CHAPTER III

MAXIMAL LOCALLY CYCLIC SUBGROUPS OF ABELIAN GROUPS

The materials of this chapter are drawn from references [1], [4], [5], [6], [7]

All groups in this chapter are presumed to be abelian. The main purpose of the chapter is to characterize the so-called maximal locally cyclic subgroups of a given group. To do this, we first prove a theorem showing that groups are direct sum of their primary components.

Definition 3.1. A group is called p-primary, where p is a prime number, if the order of each element of the given group is some power of p.

Given a group G, and a prime number p, the p-primary component of G is the set G_p consisting of all elements of G whose orders are powers of p.

For convenience, we will restate a theorem of Chapter II.

Theorem 3.2. If G is a torsion group, then G is the direct sum of its p-primary components.

Before we can prove the main theorem of this chapter, we need to develope some preliminary lemmas. Also, some definitions are needed.

Definition 3.3. Let G be a group. A subgroup of G is called a maximal locally cyclic subgroup of G if it is locally cyclic and if it is not properly contained in any other locally cyclic subgroup of G.

<u>Definition 3.4</u>. A group is said to be <u>decomposable</u> if it is a direct sum of its proper subgroups ; otherwise , the group is called <u>indecomposable</u> .

Let G be a group . A subgroup of G is said to be a <u>maximal</u> <u>indecomposable</u> (<u>decomposable</u>) if it is indecomposable (respectively , decomposable) and if it is not properly contained in any other indecomposable (respectively , decomposable) subgroup of G .

<u>Lemma 3.5</u>. A torsion group G is locally cyclic if and only if each of its p-primary components is indecomposable .

A proof of this lemma is given in [4] .

<u>Theorem 3.6</u>. Let G be a torsion group and $G = \sum_{p \in \mathbb{P}} G_p$ its p-primary decomposition . Then a subgroup M of G is a maximal locally cyclic subgroup if and only if , for each prime number p , the p-primary subgroup M_p of M is a maximal indecomposable subgroup of G_p .

<u>Proof</u>. Suppose that $M = \sum_{p \in \mathbb{P}} M_p$ is a maximal locally cyclic subgroup of G where $\mathbb{P}$ denotes the set of all prime numbers . By Lemma 3.5 , each M_p is indecomposable . If for some q in $\mathbb{P}$, M_q is a proper subgroup of an indecomposable subgroup H of G_q , then the subgroup

$$H \oplus \sum_{p \in \mathbb{P}^*} M_p$$

where $\mathbb{P}^* = \mathbb{P} \sim \{q\}$, is locally cyclic , by Lemma 3.5 and contains M as a proper subgroup . It then follows that each M_p must be maximal indecomposable as a subgroup of G_p .

Conversely, suppose $M = \sum_{p \in \mathbb{P}} M_p$, a subgroup of G such that each M_p is a maximal indecomposable subgroup of G_p. Then M is locally cyclic, by Lemma 3.5. