t}{u-s} . s + \frac{t-s}{u-s} . u\right) \\ &\leq \frac{u-t}{u-s} \varphi(s) + \frac{t-s}{u-s} \varphi(u) \end{aligned}$$

which yields,

$$(u-s)\varphi(t) \leq (u-t)\varphi(s) + (t-s)\varphi(u).$$

But $u-s = u-t+t-s$, and so

$$(u-t+t-s)\varphi(t) \leq (u-t)\varphi(s)+(t-s)\varphi(u),$$

or

$$(u-t)\varphi(t)+(t-s)\varphi(t) \leq (u-t)\varphi(s)+(t-s)\varphi(u)$$

or

$$(u-t)\big(\varphi(t)-\varphi(s)\big) \leq (t-s)\big(\varphi(u)-\varphi(t)\big).$$

Hence

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leq \frac{\varphi(u)-\varphi(t)}{u-t}$$

Conversely, assume

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leq \frac{\varphi(u)-\varphi(t)}{u-t}$$

whenever $a < s < t < u < b$. Then

$$(u-t)\big(\varphi(t)-\varphi(s)\big) \leq (t-s)\big(\varphi(u)-\varphi(t)\big)$$

or

$$\big((u-t)+(t-s)\big)\big(\varphi(t)\big) \leq (t-s)\varphi(u)+(u-t)\varphi(s)$$

and by rearranging term, we get

$$\varphi(t) \leq \frac{t-s}{u-s}\varphi(u)+\frac{u-t}{u-s}\cdot\varphi(s).$$

Substitute $\frac{u-t}{u-s}\cdot s+\frac{t-s}{u-s}\cdot u$ for t, we obtain

$$\varphi\left(\frac{u-t}{u-s}\cdot s+\frac{t-s}{u-s}\cdot u\right) = \varphi\left(\frac{u-t}{u-s}\cdot s+\left(1-\frac{u-t}{u-s}\right)\cdot u\right)$$

$$\leq \frac{u-t}{u-s}\varphi(s)+\left(1-\frac{u-t}{u-s}\right)\varphi(u)$$

This completes the proof.

1.3 Theorem. The exponential function $x \longmapsto \exp x$ is a convex function over $\mathbb{R}$.

Proof. Let $f(x) = \exp x$. By Theorem 1.2., the convexity condition $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ $(0 \leq \lambda \leq 1,\ x_1,\ x_2 \in \mathbb{R})$ is equivalent to

(1-1) $$\frac{f(b) - f(a)}{b-a} \leq \frac{f(c) - f(b)}{c-b} \quad \text{for any } a < b < c.$$

By the Mean Value Theorem for derivative, we can find $a \leq s \leq b$ and $b \leq t \leq c$ such that $\frac{f(b) - f(a)}{b-a} = f'(s)$ and $\frac{f(c) - f(b)}{c-b} = f'(t)$. Since f' is a strictly increasing function, we have $f'(s) \leq f'(t)$, and the inequality (1-1) follows immediately. Thus

$$e^{\lambda x_1 + (1-\lambda)x_2} \leq \lambda e^{x_1} + (1-\lambda) e^{x_2}.$$

Consequently, the exponential function $x \longmapsto \exp x$ is a convex function over $\mathbb{R}$.

Let p and q be positive real numbers. Then p and q are called conjugate exponents if

(1-2) $$\frac{1}{p} + \frac{1}{q} = 1.$$

Note that Eq(1-2) forces $1 < p,\ q < +\infty$, since $p,\ q > 0$. We shall consider 1 and ∞ to be conjugate exponents as well.

1.4 Theorem Let a and b be positive number and suppose p and q are conjugate exponents with $1 < p,\ q < \infty$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

<u>Proof</u>. In fact, let $x_1 = p \ln a$, $x_2 = q \ln b$, and $\lambda = \frac{1}{p}$. It follows from Theorem 1.3 that

$$e^{\frac{1}{p} \cdot p \ln a + (1-\frac{1}{p}) q \ln b} \leq \frac{1}{p} e^{p \ln a} + (1-\frac{1}{p}) e^{q \ln b}.$$

Since $1 - \frac{1}{p} = \frac{1}{q}$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

The Theorem is now proved.

1.5 <u>Theorem</u>. (Hölder's Inequality)

Let p and q be conjugate exponents ($1 < p, q < \infty$). If f and g are complex measurable functions on $\mathbb{T}$, then

$$\left|\int_{\mathbb{T}} f(\dot{x}) g(\dot{x}) d\dot{x}\right| \leq \left[\int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x}\right]^{\frac{1}{p}} \left[\int_{\mathbb{T}} |g(\dot{x})|^q d\dot{x}\right]^{\frac{1}{q}}.$$

<u>Proof</u>. Let $\alpha = \left[\int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x}\right]^{\frac{1}{p}}$ and $\beta = \left[\int_{\mathbb{T}} |g(\dot{x})|^q d\dot{x}\right]^{\frac{1}{q}}$.

If α is o, then $f = o$ a.e., so that $f(\dot{x})\, g(\dot{x}) = o$ a.e. and the inequality reduces to $o \leq o$. The same result holds if β is o. If either α or β is ∞, again the inequality reduces to $\infty \leq \infty$.

Assume then $+\infty > \alpha \neq 0 \neq \beta < +\infty$. Let $F(\dot{x}) = \frac{f(\dot{x})}{\alpha}$ and $G(\dot{x}) = \frac{g(\dot{x})}{\beta}$. By Theorem 1.4, we have

$$|F(\dot{x})\, G(\dot{x})| \leq \frac{|F(\dot{x})|^p}{p} + \frac{|G(\dot{x})|^q}{q}$$

and therefore

$$\left|\int_{\mathbb{T}} F(\dot{x})\, G(\dot{x})\, d\dot{x}\right| \leq \int_{\mathbb{T}} |F(\dot{x})\, G(\dot{x})|\, d\dot{x}$$

$$\leq \frac{1}{p} \int_{\mathbb{T}} \frac{|f(\dot{x})|^p d\dot{x}}{\alpha^p} + \frac{1}{q} \int_{\mathbb{T}} \frac{|g(\dot{x})|^q d\dot{x}}{\beta^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\left| \int_{\Upsilon} f(\dot{x})\, g(\dot{x})\, d\dot{x} \right| \leq \left[\int_{\Upsilon} |f(\dot{x})|^p d\dot{x} \right]^{\frac{1}{p}} \cdot \left[\int_{\Upsilon} |g(\dot{x})|^q d\dot{x} \right]^{\frac{1}{q}}.$$

1.6 Theorem. (Minkowski's Inequality)

Let $1 < p < +\infty$. If f and g are complex measurable function on Υ, then

$$(1\text{-}3) \quad \left[\int_{\Upsilon} |f(\dot{x}) + g(\dot{x})|^p d\dot{x} \right]^{\frac{1}{p}} \leq \left[\int_{\Upsilon} |f(\dot{x})|^p\, d\dot{x} \right]^{\frac{1}{p}} + \left[\int_{\Upsilon} |g(\dot{x})|^p d\dot{x} \right]^{\frac{1}{p}} .$$

Proof. We have

$$\left(f(\dot{x}) + g(\dot{x})\right)^p = f(\dot{x})\left(f(\dot{x}) + g(\dot{x})\right)^{p-1} + g(\dot{x})\left(f(\dot{x}) + g(\dot{x})\right)^{p-1}$$

so that

$$|f(\dot{x}) + g(\dot{x})|^p = \left| f(\dot{x})\left(f(\dot{x}) + g(\dot{x})\right)^{p-1} + g(\dot{x})\left(f(\dot{x}) + g(\dot{x})\right)^{p-1} \right|$$
$$\leq |f(\dot{x})| \cdot |f(\dot{x}) + g(\dot{x})|^{p-1} + |g(\dot{x})| \cdot |f(\dot{x}) + g(\dot{x})|^{p-1}.$$

