CHAPTER IV

SOLUTION OF $f(x \circ y)+f(x \circ y^{-1}) = 2f(x)+2f(y)$ ON VECTOR SPACES OVER Q WITH APPLICATIONS TO CERTAIN GROUPS

In this chapter, we use our main theorem to obtain all functions f from a vector space V over Q into a vector space V' over Q. This result is then applied to obtain the solution of $f(x+y)+f(x-y)=2f(x)+2f(y)$ on $\mathbb{R}$, $\mathbb{R}^n$ and to obtain the solution of $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on the multiplicative group $(\mathbb{R}^+,.)$, $(\mathbb{R}^*,.)$.

4.1 Solution of $f(x+y)+f(x-y) = 2f(x)+2f(y)$ on Vector Space over Q.

Theorem 4.1.1 Let V and V' be vector spaces over Q with $\mathcal{B} = \{v_\alpha : \alpha \in I\}$ as a basis. Let $\mathcal{B}^{(1)} = \{\{v_\alpha\} : v_\alpha \in \mathcal{B}\}$ and $\mathcal{B}^{(2)} = \{\{u,v\} : u,v \in \mathcal{B}, u \neq v\}$. A function $f : V \longrightarrow V'$ satisfies

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in V if and only if there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ such that for any $x = \sum_{i=1}^{m} \gamma_{\alpha_i} v_{\alpha_i}$ in V, where $v_{\alpha_i} \in \mathcal{B}$ and $\gamma_{\alpha_i} \in Q$, we have

$$f(x) = \sum_{1 \le i < j \le m} \gamma_{\alpha_i}\gamma_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{m} \gamma_{\alpha_i}^2 c(\{v_{\alpha_i}\}) - \left(\sum_{i=1}^{m} \gamma_{\alpha_i}\right) \sum_{i=1}^{m} \gamma_{\alpha_i} c(\{v_{\alpha_i}\}).$$

<u>Proof</u> Observe that the additive group of rational number Q has generators $a_n = \frac{1}{n}$, n = 1,2,..., with defining relations

$$a_1 - na_n = 0.$$

Let M be the set of all these generators of Q.

Let $A = \{av : a\varepsilon M, v\varepsilon \mathcal{B}\}$, $A^{(1)} = \{\{y\} : y\varepsilon A\}$ and $A^{(2)} = \{\{s,t\} : s, t\varepsilon A, s\neq t\}$.

For any $x = \sum_{i=1}^{m} \gamma_{\alpha_i} v_{\alpha_i}$ in V, then $x = \sum_{i=1}^{m} \frac{P_{\alpha_i}}{N} v_{\alpha_i}$ where $\frac{P_{\alpha_i}}{N} = \gamma_{\alpha_i}$,

P_{α_i}, i= 1,..., m and N are integers such that $N \neq 0$.

Thus $x = \sum_{i=1}^{m} \gamma_{\alpha_i} v_{\alpha_i} = \sum_{i=1}^{m} \frac{P_{\alpha_i}}{N} v_{\alpha_i} = \sum_{i=1}^{m} P_{\alpha_i}(a_N v_{\alpha_i})$ for some N.

Since $a_N v_{\alpha_i}$, i = 1,..., m are in A, therefore A is a set of generators of V with defining relations

$$a_1 v_{\alpha_i} - na_n v_{\alpha_i} = 0.$$

Hence, according to theorem 3.2.1, $f: V \longrightarrow V'$ satisfies (*) if and only if there exists a function $c : A^{(1)} \cup A^{(2)} \longrightarrow V'$ such that for any defining relation

$$\text{(n)} \qquad a_1 v_{\alpha_i} - na_n v_{\alpha_i} = 0,$$

we have

$$\text{(i-n)} \quad -nc(\{a_1 v_{\alpha_i}, a_n v_{\alpha_i}\}) + 2(c(\{a_1 v_{\alpha_i}\}) + n^2 c(\{a_n v_{\alpha_i}\})) - (1-n)(c(\{a_1 v_{\alpha_i}\}) - nc(\{a_n v_{\alpha_i}\})) = 0,$$

(ii-n) $c(\{a_1v_{\alpha_i},x_\beta\})-nc(\{a_nv_{\alpha_i},x_\beta\})-(c(\{a_1v_{\alpha_i}\})-nc(\{a_nv_{\alpha_i}\}))$

