CHAPTER V

CONTINUOUS SOLUTION OF $f(x\circ y)+f(x\circ y^{-1})= 2f(x)+2f(y)$ ON SOME TOPOLOGICAL GROUPS

In this chapter, we shall determine all continuous solution of

$$(*) \qquad f(x\circ y)+f(x\circ y^{-1}) = 2f(x)+2f(y)$$

on a vector group V into a vector group V'. This result is then applied to obtain the continuous solutions of (*) on $\mathbb{R}^n$, $\mathbb{R}^+$, $\mathbb{R}^*$ and $\mathbb{C}^*$ into $\mathbb{R}$.

5.1 Some Useful Formulae

Theorem 5.1.1 Let $(G,\circ)$ and $(G',+)$ be abelian topological groups. Let H be a subgroup of G. A function $\bar{f} : G/_H \longrightarrow G'$ is continuous and satisfies

$$(\bar{*}) \qquad \bar{f}(x\circ H\circ y\circ H)+\bar{f}(x\circ H\circ(y\circ H)^{-1}) = 2\bar{f}(x\circ H)+2\bar{f}(y\circ H)$$

for all $x\circ H$, $y\circ H$ in $G/_H$ if and only if $\bar{f}(x\circ H) = f(x)$ for some continuous function $f : G \longrightarrow G'$ satisfies

$$(*) \qquad f(x\circ y)+f(x\circ y^{-1}) = 2f(x)+2f(y)$$

for all x, y in G and f is constant on each coset of H.

Proof Assume that $f : G \longrightarrow G'$ is continuous, satisfies (*) and is constant on each coset of H.

Let $\bar{f}$ be defined by

$$\bar{f}(x\circ H) = f(x).$$

Hence, by theorem 3.1.4, $\bar{f} : G/_H \longrightarrow G'$ satisfies

$$(\bar{*}) \qquad \bar{f}(x \circ H \circ y \circ H) + \bar{f}(x \circ H \circ (y \circ H)^{-1}) = 2\bar{f}(x \circ H) + 2\bar{f}(y \circ H),$$

for all $x \circ H$, $y \circ H$ in $G/_H$.

Observe that $f = \bar{f} \circ \Psi$, where $\Psi : G \longrightarrow G/_H$ is the canonical homomorphism of G into $G/_H$, i.e. Ψ defined by $\Psi(x) = x \circ H$ for all x in G.

Hence, by theorem 2.3.3, $\bar{f}$ is continuous.

Conversely, assume that $\bar{f} : G/_H \longrightarrow G'$ is continuous and satisfies $(\bar{*})$.

Let $f(x) = \bar{f}(x \circ H)$ for all x in G.

Hence, by theorem 3.1.4, f satisfies (*) on G.

Again, the continuity of f follows from theorem 2.3.3.

Remark 5.1.2 Let $(G_1,\circ)$, $(G_2,\circ)$ and $(G',+)$ be topological groups. Let $\nu : G_1 \longrightarrow G_2$ be an isomorphism and $f_2 : G_2 \longrightarrow G'$ be any function. Then function $f_1 = f_2 \circ \nu : G_1 \longrightarrow G'$ is continuous and satisfies

$$(*_1) \qquad f_1(x_1 \circ y_1) + f_1(x_1 \circ y_1^{-1}) = 2f_1(x_1) + 2f_1(y_1)$$

for all x_1, y_1 in G_1 if and only if f_2 is continuous and satisfies

$$(*_2) \qquad f_2(x_2 \circ y_2) + f_2(x_2 \circ y_2^{-1}) = 2f_2(x_2) + 2f_2(y_2)$$

for all x_2, y_2 in G_2.

Proof Assume that $f_2 : G_2 \longrightarrow G'$ is continuous and satisfies $(*_2)$. Then $f_1 = f_2 \circ \nu$, being the composition of two continuous function, is also continuous. Since f_2 satisfies $(*_2)$ and ν is an isomorphism of G_1 onto G_2. Hence, by Remark 3.1.5, f_1 satisfies $(*_1)$.

Conversely, assume that $f_1 : G_1 \longrightarrow G'$ is continuous and satisfies $(*_1)$. Then $f_2 = f_1 \circ \nu^{-1}$. By the same arguments, we see that f_2 is continuous and satisfies $(*_2)$.

5.2 Continuous Solution of $f(x+y)+f(x-y) = 2f(x)+2f(y)$ on Vector Group.

Theorem 5.2.1 Let V, V' be vector groups with $\mathcal{B} = \{v_\alpha : \alpha \in I\}$ as a basis of V and let $\mathcal{B}^{(1)} = \{\{v_\alpha\} : v_\alpha \in \mathcal{B}\}$, $\mathcal{B}^{(2)} = \{\{u,v\} : u,v \in \mathcal{B}, u \neq v\}$. Let $f : V \longrightarrow V'$ be a continuous function satisfying

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x,y in V. Then there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ such that for any $x = \sum_{i=1}^{n} y_{\alpha_i} v_{\alpha_i}$ in V, where $y_{\alpha_i} \in \mathbb{R}$ and $v_{\alpha_i} \in \mathcal{B}$, we have

$$(5.2.1.1) \quad f(x) = \sum_{1 \leq i < j \leq n} y_{\alpha_i} y_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{n} y_{\alpha_i}^2 c(\{v_{\alpha_i}\}) - \left(\sum_{i=1}^{n} y_{\alpha_i}\right) \sum_{i=1}^{n} y_{\alpha_i} c(\{v_{\alpha_i}\}).$$

Furthermore, if V is of finite dimensional, then any function f of the form (5.2.1.1) is continuous and satisfies (*).

Proof Assume that f is continuous and satisfies (*). Let G consist of elements x of V of the form $x = \sum_{i=1}^{n} \gamma_{\alpha_i} v_{\alpha_i}$, where $\gamma_{\alpha_i} \in Q$. Observe that the subset G of V is a vector space over Q and the restriction of f on G satisfies (*). Hence by theorem 4.1.1, there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ such that for any

$x = \sum_{i=1}^{n} \gamma_{\alpha_i} v_{\alpha_i}$ in G, where $\gamma_{\alpha_i} \in Q$ and $v_{\alpha_i} \in \mathcal{B}$, we have

$$(5.2.1.2)\quad f(x) = \sum_{1\le i<j\le n} \gamma_{\alpha_i}\gamma_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{n} \gamma_{\alpha_i}^2 c(\{v_{\alpha_i}\}) - \left(\sum_{i=1}^{n}\gamma_{\alpha_i}\right)\sum_{i=1}^{n}\gamma_{\alpha_i} c(\{v_{\alpha_i}\}).$$

Let x be any element in V.

Therefore $x = \sum_{i=1}^{n} y_{\alpha_i} v_{\alpha_i}$, where $y_{\alpha_i} \in \mathbb{R}$ and $v_{\alpha_i} \in \mathcal{B}$.

Since the set of rational numbers, Q is dense in $\mathbb{R}$, hence for each α_i, choose a sequence $\{\gamma_{\alpha_i}^{(m)}\}$ in Q such that $\lim_{m\to\infty} \gamma_{\alpha_i}^{(m)} = y_{\alpha_i}$.

Since f, addition and scalar multiplication are continuous, then

$$\begin{aligned}
f\left(\sum_{i=1}^{n} y_{\alpha_i} v_{\alpha_i}\right) &= f\left(\sum_{i=1}^{n} \lim_{m\to\infty} \gamma_{\alpha_i}^{(m)} v_{\alpha_i}\right), \\
&= f\left(\lim_{m\to\infty} \sum_{i=1}^{n} \gamma_{\alpha_i}^{(m)} v_{\alpha_i}\right), \\
&= \lim_{m\to\infty} f\left(\sum_{i=1}^{n} \gamma_{\alpha_i}^{(m)} v_{\alpha_i}\right), \\
&= \lim_{m\to\infty} \Big[\sum_{1\le i<j\le n} \gamma_{\alpha_i}^{(m)}\gamma_{\alpha_j}^{(m)} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{n} (\gamma_{\alpha_i}^{(m)})^2 c(\{v_{\alpha_i}\}) - \\
&\quad - \left(\sum_{i=1}^{n}\gamma_{\alpha_i}^{(m)}\right)\sum_{i=1}^{n}\gamma_{\alpha_i}^{(m)} c(\{v_{\alpha_i}\})\Big],
\end{aligned}$$

$$= \sum_{1\leq i< \leq n} y_{\alpha_i} y_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{n} y_{\alpha_i}^{2} c(\{v_{\alpha_i}\})$$

$$-\left(\sum_{i=1}^{n} y_{\alpha_i}\right) \sum_{i=1}^{n} y_{\alpha_i} c(\{v_{\alpha_i}\}).$$

Therefore $f(x) = \sum_{1\leq i<j\leq n} y_{\alpha_i} y_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{n} y_{\alpha_i}^{2} c(\{v_{\alpha_i}\})$

$$-\left(\sum_{i=1}^{n} y_{\alpha_i}\right) \sum_{i=1}^{n} y_{\alpha_i} c(\{v_{\alpha_i}\}),$$

for all $x = \sum_{i=1}^{n} y_{\alpha_i} v_{\alpha_i}$ in V.

In case V is of finite demensional, we may assume that $\mathcal{B} = \{v_1, \ldots, v_n\}$. Let $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ be given. For each $x = \sum_{i=1}^{n} y_i v_i$, let

$$f(x) = \sum_{1\leq i<j\leq n} y_i y_j c(\{v_i, v_j\}) + 2\sum_{i=1}^{n} y_i^2 c(\{v_i\}) - \left(\sum_{i=1}^{n} y_i\right) \sum_{i=1}^{n} y_i c(\{v_i\})$$

By a straightforward verification, it can be shown that f satisfies (*). To see that f is continuous, we define $p_i : V \longrightarrow \mathbb{R}$, $c_i : V \longrightarrow V'$, $c_{ij} : V \longrightarrow V'$ as follows :

$$p_i(x) = y_i \quad , \qquad \text{if } x = \sum_{i=1}^{n} y_i v_i ,$$

$$c_i(x) = c(\{v_i\}) \qquad \text{for all } x,$$

$$c_{ij}(x) = c(\{v_i, v_j\}) \qquad \text{for all } x.$$

It can be verified that p_i, c_i, c_{ij} are continuous. Observe that

$$f(x) = \sum_{1 \leq i < j \leq n} p_i(x)p_j(x)c_{ij}(x) + 2\sum_{i=1}^{n} (p_i(x))^2 c_i(x) - (\sum_{i=1}^{n} p_i(x)) \sum_{i=1}^{n} p_i(x)c_i(x).$$

It can be shown that for any continuous function $p : V \longrightarrow \mathbb{R}$ and any continuous function $c : V \longrightarrow V'$, the function $(p,c) : V \longrightarrow \mathbb{R} \times V'$ defined by

$$(p,c)(x) = (p(x), c(x)),$$

is continuous. Let $s : \mathbb{R} \times V' \longrightarrow V'$ denote the scalar multiplication on V'. Then

$$p_i(x)p_j(x)c_{ij}(x) = [s \circ (p_i, s \circ (p_j, c_{ij}))](x),$$

$$p_i(x)p_i(x)c_i(x) = [s \circ (p_i, s \circ (p_i, c_i))](x),$$

$$p_i(x)p_j(x)c_j(x) = [s \circ (p_i, s \circ (p_j, c_j))](x).$$

Therefore, f may be written in terms of a finite sum of compositions of continuous functions. Hence f is continuous.

Note If V is of infinite dimensional, then any function f of the form (5.2.1.1) is not necessary continuous. The following is a counterexample.

For each n = 1,2,3,..., let $X_n = \mathbb{R}$ with the usual topology.

Let $$X = \prod_{n=1}^{\infty} X_n .$$

Let $V = \{v | v \in X \text{ and } v(n) = 0 \text{ for all but a finite number of } n\}$.

Define addition and scalar multiplication on V by

$$(v_1 + v_2)(n) = v_1(n) + v_2(n),$$

$$(\lambda v)(n) = \lambda(v(n)).$$

Then V, as a subspace of X, is a vector group. Let

$$e_i(n) = \begin{cases} 1 \ , & \text{if } i = n \\ 0 \ , & \text{if } i \neq n \end{cases}$$

Then $\mathcal{B} = \{e_i : i = 1,2,\ldots\}$ is a basis of V.

