CHAPTER II

ON THE SUBHARMONICITY OF $|F(Z)|^p$

2.1 Notation

As in the case of two variables, we may characterize a system of conjugate harmonic functions in term of differential equations. The n-tuple $F = (u_1, u_2, \ldots, u_n)$ of harmonic functions forms a system of conjugate harmonic functions if and only if it satisfies the analogue of the Cauchy-Riemann equations

$$\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \, , \quad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \, , \quad i \neq j \, . \tag{2.1.1}$$

We first find a result that will replace the two dimensional result that $|F(Z)|^p$, $p > 0$ is subharmonic when $F(Z)$ is analytic

Let F be a system of conjugate harmonic functions and denote by $|F|$ the norm $(u_1^2 + u_2^2 + \ldots + u_n^2)^{\frac{1}{2}}$. We thus begin by asking the question : Is the function $|F|^p$ a subharmonic function of variables $x_1, x_2, \ldots, x_n$?

2.2 In this section we answer the above question by the following theorem.

<u>Theorem 2.2.1.</u> $|F|^p$ is subharmonic if $p \geq \frac{n-2}{n-1}$.

<u>Proof</u> Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator.

By theorem 1.3, it suffices to show that $\Delta(|F|^p) \geq 0$.

And toward this end, we begin by calculating $\Delta(|F|^p)$ and expressing our result in vector notation.

If $G = (v_1, v_2, \ldots, v_n)$ is another function, we let $F.G = u_1v_1 + u_2v_2 + \ldots + u_nv_n$ be the inner product of F and G. We note that $F.G = G.F$.

For $k = 1,2,\ldots,n$, we let

$$G_{x_k} = \left(\frac{\partial v_1}{\partial x_k}, \frac{\partial v_2}{\partial x_k}, \ldots, \frac{\partial v_n}{\partial x_k}\right), \quad \text{then}$$

$$\frac{\partial}{\partial x_k}(G.F) = G_{x_k}.F + G.F_{x_k}.$$

Thus

$$\frac{\partial}{\partial x_k}(|F|^p) = \frac{\partial}{\partial x_k}(F.F)^{\frac{1}{2}p}$$

$$= \frac{p}{2}(F.F)^{\frac{1}{2}(p-2)}\frac{\partial}{\partial x_k}(F.F)$$

$$= \frac{p}{2}|F|^{p-2}(2.F.F_{x_k})$$

$$= p\,|F|^{p-2}(F_{x_k}.F)$$

hence

$$\frac{\partial^2}{\partial x_k^2}(|F|^p) = \frac{\partial}{\partial x_k}(p|F|^{p-2}(F_{x_k}.F))$$

$$= \frac{1}{2}p(p-2)(F.F)^{\frac{p}{2}-2}(F_{x_k}.F)\frac{\partial}{\partial x_k}(F.F) +$$

$$p(F.F)^{\frac{p}{2}-1}\frac{\partial}{\partial x_k}(F_{x_k}.F)$$

$$= p(p-2)|F|^{p-4}(F.F_{x_k})^2 + p|F|^{p-2}\{|F_{x_k}|^2 + F.F_{x_kx_k}\}$$

Summing over k we get

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}(|F|^p) = p(p-2)|F|^{p-4} \sum_{k=1}^{n} (F.F_{x_k})^2 + p|F|^{p-2} \sum_{k=1}^{n} (|F_{x_k}|^2 + F.F_{x_k x_k}) .$$

Consider the term $\sum_{k=1}^{n} F.F_{x_k x_k}$

$$\sum_{k=1}^{n} F.F_{x_k x_k} = \sum_{k=1}^{n} [u_1 \frac{\partial^2 u_1}{\partial x_k^2} + u_2 \frac{\partial^2 u_2}{\partial x_k^2} + \dots + u_n \frac{\partial^2 u_n}{\partial x_k^2}]$$

$$= u_1 \sum_{k=1}^{n} \frac{\partial^2 u_1}{\partial x_k^2} + u_2 \sum_{k=1}^{n} \frac{\partial^2 u_2}{\partial x_k^2} + \dots + u_n \sum_{k=1}^{n} \frac{\partial^2 u_n}{\partial x_k^2} .$$

Since the components of F are harmonic, $\sum_{k=1}^{n} \frac{\partial^2 u_i}{\partial x_k^2} = 0$

for all $i = 1,2,\dots, n.$, then

$$\sum_{k=1}^{n} F.F_{x_k x_k} = 0 .$$

Therefore, $$\Delta(|F|^p) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}(|F|^p)$$

$$= p(p-2)|F|^{p-4} \sum_{k=1}^{n} (F.F_{x_k})^2 +$$

$$p|F|^{p-2} \sum_{k=1}^{n} |F_{x_k}|^2 . \qquad (2.2.2)$$

We see that, from the equation (2.2.2), $\Delta(|F|^p)$ fails to be defined only when $F(X) = 0$ (for $p < 4$). But, if $F(X) = 0$ at some point X, since $|F|^p \geq 0$, the mean value property of subharmonic function must hold at X. Thus inorder to establish the subharmonicity of $|F|^p$ it suffices to show $\Delta(|F|^p) \geq 0$ whenever the latter is defined. Thus we may assume that F is never be zero vector.