Which implie

$$\int_{\Upsilon} |f(\dot{x}) + g(\dot{x})|^p \leq \int_{\Upsilon} |f(\dot{x})|\, |f(\dot{x}) + g(\dot{x})|^{p-1}\, d\dot{x} + \int_{\Upsilon} |g(\dot{x})|\, |f(\dot{x}) + g(\dot{x})|^{p-1}\, d\dot{x} .$$

Let q be the conjugate exponent of p. Then by Hölder's Inequality, we have

$$\int_{\Upsilon} |f(\dot{x}) + g(\dot{x})|^p d\dot{x} \leq \left[\int_{\Upsilon} |f(\dot{x})|^p d\dot{x} \right]^{\frac{1}{p}} \left[\int_{\Upsilon} |f(\dot{x}) + g(\dot{x})|^{(p-1)q} d\dot{x} \right]^{\frac{1}{q}} + \left[\int_{\Upsilon} |g(\dot{x})|^p d\dot{x} \right]^{\frac{1}{p}} \left[\int_{\Upsilon} |f(\dot{x}) + g(\dot{x})|^{(p-1)q} d\dot{x} \right]^{\frac{1}{q}}$$

and therefore,

$$(1\text{-}4)\quad \int_{\mathbb{T}} |f(\dot{x})+g(\dot{x})|^p d\dot{x} \leq \left[\int_{\mathbb{T}} |f(\dot{x})+g(\dot{x})|^p d\dot{x}\right]^{\frac{1}{q}} \left\{ \left[\int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x}\right]^{\frac{1}{p}} + \left[\int_{\mathbb{T}} |g(\dot{x})|^p d\dot{x}\right]^{\frac{1}{p}} \right\}$$

since $\frac{1}{p}+\frac{1}{q}=1$ **implies** $(p-1)\,q = p.$

Moreover, Ineq (1-3) is obvious if either its left hand members is o or either one of its right hand summands is $+\infty$. Thus assume the left-hand member of Ineq (1-3) to be different from o and both of the right-hand summands different from $+\infty$. We can then divide both sides of Ineq (1-4) by $\left[\int_{\mathbb{T}} |f(\dot{x})+g(\dot{x})|^p d\dot{x}\right]^{\frac{1}{q}}$ to get Ineq (1-3) with the observation that $1-\frac{1}{q}=\frac{1}{p}$.

2. The L^p - spaces

2.1 Definition. If $o<p<\infty$ and if f is a complex measurable function on $\mathbb{T}$, define

$$||f||_p = \left\{ \int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x} \right\}^{\frac{1}{p}}$$

and let $L^p(\mathbb{T})$ consist of all f for which $||f||_p<\infty$. We call $||f||_p$ the L^p - norm of f.

2.2 Definition. Suppose $g:\mathbb{T}\longrightarrow[o,\infty]$ is measurable. Let S be the set of all real α such that $\mu\left(g^{-1}(\alpha,\infty]\right) = o$, where μ is the "Lebesgue measure" on $\mathbb{T}$ induced by the Lebesgue measure on $(0,1)$.

If $S = \emptyset$, put $\beta = \infty$. If $S \neq \emptyset$, put $\beta = \inf S$. Since $g^{-1}(\beta, \infty] = \bigcup_{n=1}^{\infty} g^{-1}\left((\beta + \frac{1}{n}, \infty]\right)$ and since the union of a countable collection of sets of measure o has measure o, we see that $\beta \in S$.

We call β <u>the essential supremum of g</u>.

If f is a complex measurable function on $\mathbb{T}$, we define $\| f \|_\infty$ to be the essential supremum of $|f|$ and we let $L^\infty(\mathbb{T})$ consist of all f for which $\| f \|_\infty < \infty$.

2.3 <u>Theorem</u>. If p and q are conjugate exponents, $1 \leq p \leq \infty$, and if $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, then $fg \in L^1(\mathbb{T})$ and $\| fg \|_1 \leq \| f \|_p . \| g \|_q$.

