CHAPTER I

INTRODUCTION

This thesis is concerned with a class of functions F of several variables. Our work is to extend to n variables some of the deeper properties known to hold in the case of two variables.

As it is well-known that two harmonic functions u(x,y) and v(x,y), satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1.1)$$

in a region if and only if there exists a harmonic function h(x,y), such that the pair (v,u) is the gradient of the function h; i.e. $v = \frac{\partial h}{\partial x}$ and $u = \frac{\partial h}{\partial y}$. Thus, analytic functions of one complex variable are in one-to-one correspondence with gradient of harmonic functions of two variables.

We so now introduce the notion of conjugacy. We say that an n-tuple, $F = (u_1, u_2, \ldots, u_n)$ of real valued harmonic functions of n-variables, $X = (x_1, x_2, \ldots, x_n)$, forms a system of conjugate harmonic functions in a region, if, in this region, it is the gradient of a harmonic function h(X), i.e.,

$$\frac{\partial h}{\partial x_i} = u_i(X) ,$$

for example,

$$F(X) = \left(\frac{x_1}{r^n}, \frac{x_2}{r^n}, \ldots, \frac{x_n}{r^n}\right) \qquad \text{where}$$

$$r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \qquad \text{and } n \geq 3 .$$

Then F is the gradient of the harmonic function $\frac{r^{2-n}}{2-n}$.

In the case of two variables $|F(Z)|^p$, $p > 0$, is subharmonic when F(Z) is analytic. So we will show in the case of several variables that $|F(X)|^p$ is subharmonic if $p \geq \frac{n-2}{n-1}$, where F is a system of conjugate harmonic functions.

This property of general system of conjugate harmonic functions is then the basic tool we will use in constructing theory of functions of several variables in H^p spaces.

Instead of extending the more familiar case of H^p spaces of functions defined in the interior of unit disc we will generalize the somewhat more difficult case of functions defined in the upper half plane. That is, we will extend to n dimensions the boundary-value results known for functions $F(Z)$, $Z = x + iy$, analytic for $y > 0$ and satisfying

$$\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx \leq A < \infty$$

for all $y > 0$ (see Rudin [8]). As in the case of the circle boundary values $F(x) = \lim_{y \to 0} F(x+iy)$ exists both almost everywhere and in the norm. In this situation the role played by the variables x and y are abviously different. This difference persists in higher dimensions and we now change our notation slightly in order to reflect better this distinct roles. We shall consider n+1 variables, $(X,y) = (x_1,x_2,\ldots,x_n,y)$ and if the system of conjugate harmonic functions $F(X,y)$ arises as the gradient of harmonic function $h(X,y)$, we shall denote by $u(X,y)$ the partial derivative of h with respect to the distinguished variable y and by V the vector $(\frac{\partial h}{\partial x_1},\frac{\partial h}{\partial x_2},\ldots,\frac{\partial h}{\partial x_n})$.

This notation, then reflects the fact that we consider the function $F(X,y)$ as defined in the region $y > 0$ and the boundary values $F(X,0) = F(X)$ will be assumed in the hyperplane $y = 0$ we also write v_k instead of $\frac{\partial h}{\partial x_k}$, $k = 1,2,\ldots,n$ and refer to $v_1,v_2,\ldots, v_n$ as the n conjugate of u. Thus, we see that the notation $F(X,y) = (u(X,y), V(X,y))$ is a natural extension of that use in the two dimensional case.

With this notation, the generalized Cauchy-Riemann equations become

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0 \quad , \frac{\partial u}{\partial x_i} = \frac{\partial v_i}{\partial y} \; , \; i = 1,2,\ldots,n \; .$$

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \; , \quad i \neq j \; , \; 1 \leq i, j \leq n \; . \tag{1.2}$$

These equations are assumed to hold in the region $y > 0$.

We then, for each $p > 0$, define the class H^p to consists of all those systems of conjugate harmonic functions $F(X,y)$ defined on

$$\mathbb{R}^{+}_{n+1} = \{(X,y) : X \in \mathbb{R}_n , \ y > 0\} ,$$

satisfying

$$\int_{\mathbb{R}_n} |F(X,y)|^p \, dX \leq A < \infty, \forall \ y > 0.$$

We also define; for each $p > 0$, the class G^p of all those systems of conjugate harmonic functions on $\mathbb{R}^{+}_{n+1}$ satisfying

$$\int_{\mathbb{R}_n} \{X^2+(1+y)^2\}^{-\frac{1}{2}(n+1)} |F(X,y)|^p dX \leq A < \infty$$

for $0 < y < \infty$.

It is clear that $H^p \subseteq G^p$.

By means of the integral

$$K(|s|^p, y) = \int_{\mathbb{R}_n} \{X^2+(1+y)^2\}^{-\frac{1}{2}(n+1)} |s(X,y)|^p dX$$

we will show, in chapter 3, that there exist boundary values $F(X) = F(X,0)$ such that $F(X,y)$ converges in the norm to $F(X)$ as y tend to zero, for $p > 1$ and $F(X,y)$ tend to $F(X)$ for almost everywhere for $p \geqslant 1$. We will prove this result by using some theorems of subharmonic function in half-spaces and of their harmonic majorants.

By the term subharmonic function we mean an extended real valued function s on Ω , Ω be an open subset of $\mathbb{R}_n$, satisfying the following properties :

i) $s \not\equiv -\infty$ on any component of Ω

ii) $s < \infty$ on Ω

iii) s is upper semicontinuous on Ω

iv) for any X in Ω and any $\rho > 0$ with $\overline{B(X,\rho)} \subset \Omega$ where $\overline{B(X,\rho)}$ denotes the closure of the ball with radius ρ and center at X, we have

$$s(X) \leq \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B(X,\rho)} s(Z)\,d\sigma(Z)$$

where σ_n = surface area of unit ball in $\mathbb{R}_n$

σ = surface area measure on $\mathbb{R}_n$

(Helms [4]).

We say that m is a harmonic majorant of s in $\mathbb{R}_n$ if $m(X) \geq s(X)$ in $\mathbb{R}_n$ and m is harmonic. And the function m is a least harmonic majorant of s if m is a harmonic majorant of s and for any harmonic majorant $\tilde{m}$ of s, $m(X) \leq \tilde{m}(X)$ in $\mathbb{R}_n$.

<u>Theorem 1.3</u> Let s be a function defined on an open set $\Omega \subset \mathbb{R}_n$ having continuous partials. Then s is subharmonic if and only if $\Delta s \geq 0$ on Ω, where $\Delta s = \sum_{i=1}^{n} \frac{\partial^2 s}{\partial x_i^2}$; the Laplacian of s.

(See in Helms [4], page 63).