Note that $\Delta(|F|^p) \geq 0$ is obviously true if $p \geq 2$. If $1 \leq p < 2$, using Schwarz's inequality, $(F_{x_k} \bullet F)^2 \leq |F_{x_k}|^2|F|^2$ we have

$$\Delta(|F|^p) \geq p(p-2)|F|^{p-4} \sum_{k=1}^{n} |F_{x_k}|^2|F|^2 + p|F|^{p-2} \sum_{k=1}^{n} |F_{x_k}|^2$$

$$\geq p(p-1)|F|^{p-2} \sum_{k=1}^{n} |F_{x_k}|^2 \geq 0 \quad .$$

The result that $\Delta(|F|^p) \geq 0$ for value of p less than 1 depends on the following lemma.

<u>Lemma 2.2.4</u> Suppose that

$$\mathfrak{M} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is a symmetric matrix with trace ($= \sum_{i=1}^{n} a_{ii}$) zero. Let $\|\mathfrak{M}\|$ be the norm of $\mathfrak{M}$, which is defined by

$\|\mathfrak{M}\| = \sup|\mathfrak{M}A|$ where the supremum is taken over all vector $A = (a_1, a_2, \ldots, a_n)$ such that $|A| = (|a_1|^2+|a_2|^2+\ldots+|a_n|^2)^{\frac{1}{2}} \leq 1$

$$|||\mathcal{M}||| = \sqrt{\sum_{i,j} |a_{ij}|^2}, \text{ the Hilbert-Schmidt norm of } \mathcal{M}.$$

Then $\|\mathcal{M}\|^2 \leq \frac{n-1}{n} |||\mathcal{M}|||^2$.

Proof of the lemma. By [5] pages 173-175, every symmetric matrix can be reduced to a diagonal form by using a unitary transformation. Thus we can reduce $\mathcal{M}$ to a diagonal matrix.

Since $\|\mathcal{M}\|$ and $|||\mathcal{M}|||$ are unitary invariant and $\mathcal{M}$ is symmetric, we can assume that $\mathcal{M}$ is a diagonal matrix. Thus we have

$$\mathcal{M} = \begin{bmatrix} \lambda_1 & 0 & 0 \dots 0 \\ 0 & \lambda_2 & 0 \dots 0 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 0 & 0 & \dots 0 \quad \lambda_n \end{bmatrix}.$$

Then $\|\mathcal{M}\| = \sup_{|A|\leq 1} |\mathcal{M}A| = \sup_{(a_1^2+a_2^2+\dots+a_n^2)\leq 1} (\lambda_1^2 a_1^2 + \lambda_2^2 a_2^2 + \dots + \lambda_n^2 a_n^2)^{\frac{1}{2}}$.

Therefore, $\|\mathcal{M}\|^2 = \sup_{|A|=1} (\lambda_1^2 a_1^2 + \lambda_2^2 a_2^2 + \dots + \lambda_n^2 a_n^2) \geq \lambda_k^2, \forall k$.

That is $\|\mathcal{M}\|^2 \geq \max \{\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2\}$ (2.2.5)

and $(a_1^2\lambda_1^2 + a_2^2\lambda_2^2 + \dots + a_n^2\lambda_n^2) \leq (a_1^2 + a_2^2 + \dots + a_n^2) \max_{1\leq k\leq n} \{\lambda_k^2\}$

$$\leq \max \{\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2\}.$$

Therefore, $\|\mathcal{M}\|^2 \leq \max \{\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2\}$. (2.2.6)

By (2.2.5) and (2.2.6) we have $\| \mathcal{M} \|^2 = \max \{\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\}$

and $$|||\mathcal{M}|||^2 = \sum_{i=1}^{n} \lambda_i^2 .$$

Since the trace of a matrix is also invariant under unitary transformation, we have $\sum_{i=1}^{n} \lambda_i = 0$.

We now show that for $k = 1,2,3,\ldots,n$

$$\lambda_k^2 \leq \frac{n-1}{n} \left(\sum_{i=1}^{n} \lambda_i^2\right).$$

By Schwarz's inequality $|x \cdot y| \leq |x||y|$

$$\left| \sum_{i \neq k} \lambda_i \right| = \left| \sum_{i \neq k} 1.\lambda_i \right| \leq (n-1)^{\frac{1}{2}} \left(\sum_{i \neq k} \lambda_i^2 \right)^{\frac{1}{2}} .$$

Since

$$\sum_{i=1}^{n} \lambda_i = \sum_{i \neq k} \lambda_i + \lambda_k = 0$$

$$\lambda_k = -\sum_{i \neq k} \lambda_i$$

$$\lambda_k^2 = \left(\sum_{i \neq k} \lambda_i \right)^2 ;$$

Therefore $$\lambda_k^2 = \left(\sum_{i \neq k} \lambda_i \right)^2 \leq (n-1)\left(\sum_{i \neq k} \lambda_i^2 \right)$$

$$= (n-1) \sum_{i=1}^{n} \lambda_i^2 - (n-1)\lambda_k^2 .$$

That is $$\lambda_k^2 \leq \frac{n-1}{n} \sum_{i=1}^{n} \lambda_i^2 .$$

Thus $$\|\mathcal{M}\|^2 \leq \frac{n-1}{n} |||\mathcal{M}|||^2$$

and the lemma is proved.