<u>Proof</u>. For $1 < p < +\infty$, it is simply Hölder inequality,

$$\begin{aligned} \| fg \|_1 &= \int_{\mathbb{T}} | f(\dot{x})\, g(\dot{x}) |\, d\dot{x} \\ &= \int_{\mathbb{T}} | f(\dot{x}) |\, | g(\dot{x}) |\, d\dot{x} \\ &\leq \left(\int_{\mathbb{T}} | f(\dot{x}) |^p d\dot{x} \right)^{\frac{1}{p}} \left(\int_{\mathbb{T}} | g(\dot{x}) |^q d\dot{x} \right)^{\frac{1}{q}} \end{aligned}$$

If $p = +\infty$, note that

$$\begin{aligned} | f(\dot{x})\, g(\dot{x}) | &= | f(\dot{x}) |\, | g(\dot{x}) | \\ &\leq \| f \|_\infty \; | g(\dot{x}) | \quad \text{for almost all} \end{aligned}$$

$\dot{x}$ in $\mathbb{T}$, so that

$$\int_{\mathbb{T}} | f(\dot{x})\, g(\dot{x}) |\, d\dot{x} \leq \| f \|_\infty \int_{\mathbb{T}} | g(\dot{x}) |\, d\dot{x}.$$

Hence

$$\|fg\|_1 \leq \|f\|_\infty \; \|g\|_1 .$$

Similarly for $p = 1$, we have

$$\|fg\|_1 \leq \|f\|_1 \; \|g\|_\infty .$$

2.4 Theorem. Let $1 \leq p \leq +\infty$ and $f \in L^p(\mathbb{T})$, $g \in L^p(\mathbb{T})$. Then $f+g \in L^p(\mathbb{T})$, and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof. For $1 < p < +\infty$, it follows from Minkowski's inequality. For $p=1$ or $p=+\infty$, it follows from $|f+g| \leq |f| + |g|$. For $p=1$, we have

$$\int_{\mathbb{T}} |f(\dot{x})+g(\dot{x})|\, d\dot{x} \leq \int_{\mathbb{T}} |f(\dot{x})|\, d\dot{x} + \int_{\mathbb{T}} |g(\dot{x})|\, d\dot{x}.$$

Hence $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$. For $p = +\infty$, we have

$$|f+g| \leq \|f\|_\infty + \|g\|_\infty$$

and so

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty .$$

2.5 Remark. Fix p, $1 \leq p \leq +\infty$. If $f \in L^p(\mathbb{T})$, α is a real number, then $\alpha f \in L^p(\mathbb{T})$.

Proof. Observe that for $1 \leq p < +\infty$.

$$\left(\int_{\mathbb{T}} |\alpha f(\dot{x})|^p d\dot{x}\right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}} |\alpha|^p |f(\dot{x})|^p d\dot{x}\right)^{\frac{1}{p}} = |\alpha| \left(\int_{\mathbb{T}} |f(\dot{x})|^p d\dot{x}\right)^{\frac{1}{p}} .$$

On the other hand $\|\alpha f\|_\infty = |\alpha| \; \|f\|_\infty$. Thus we have proved the Remark.

By Theorem 2.4 and Remark 2.5 we can easily see that $L^p(\mathcal{T})$ is a linear space.

For $1 \leq p \leq +\infty$, we defined

$$d(f,g) = \|f - g\|_p \quad (f,g \in L^p(\mathcal{T})).$$

Then it follows from Theorem 2.4 and Remark 2.5 that d satisfies all the axioms of a metric excepting that $d(f,g) = 0$ does not necessarily imply that $f \equiv g$. To remedy this situation, we define a relation on $L^p(\mathcal{T})$:

$f \sim g$ if and only if $d(f,g) = \|f-g\|_p = 0$. It is easy to see that $\sim$ is an equivalence relation on $L^p(\mathcal{T})$ and, therefore, partitioned $L^p(\mathcal{T})$ into equivalence classes. If $[f]$ and $[g]$ are equivalence classes and if α, β are complex numbers, we define

$$(1)\ \tilde{d}([f], [g]) = d(f,g) = \|f-g\|_p ;$$

$$(2)\ \alpha[f] + \beta[g] = [\alpha f + \beta g].$$

(1) is well defined since if $f_1 \sim f$, $g_1 \sim g$, then $d(f_1, f) = 0$, $d(g_1, g) = 0$. So that

$$\begin{aligned} d(f_1, g_1) &\leq d(f_1, f) + d(f, g_1) \\ &\leq d(f, g) + d(g, g_1) \\ &= d(f, g) \quad . \end{aligned}$$

Similarly, we have $d(f,g) \leq d(f_1, g_1)$. Hence $d(f, g) = d(f_1, g_1)$.