Put $x_n = \frac{1}{n} e_n$. The sequence $\{x_n\}$ is a convergent sequence in V. It converges to $\bar{0} = (0,0,\ldots.)$. Let $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow R$ be given by

$$c(\{e_n\}) = n^2,$$

$$c(\{e_m, e_n\}) = 0.$$

For this choice of c, we have $f(x_n) = \frac{1}{n^2} c(\{e_n\}) = 1$. Observe that $\lim_{n \to \infty} f(x_n) = 1$. But $f(\bar{0}) = 0 \neq 1$. Therefore f is not continuous.

Corollary 5.2.2 Let $e_k = (\delta_{k1}, \ldots, \delta_{kn})$, $k = 1,\ldots,n$, where $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ if $k \neq j$. Let $\mathcal{B} = \{e_k : k = 1,\ldots, n\}$, $\mathcal{B}^{(1)} = \{\{e_k\} : e_k \in \mathcal{B}\}$ and $\mathcal{B}^{(2)} = \{\{e_i, e_j\} : e_i, e_j \in \mathcal{B}, i \neq j\}$. A continuous function f from $\mathbb{R}^n$ into a vector group V' satisfies

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}^n$ if and only if there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ such that for any $x = (x_1,\ldots, x_n)$ in $\mathbb{R}^n$ we have

$$f(x) = \sum_{1 \leq i<j \leq n} x_i x_j c(\{e_i, e_j\}) + 2\sum_{i=1}^{n} x_i^2 c(\{e_i\}) - (\sum_{i=1}^{n} x_i) \sum_{i=1}^{n} x_i c(\{e_i\}).$$

<u>Proof</u> Since $\mathbb{R}^n$ is a finite dimensional vector group having $\mathcal{B}$ as a basis, hence the theorem follows from theorem 5.2.1.

<u>Corollary 5.2.3</u> Let $\mathcal{B}$, $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ be as in the corollary 5.2.2. A continuous function f from $\mathbb{R}^n$ into $\mathbb{R}$ satisfies

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y),$$

for all x, y in $\mathbb{R}^n$ if and only if there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow \mathbb{R}$ such that for any $x = (x_1,\dots,x_n)$ in $\mathbb{R}^n$, we have

$$f(x) = \sum_{1\leq i<j\leq n} x_i x_j c(\{e_i,e_j\}) + 2\sum_{i=1}^{n} x_i^2 c(\{e_i\}) - \left(\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i c(\{e_i\}).$$

<u>Proof</u>. Since $\mathbb{R}$ is a vector group, hence the theorem follows immediately from corollary 5.2.2.

<u>Corollary 5.2.4</u> A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}$ if and only if $f(x) = Ax^2$ for all $x \in \mathbb{R}$, where A is a real number.

<u>Proof</u> It follows from corollary 5.2.3, that a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies (*) if and only if there exists a function $c : \mathcal{B}^{(1)} \longrightarrow \mathbb{R}$ where $\mathcal{B}^{(1)} = \{\{1\}\}$, such that for any x in $\mathbb{R}$, we have

$$f(x) = x^2 c(\{1\}).$$

Let $A = c(\{1\})$, hence

$$f(x) = Ax^2.$$

5.3 Continuous Solution $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on $\mathbb{R}^+$ into $\mathbb{R}$.

Theorem 5.3.1 Let $f : (\mathbb{R}^+,.) \longrightarrow (\mathbb{R},+)$. f is continuous and satisfies

$$(*) \qquad f(xy) + f(\frac{x}{y}) = 2f(x) + 2f(y)$$

for all x, y in $\mathbb{R}^+$ if and only if $f(y) = A(\ell n\ y)^2$ for all y in $\mathbb{R}^+$, where A is a real number.

Proof Let $\nu : (\mathbb{R}^+,.) \longrightarrow (\mathbb{R},+)$ be given by $\nu(x) = \ell n\ x$. Hence ν is an isomorphism from $\mathbb{R}^+$ onto $\mathbb{R}$.

Put $g = f \circ \nu^{-1}$, then $g : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$.

Hence, by remark 5.1.2, $f = g \circ \nu$ is continuous and satisfies (*) if and only if g is continuous and satisfies

$$(*_g) \qquad g(x+y)+g(x-y) = 2g(x)+2g(y)$$

for all x, y in $\mathbb{R}$.