$-(1-n)c(\{x_\beta\}) = 0,$

for all $x_\beta \neq a_1v_{\alpha_i}, a_nv_{\alpha_i}$ and

(iii-n)
$$\begin{cases} -nc(\{a_nv_{\alpha_i},a_1v_{\alpha_i}\})-(-n)c(\{a_nv_{\alpha_i}\})-(-n)c(\{a_1v_{\alpha_i}\})+2c(\{a_1v_{\alpha_i}\})= 0, \\ c(\{a_1v_{\alpha_i},a_nv_{\alpha_i}\})-c(\{a_1v_{\alpha_i}\})-c(\{a_nv_{\alpha_i}\})+2(-n)c(\{a_nv_{\alpha_i}\}) = 0, \end{cases}$$

and for any $x = \sum_{i=1}^{m} p_{\alpha_i}(a_N V_{\alpha_i})$ in V,

(4.1.1.1) $$f(x) = \sum_{1\leq i<j\leq m} p_{\alpha_i}p_{\alpha_j}c(\{a_Nv_{\alpha_i},a_Nv_{\alpha_j}\})+2\sum_{i=1}^{m} p_{\alpha_i}^2c(\{a_Nv_{\alpha_i}\})-$$

$$-(\sum_{i=1}^{m} p_{\alpha_i})\sum_{i=1}^{m} p_{\alpha_i}\, c(\{a_Nv_{\alpha_i}\}).$$

From the second equation in (iii-n) we find that

(4.1.1.2) $$c(\{a_1v_{\alpha_i},a_nv_{\alpha_i}\}) = c(\{a_1v_{\alpha_i}\})+(2n+1)c(\{a_nv_{\alpha_i}\}).$$

By replacing the value of $c(\{a_1v_{\alpha_i}, a_nv_{\alpha_i}\})$ from (4.1.1.2) in (i-n) and simplify the result we have

(4.1.1.3) $$c(\{a_nv_{\alpha_i}\}) = \frac{1}{n^2}\, c(\{a_1v_{\alpha_i}\}).$$

Substituting the value of $c(\{a_nv_{\alpha_i}\})$ from (4.1.1.3) in (4.1.1.2), we get

(4.1.1.4) $$c(\{a_1v_{\alpha_i},a_nv_{\alpha_i}\}) = \frac{1}{n^2}(1+n)^2\, c(\{a_1v_{\alpha_i}\}).$$

In case $x_\beta = a_1 v_\gamma$, where $v_\gamma \neq v_{\alpha_i}$, it follows from (ii-n) that

$$c(\{a_1 v_{\alpha_i}, a_1 v_\gamma\}) - nc(\{a_n v_{\alpha_i}, a_1 v_\gamma\}) - c(\{a_1 v_{\alpha_i}\}) + nc(\{a_n v_{\alpha_i}\}) - (1-n)c(\{a_1 v_\gamma\}) = 0.$$

Hence

$$(4.1.1.5)\quad c(\{a_n v_{\alpha_i}, a_1 v_\gamma\}) = \frac{1}{n}[c(\{a_1 v_{\alpha_i}, a_1 v_\gamma\}) - c(\{a_1 v_{\alpha_i}\}) + nc(\{a_n v_{\alpha_i}\}) - (1-n)c(\{a_1 v_\gamma\})].$$

Substituting the value of $c(\{a_n v_{\alpha_i}\})$ from (4.1.1.3) in (4.1.1.5) we find that

$$(4.1.1.6)\quad c(\{a_n v_{\alpha_i}, a_1 v_\gamma\}) = \frac{1}{n}c(\{a_1 v_{\alpha_i}, a_1 v_\gamma\}) + \frac{1}{n}(\frac{1}{n} - 1)c(\{a_1 v_{\alpha_i}\}) + (1 - \frac{1}{n})c(\{a_1 v_\gamma\}).$$

In case $x_\beta = a_\delta v_\gamma$, where $\delta \neq 1$ and $v_\gamma \neq v_{\alpha_i}$, it follows from (ii-n) that

$$c(\{a_1 v_{\alpha_i}, a_\delta v_\gamma\}) - nc(\{a_n v_{\alpha_i}, a_\delta v_\gamma\}) - c(\{a_1 v_{\alpha_i}\}) + nc(\{a_n v_{\alpha_i}\}) - (1-n)c(\{a_\delta v_\gamma\}) = 0.$$