(2) is also well defined since if $f \sim f_1$ and $g \sim g_1$ implie $\alpha f + \beta g \sim \alpha f_1 + \beta g_1$.

With these operations, the set of all equivalence classes of $L^p(\mathcal{T})$ by $\sim$ forms a linear space with a metric $\tilde{d}$ which is compatible with its structure. From now on we shall also use the symbol <u>$L^p(\mathcal{T})$ for the metrizable linear space of equivalence classes</u>.

2.6 <u>Theorem</u>. $L^p(\mathcal{T})$ is a complete metric space for $1 \leq p \leq \infty$.

<u>Proof</u>. Consider $1 \leq p < \infty$.

Let $\{f_n\}$ be a Cauchy sequence in $L^p(\mathcal{T})$. Take $\varepsilon = \frac{1}{2}$, there exists $n_1 \in \mathbb{Z}\ (> 0)$ such that $\|f_n - f_{n_1}\|_p < \frac{1}{2}$ for all $n \geq n_1$. Suppose we have obtained a sequence $n_1 \leq n_2 \leq \ldots\ldots \leq n_k$. Then letting $\varepsilon = \frac{1}{2^k}$, there exists $n_k \geq n_{k-1}$ in $\mathbb{Z}\ (> 0)$ such that $\|f_n - f_{n_k}\|_p < \frac{1}{2^k}$ for all $n \geq n_k$. Hence we obtain a sequence $\{f_{n_i}\}$, $n_1 \leq n_2 \leq \ldots$, such that

$$(2\text{-}1) \qquad \|f_{n_{i+1}} - f_{n_i}\|_p \leq 2^{-i}, \quad \text{for } i = 1, 2, \ldots\ldots .$$

Define

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, \quad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

Since (2-1) holds, the Minkowski's inequality shows that, for any $k \in \mathbb{Z}\ (> 0)$,

$$\|g_k\|_p = \left(\int_{\mathcal{T}} |g_k|^p d\mu\right)^{\frac{1}{p}} = \left(\int_{\mathcal{T}} g_k{}^p d\mu\right)^{\frac{1}{p}}$$

$$\leq \sum_{i=1}^{k} \left(\int_{\mathcal{T}} |f_{n_{i+1}} - f_{n_i}|^p d\mu\right)^{\frac{1}{p}}$$

$$= \sum_{i=1}^{k} \| f_{n_{i-1}} - f_{n_i} \|_p < \sum_{i=1}^{k} 2^{-i} < \sum_{i=1}^{\infty} 2^{-i} = 1.$$

Hence an application of Fatou's lemma to $\{ g_k^{\,p} \}$ gives

$$\| g \|_p = \left(\int_{\mathcal{T}} g^p d\mu \right)^{\frac{1}{p}} = \left(\int_{\mathcal{T}} \lim_{k \to \infty} g_k^{\,p} d\mu \right)^{\frac{1}{p}}$$

$$\leq \lim_{k \to \infty} \left(\int_{\mathcal{T}} g_k^{\,p} d\mu \right)^{\frac{1}{p}} \leq 1.$$