Therefore by corollary 5.2.4, $g(x) = Ax^2$ for all $x \in \mathbb{R}$, where A is a real number. Thus for all y in $\mathbb{R}^+$,

$$\begin{aligned} f(y) &= g(\nu(y)), \\ &= g(\ell n\ y), \\ &= A(\ell n\ y)^2. \end{aligned}$$

Hence, f is continuous and satisfies (*) if and only if $f(y) = A(\ell n\ y)^2$ for all $y \in \mathbb{R}^+$, where A is a real number

5.4 Continuous Solution of $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on $\mathbb{R}^*$ into $\mathbb{R}$.

Theorem 5.4.1 Let $f : (\mathbb{R}^*,.) \longrightarrow (\mathbb{R},+)$. A function f is continuous and satisfies

$$(*) \qquad f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}^*$ if and only if $f(y) = A(\ln|y|)^2$ for all y in $\mathbb{R}^*$, where A is a real number.

Proof Assume that $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ is given by $f(y) = A(\ln|y|)^2$, for some real number A. It can be verified directly that f is continuous and satisfies (*).

Conversely, assume that f is continuous and satisfies (*).

Let $T = \{1,-1\}$. Hence T is a subgroup of $\mathbb{R}^*$. It can be verified as in the proof of theorem 4.5.1, that f is constant on each coset of T. Hence, by theorem 5.1.1, $\bar{f} : \mathbb{R}^*/_T \longrightarrow \mathbb{R}$ defined by

$$\bar{f}(xT) = f(x)$$

is continuous and satisfies

$$(\bar{*}) \qquad \bar{f}(xTyT)+\bar{f}(xT(yT)^{-1}) = 2\bar{f}(xT)+2\bar{f}(yT)$$

for all xT, yT in $\mathbb{R}^*/_T$.

Let $\nu : \mathbb{R}^*/_T \longrightarrow \mathbb{R}^+$ be defined by

$$\nu(xT) = |x|.$$

Then ν is an isomorphism from $\mathbb{R}^*/_T$ onto $\mathbb{R}^+$.

Set $g = \bar{f} \circ \nu^{-1}$, hence $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$

By remark 5.1.2, g is continuous and satisfies

$(*_g)$ $$g(xy)+g(\frac{x}{y}) = 2g(x)+2g(y)$$

for all x, y in $\mathbb{R}^+$. Therefore $g(x) = A(\ell n\, x)^2$ for all $x \in \mathbb{R}^+$, where A is a real number.

Also, from $g = \bar{f} \circ \nu^{-1}$, we have $\bar{f} = g \circ \nu$, hence $f(y) = \bar{f}(yT) = g(\nu(yT))$ $= g(|y|) = A(\ell n|y|)^2$.

5.5 Continuous Solution of $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on $\mathbb{C}^*$ into $\mathbb{R}$.

Lemma 5.5.1 Let $f : (\mathbb{R}^2,+) \longrightarrow (\mathbb{R},+)$ be continuous function satisfying

$(*)$ $$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}^2$. If f is constant on each coset of $H = \{(0,n):n \in \mathbb{Z}\}$, then $f(x_1,x_2) = Ax_1^2$ for all $(x_1,x_2) \in \mathbb{R}^2$, where A is a real number.

Proof Since f is continuous and satisfies $(*)$, hence by corollary 5.2.3, there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow \mathbb{R}$ where $\mathcal{B}^{(1)} = \{\{e_1\},\{e_2\}\}$, $\mathcal{B}^{(2)} = \{\{e_1,e_2\}\}$, $e_1 = (1,0)$, $e_2 = (0,1)$, such that for any $(x_1, x_2) \in \mathbb{R}^2$ we have

(5.5.1.1) $$f(x_1,x_2) = x_1x_2c(\{e_1,e_2\})+x_1(x_1-x_2)c(\{e_1\})+x_2(x_2-x_1)c(\{e_2\}).$$

Since f is constant on each coset of H, hence, for any $(y_1,y_2) \in \mathbb{R}^2$, we have

(5.5.1.2) $$f(y_1,y_2) = f(y_1,y_2+n).$$

By applying (5.5.1.1) to both sides of (5.5.1.2), we have

$y_1y_2c(\{e_1,e_2\})+y_1(y_1-y_2)c(\{e_1\})+y_2(y_2-y_1)c(\{e_2\}) = y_1(y_2+n)c(\{e_1,e_2\})+$

$y_1(y_1-y_2-n)c(\{e_1\})+(y_2+n)(y_2+n-y_1)c(\{e_2\})$.