Hence

$$c(\{a_n v_{\alpha_i}, a_\delta v_\gamma\}) = \frac{1}{n}[c(\{a_1 v_{\alpha_i}, a_\delta v_\gamma\}) - c(\{a_1 v_{\alpha_i}\}) + nc(\{a_n v_{\alpha_i}\}) - (1-n)c(\{a_\delta v_\gamma\})]$$

Applying (4.1.1.3) and (4.1.1.5) to the right hand side of this equation, we get

$$c(\{a_n v_{\alpha_i}, a_\delta v_\gamma\}) = \frac{1}{n}[\frac{1}{\delta}\{c(\{a_1 v_{\alpha_i}, a_1 v_\gamma\}) - c(\{a_1 v_\gamma\}) + \delta c(\{a_\delta v_\gamma\}) - (1-\delta)c(\{a_1 v_{\alpha_i}\})\} - c(\{a_1 v_{\alpha_i}\}) + nc(\{a_n v_{\alpha_i}\}) - (1-n)c(\{a_\delta v_\gamma\})],$$

$$= \frac{1}{n}[\frac{1}{\delta}c(\{a_1v_{\alpha_i}, a_1v_\gamma\}) - \frac{1}{\delta}c(\{a_1v_\gamma\}) + \frac{1}{\delta^2}c(\{v_\gamma\}) - \frac{1}{\delta}(1-\delta)c(\{a_1v_{\alpha_i}\})$$

$$-c(\{a_1v_{\alpha_i}\}) + \frac{1}{n}c(\{v_{\alpha_i}\}) - \frac{1}{\delta^2}(1-n)c(\{a_1v_\gamma\})].$$

Therefore

$$(4.1.1.7) \quad c(\{a_nv_{\alpha_i}, a_\delta v_\gamma\}) = \frac{1}{n\delta}c(\{a_1v_{\alpha_i}, a_1v_\gamma\}) + \frac{1}{n}(\frac{1}{n} - \frac{1}{\delta})c(\{a_1v_{\alpha_i}\})$$

$$+ \frac{1}{\delta}(\frac{1}{\delta} - \frac{1}{n})c(\{a_1v_\gamma\})$$

In case $x_\beta = a_\beta v_{\alpha_i}$, where $\beta \neq 1$, n, it follows from (ii-n) that

$$c(\{a_1v_{\alpha_i}, a_\beta v_{\alpha_i}\}) - nc(\{a_nv_{\alpha_i}, a_\beta v_{\alpha_i}\}) - c(\{a_1v_{\alpha_i}\}) + nc(\{a_nv_{\alpha_i}\}) - (1-n)c(\{a_\beta v_{\alpha_i}\}) = 0.$$

Hence

$$c(\{a_nv_{\alpha_i}, a_\beta v_{\alpha_i}\}) = \frac{1}{n}[c(\{a_1v_{\alpha_i}, a_\beta v_{\alpha_i}\}) - c(\{a_1v_{\alpha_i}\}) + nc(\{a_nv_{\alpha_i}\}) - (1-n)c(\{a_\beta v_{\alpha_i}\})]$$

Applying (4.1.1.3) and (4.1.1.4) to the right hand side of this equation, we get

$$(4.1.1.8) \qquad c(\{a_nv_{\alpha_i}, a_\beta v_{\alpha_i}\}) = \frac{1}{n^2\beta^2}(n+\beta)^2 c(\{a_1v_{\alpha_i}\}),$$

for all n and all $\beta \neq 1$, n.

It can be verified that the values of $c(\{a_nv_{\alpha_i}\})$, $c(\{a_1v_{\alpha_i}, a_nv_{\alpha_i}\})$, $c(\{a_nv_{\alpha_i}, a_1v_\gamma\})$, $c(\{a_nv_{\alpha_i}, a_\delta v_\gamma\})$, $c(\{a_nv_{\alpha_i}, a_\beta v_{\alpha_i}\})$ given in (4.1.1.3), (4.1.1.4), (4.1.1.6), (4.1.1.7), (4.1.1.8) satisfy (i-n), (ii-n) and (iii-n). Substituting these values in (4.1.1.1), we get

$$f(x) = \sum_{1\le i<j\le m} \frac{p_{\alpha_i} p_{\alpha_j}}{N^2} c(\{a_1 v_{\alpha_i}, a_1 v_{\alpha_j}\}) + 2\sum_{i=1}^{m} \frac{p_{\alpha_i}^2}{N^2} c(\{a_1 v_{\alpha_i}\}) -$$

$$-\left(\sum_{i=1}^{m} p_{\alpha_i}\right) \sum_{i=1}^{m} \frac{p_{\alpha_i}}{N^2} c(\{a_1 v_{\alpha_i}\}).$$