And $g \in L^p(\mathcal{T})$ implies g is finite a.e on $\mathcal{T}$, so that the series $\sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i})$ converges absolutely a.e. on $\mathcal{T}$. Then the series

$$(2\text{-}2) \quad f_{n_1}(\dot{x}) + \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(\dot{x}) - f_{n_i}(\dot{x}) \right)$$

converges absolutely a.e on $\mathcal{T}$. We denote the sum of (2-2) by $f(\dot{x})$ for those $\dot{x}$ at which (2-2) converges, put $f(\dot{x}) = 0$ on the remaining set of measure zero. Since

$$f_{n_1}(\dot{x}) + \sum_{i=1}^{k-1} \left(f_{n_{i+1}}(\dot{x}) - f_{n_i}(\dot{x}) \right) = f_{n_k}(\dot{x}),$$

we see that

$$f(\dot{x}) = \lim_{k \to \infty} f_{n_k}(\dot{x}) \text{ a.e.}$$

Since $\{ f_n \}$ is a Cauchy sequence in $L^p(\mathcal{T})$. For any given $\varepsilon > 0$, there exists $N \in \mathbb{Z}\ (> 0)$ such that $\int_{\mathcal{T}} | f_n - f_m |^p d\mu < \varepsilon^p$ if $n \geq N$, $m \geq N$. For every $m \geq N$ and for some i onwards we have $n_i \geq N$ so that $\int_{\mathcal{T}} | f_{n_i} - f_m |^p d\mu < \varepsilon^p$. For every $f \geq N$, Fatou's lemma therefore shows that

$$(2\text{-}3) \quad \int_{\mathfrak{T}} |f-f_m|^p d\mu = \int_{\mathfrak{T}} \lim_{i\to\infty} |f_{n_i} - f_m|^p d\mu$$

$$\leq \varliminf_{i\to\infty} \int_{\mathfrak{T}} |f_{n_i} - f_m|^p d\mu$$

$$\leq \varepsilon^p .$$

We conclude from (2-3) that $f-f_m \in L^p(\mathfrak{T})$, hence that $f \in L^p(\mathfrak{T})$, and finally that $\|f_n - f_m\|_p$ tends to zero as m tends to ∞. This completes the proof for the case $1 \leq p < \infty$.

In $L^\infty(\mathfrak{T})$, suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mathfrak{T})$, let A_k and $B_{m,n}$ be the sets, respectively, where $|f_k(\dot{x})| > \|f_k\|_\infty$ and $|f_n(\dot{x}) - f_m(\dot{x})| > \|f_n - f_m\|_\infty$, and let E be the union of these sets, for $k, m, n = 1,2, \ldots\ldots$. Then $\mu(E) = 0$, and we show that on the complement of E the sequence $\{f_n\}$ converges uniformly to a bounded function. For any $\dot{x} \in E^c$, $\{f_n(\dot{x})\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete, so that $\lim_{n\to\infty} f_n(\dot{x}) = f(\dot{x})$. For any $\varepsilon > 0$, there exist n_0, $n_1 \in \mathbb{Z}$ (> 0) such that for all $n \geq n_0$, $|f_n(\dot{x}) - f(\dot{x})| < \frac{\varepsilon}{3}$ and for all $m \geq n_1$, $n \geq n_1$ $\|f_n - f_m\|_\infty < \frac{\varepsilon}{3}$. Let $n^* = \max(n_0, n_1)$. For any $n \geq n^*$ there is an $\dot{x}_0 \in E^c$ such that for $\dot{x} \in E^c$,

$$\sup_{x \in E^c} |f_n(\dot{x}) - f(\dot{x})| \leq |f_n(\dot{x}_0) - f(\dot{x}_0)| + \frac{\varepsilon}{3}$$

$$\leq |f_n(\dot{x}_0) - f_{n^*}(\dot{x}_0)| + |f_{n^*}(\dot{x}_0) - f(\dot{x}_0)| + \frac{\varepsilon}{3}$$

$$|f(\dot{x})| \leq |f(\dot{x}) - f_{n_o}(\dot{x})| + |f_{n_o}(x)| < \varepsilon + |f_{n_o}(\dot{x})|$$

$$\leq |f_{n_o}(\dot{x})| \leq ||f_{n_o}||_\infty < \infty.$$

Define $f(\dot{x}) = o$ for $\dot{x} \in E$. Then $f \in L^\infty(\Psi)$ and $||f_n - f||_\infty \longrightarrow o$ as $n \longrightarrow \infty$.

Hence the Theorem is now proved.