To apply lemma 2.2.4 to our problem we let

$$\mathcal{M} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{bmatrix}.$$

The trace of the matrix is now $\sum_{i=1}^{n} (\frac{\partial u_i}{\partial x_i}) = 0$ by (2.1.1), since F is a system of conjugate harmonic functions. And also $\mathcal{M}$ is a symmetric matrix by (2.1.1).

Since $$F.F_{x_k} = u_1\frac{\partial u_1}{\partial x_k} + u_2\frac{\partial u_2}{\partial x_k} + \ldots + u_n \frac{\partial u_n}{\partial x_k}$$

$$\sum_{k=1}^{n} (F.F_{x_k})^2 = \sum_{k=1}^{n} (u_1\frac{\partial u_1}{\partial x_k} + u_2\frac{\partial u_2}{\partial x_k} + \ldots + u_n \frac{\partial u_n}{\partial x_k})^2$$

$$\mathcal{M}F = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$|\mathcal{M}F|^2 = \sum_{k=1}^{n} (u_1\frac{\partial u_1}{\partial x_k} + u_2\frac{\partial u_2}{\partial x_k} + \ldots + u_n\frac{\partial u_n}{\partial x_k})^2 .$$

We then see that $|\mathcal{M}F|^2 = \sum_{k=1}^{n} (F.F_{x_k})^2$

$$|F_{x_k}|^2 = (\frac{\partial u_1}{\partial x_k})^2 + (\frac{\partial u_2}{\partial x_k})^2 + \dots + (\frac{\partial u_n}{\partial x_k})^2$$

$$\sum_{k=1}^{n} |F_{x_k}|^2 = \sum_{k=1}^{n} [(\frac{\partial u_1}{\partial x_k})^2 + (\frac{\partial u_2}{\partial x_k})^2 + \dots + (\frac{\partial u_n}{\partial x_k})^2]$$

$$= |||\mathcal{M}|||^2.$$

The equation (2.2.2) can be reduced to

$$\Delta(|F|^p) = p(p-2)|F|^{p-4}|\mathcal{M}F|^2 + p|F|^{p-2}|||\mathcal{M}|||^2. \quad (2.2.7)$$

If $p = 0$, it is clear that $\Delta(|F|^p) = 0$.

Assume $p \neq 0$, the inequality $\Delta(|F|^p) \geqslant 0$ is equivalent to

$$|F|^{p-2}|||\mathcal{M}|||^2 \geqslant (2-p)|F|^{p-4}\,|\mathcal{M}F|^2$$

which can be reduced to

$$|\mathcal{M}F|^2 \leqslant \frac{1}{2-p}|||\mathcal{M}|||^2\,|F|^2. \quad (2.2.8)$$

Since $|\mathcal{M}F| = |F|\,|\mathcal{M}(\frac{F}{|F|})|$

$$\leqslant |F|\,||\mathcal{M}||$$

therefore $|\mathcal{M}F|^2 \leqslant ||\mathcal{M}||^2|F|^2$. (2.2.9)

Since $\frac{n-2}{n-1} \leqslant p < 1$ we note that $\frac{n-1}{n} \leqslant \frac{1}{2-p}$.

Thus from lemma (2.2.4) we get

$$||\mathcal{M}||^2 \leqslant \frac{1}{2-p}|||\mathcal{M}|||^2.$$

Then $||\mathcal{M}||^2|F|^2 \leqslant \frac{1}{2-p}|||\mathcal{M}|||^2|F|^2$. (2.2.10).

Then by (2.2.9) and (2.2.10) we get (2.2.8) and the theorem is now proved.

Using the subharmonic character of $|F|^p$ we obtain the extension of the basie theorem of the classical H^p spaces to the n-dimensional spaces, whenever $p \geq \frac{n-2}{n-1}$.

Instead of considering the n-dimensional spaces, from now on we will slightly change some notation about F(X). We will consider $F(X,y) = (u(X,y), V(X,y)) = (u(X,y), v_1(X,y), v_2(X,y), \ldots, v_n(X,y))$ satisfying (1.2) and so in this case we have a result that $|F(X,y)|^p$ is subharmonic whenever $p \geq \frac{(n+1)-2}{(n+1)-1} = \frac{n-1}{n}$.