Hence

$$f(x) = \sum_{1\le i<j\le m} \frac{p_{\alpha_i}}{N} \cdot \frac{p_{\alpha_j}}{N} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{m} \left(\frac{p_{\alpha_i}}{N}\right)^2 c(\{v_{\alpha_i}\}) -$$

$$-\left(\sum_{i=1}^{m} \frac{p_{\alpha_i}}{N}\right) \sum_{i=1}^{m} \frac{p_{\alpha_i}}{N} c(\{v_{\alpha_i}\})$$

Therefore

$$f(x) = \sum_{1\le i<j\le m} \gamma_{\alpha_i}\gamma_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{m} \gamma_{\alpha_i}^2 c(\{v_{\alpha_i}\}) - \left(\sum_{i=1}^{m} \gamma_{\alpha_i}\right) \sum_{i=1}^{m} \gamma_{\alpha_i} c(\{v_{\alpha_i}\}).$$

4.2 Solution of $f(x+y)+f(x-y) = 2f(x)+2f(y)$ on $\mathbb{R}$ into Vector Space over Q

Theorem 4.2.1 Let V' be any vector space over Q and $\mathcal{H} = \{v_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. Let $\mathcal{H}^{(1)} = \{\{v_\alpha\} : v_\alpha \in \mathcal{H}\}$, $\mathcal{H}^{(2)} = \{\{u,v\} : u,v \in \mathcal{H},\ u \ne v\}$. Then $f : \mathbb{R} \longrightarrow V'$ satisfies

(*) $\qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$

for all x, y in $\mathbb{R}$ if and only if there exists a function $c : \mathcal{H}^{(1)} \cup \mathcal{H}^{(2)} \longrightarrow V'$ such that for any $x = \sum_{i=1}^{m} \gamma_{\alpha_i} v_{\alpha_i}$ in $\mathbb{R}$, where $\gamma_{\alpha_i} \in Q$ and $v_{\alpha_i} \in \mathcal{H}$, we have

$$f(x) = \sum_{1\le i<j\le m} \gamma_{\alpha_i}\gamma_{\alpha_j} c(\{v_{\alpha_i}, v_{\alpha_j}\}) + 2\sum_{i=1}^{m} \gamma_{\alpha_i}^2 c(\{v_{\alpha_i}\}) - \left(\sum_{i=1}^{m} \gamma_{\alpha_i}\right) \sum_{i=1}^{m} \gamma_{\alpha_i} c(\{v_{\alpha_i}\}).$$

Proof   Since $\mathbb{R}$ is a vector space over Q having $\mathcal{H}$ as a basis, hence the theorem follows immediately from theorem 4.1.1.

4.3 Solution of $f(x+y)+f(x-y) = 2f(x)+2f(y)$ on $\mathbb{R}^n$ into Vector Space over Q.

Theorem 4.3.1   Let $\mathcal{H} = \{v_\alpha : \alpha\varepsilon I\}$ be a Hamel basis of $\mathbb{R}$ over Q. Let $e_k = (\delta_{k1},\ldots,\delta_{kn})$, $k = 1,\ldots,n$, where $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ if $k \neq j$. Let $\mathcal{B} = \{v_\alpha e_k : \alpha\varepsilon I, k = 1,\ldots,n\}$, $\mathcal{B}^{(1)} = \{\{b\} : b\varepsilon\mathcal{B}\}$ and $\mathcal{B}^{(2)} = \{\{a,b\} : a,b\varepsilon\ \mathcal{B}, a \neq b\}$.

A function f from $\mathbb{R}^n$ into a vector space V' over Q satisfies

$$(*)\qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}^n$ if and only if there exists a function $c : \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} \longrightarrow V'$ such that for any $x = (x_1,\ldots,x_n)$, where $x_k = \sum_{i=1}^{m} \gamma_{ki} v_{\alpha_i}$, $\gamma_{ki}\varepsilon\ Q$, $k = 1,\ldots,n$, we have

$$f(x) = \sum_{\substack{i,i'=1,\ldots,m\\ k,k'=1,\ldots,n\\ (i,k)\neq(i',k')}} \gamma_{ki}\gamma_{k'i'} c(\{v_{\alpha_i}e_k, v_{\alpha_{i'}}e_{k'}\}) + 2\sum_{k=1}^{n}\sum_{i=1}^{m}\gamma_{ki}^2 c(\{v_{\alpha_i}e_k\})$$

$$-\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\gamma_{ki}\right)\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\gamma_{ki}\, c(\{v_{\alpha_i}e_k\})\right).$$

Proof   Observe that $\mathbb{R}^n$ is a vector space over Q having $\mathcal{B}$ as a basis. Hence the theorem follows immediately from theorem 4.1.1.