After simplication, we have

$$n[y_1c(\{e_1,e_2\})-y_1c(\{e_1\})+y_2c(\{e_2\})+(y_2+n-y_1)c(\{e_2\})] = 0,$$

for all n and all y_1, $y_2 \in \mathbb{R}$. In particular, when n = 1, we have

(5.5.1.3) $y_1[c(\{e_1,e_2\})-c(\{e_1\})-c(\{e_2\})]+y_2[2c(\{e_2\})]+c(\{e_2\}) = 0,$

for all y_1, y_2 in $\mathbb{R}$. when $y_1 = y_2 = 0$, it follows from (5.5.1.3) that

(5.5.1.4) $$c(\{e_2\}) = 0.$$

Hence (5.5.1.3) becomes

$$y_1[c(\{e_1,e_2\})-c(\{e_1\})] = 0,$$

for all y_1 in $\mathbb{R}$. Therefore, we have

(5.5.1.5) $$c(\{e_1,e_2\}) = c(\{e_1\}).$$

Substituting the values $c(\{e_2\})$ from (5.5.1.4) and $c(\{e_1,e_2\})$ from (5.5.1.5) in (5.5.1.1) we get

$$f(x_1,x_2) = x_1^2\, c(\{e_1\}).$$

Let $A = c(\{e_1\})$. Hence

$f(x_1,x_2) = Ax_1^2$ for all $(x_1,x_2) \in \mathbb{R}^2$, where A is a real number.

Theorem 5.5.2 Let $f : (\mathbb{C}^*,\circ) \to (\mathbb{R},+)$. Then f is continuous and satisfies

(*) $$f(wz)+f(\tfrac{w}{z}) = 2f(w)+2f(z)$$

for all w, z in $\mathbb{C}^*$ if and only if $f(z) = A(\ell n|z|)^2$ for all $z \in \mathbb{C}^*$, where A is a real number.

Proof Assume that $f : (\mathbb{C}^*,.) \to (\mathbb{R},+)$ is given by $f(z) = A(\ell n|z|)^2$ for all z in $\mathbb{C}^*$, where A is a real number. It can be verified that f is continuous and satisfies (*).

Conversely, assume that f is continuous and satisfies (*).

Let $\nu : \mathbb{R}^2/_H \longrightarrow \mathbb{C}^*$ be defined by

$$\nu((x,y) + H) = e^{x+2\pi iy} ,$$

where $H = \{(0,n): n \in \mathbb{Z}\}$. Then ν is an isomorphism. Its inverse is given by

$$\nu^{-1}(z) = (\ell n|z|, \theta) + H ,$$

where θ is such that $|z|\, e^{2\pi i\theta} = z$.

Let $\bar{f} = f \circ \nu$.

Hence by remark 5.1.2, $\bar{f} : \mathbb{R}^2/_H \longrightarrow \mathbb{R}$ is continuous and satisfies

$$(\bar{*})\quad \bar{f}((x_1,y_1)+H+(x_2,y_2)+H)+\bar{f}((x_1,y_1)+H-((x_2,y_2)+H)) = 2\bar{f}((x_1,y_1)+H) +2\bar{f}((x_2,y_2)+H)$$

for all $(x_1,y_1)+H$, $(x_2,y_2)+H$ in $\mathbb{R}^2/_H$.

Let $f' : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f'(x,y) = \bar{f}((x,y)+H).$$

By theorem 5.1.1., f' is continuous and satisfies

$$(*')\quad f'((x_1,y_1)+(x_2,y_2))+f'((x_1,y_1)-(x_2,y_2)) = 2f'((x_1,y_1))+2f'((x_2,y_2)),$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ and f' is constant on each coset of H.

Hence, by lemma 5.5.1, we have

$$f'(x,y) = Ax^2 \text{ for all } (x,y) \in \mathbb{R}^2, \text{ where } A \text{ is a real number.}$$

Therefore, for any z in $\mathbb{C}^*$, we have

$$f(z) = \bar{f}(\nu^{-1}(z)) = \bar{f}((\ell n|z|,\ \theta)+H)) = f'(\ell n|z|,\theta) = A(\ell n|z|)^2.$$