4.4 Solution of $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on $\mathbb{R}^+$ into Vector Space V' over Q.

Theorem 4.4.1 A function f from the multiplication group $\mathbb{R}^+$ into a vector space V' over Q satisfies

$$(*) \qquad f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$$

for all x, y in $\mathbb{R}^+$ if and only if there exists a function F on $\mathbb{R}$ into V' satisfying

$$(*_F) \qquad F(x+y)+F(x-y) = 2F(x)+2F(y)$$

for all x, y in $\mathbb{R}$ such that $f = F_{o}\ell n$.

Proof Since ℓn is an isomorphism from $(\mathbb{R}^+,.)$ onto $(\mathbb{R},+)$, hence the theorem is follows from remark 3.1.5.

4.5 Solution of $f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$ on $\mathbb{R}^*$ into vector Space V' over Q.

Theorem 4.5.1 A function f from the multiplicative group $\mathbb{R}^*$ into vector space V' over Q. satisfies

$$(*) \qquad f(xy)+f(\frac{x}{y}) = 2f(x)+2f(y)$$

for all x,y in $\mathbb{R}^*$ if and only if there exists a function $g : (\mathbb{R}^+,\cdot) \longrightarrow (V',+)$ satisfying

$$(*_g) \qquad g(xy)+g(\frac{x}{y}) = 2g(x)+2g(y)$$

for all x, y in $\mathbb{R}^+$ such that $f(x) = g(|x|)$ for all $x \in \mathbb{R}^*$.

Proof Assume that $f : \mathbb{R}^* \longrightarrow V'$ satisfies (*).

Let $T = \{1,-1\}$. Hence T is a subgroup of $\mathbb{R}^*$.

We claim that f is constant on each coset of T.

Let $y \in xT$. Then $y = xt$, $t \in T$. Hence $t^2 = 1$.

By (3.1.1.3) of proposition 3.1.1, we have

$$4f(y) = f(y^2) = f((xt)^2) = f((xt)(xt)) = f(x^2t^2) = f(x^2) = 4f(x).$$

Thus

$$4[f(y) - f(x)] = 0$$

Hence

$$f(y) = f(x).$$

Therefore f is constant on each coset of T.

Hence, by theorem 3.1.4, there exists $\bar{f} : \mathbb{R}^*/_T \longrightarrow V'$ such that

$$(\bar{*}) \qquad \bar{f}(xTyT)+\bar{f}(xT/yT) = 2\bar{f}(xT)+2\bar{f}(yT),$$

for all xT, yT in $\mathbb{R}^*/_T$ and $\bar{f}(xT) = f(x)$.

Let $\nu : \mathbb{R}^*/_T \longrightarrow \mathbb{R}^+$ be given by

$$\nu(xT) = |x|.$$

Then ν is an isomorphism from $\mathbb{R}^*/_T$ onto $\mathbb{R}^+$.

Set $g = \bar{f} \circ \nu^{-1}$, hence $g : \mathbb{R}^+ \longrightarrow V'$.

By remark 3.1.5, g satisfies

$$(*_g) \qquad g(xy)+g(\tfrac{x}{y}) = 2g(x)+2g(y)$$

for all x, y in $\mathbb{R}^+$.

Also, from $g = \bar{f} \circ \nu^{-1}$, we have $\bar{f} = g \circ \nu$, hence $f(x) = \bar{f}(xT) = g(\nu(xT))$ $= g(|x|)$.

Conversely, assume that we are given $g : \mathbb{R}^{+} \longrightarrow V'$ satisfying $(*_g)$.
Let $f : \mathbb{R}^{*} \longrightarrow V'$ be defined by

$$f(x) = g(|x|).$$

For any x, y in $\mathbb{R}^{*}$ we have

$$\begin{aligned} f(xy) + f(\tfrac{x}{y}) &= g(|xy|) + g(|\tfrac{x}{y}|) , \\ &= g(|x||y|) + g(\tfrac{|x|}{|y|}) , \\ &= 2g(|x|) + 2g(|y|), \\ &= 2f(x) + 2f(y). \end{aligned}$$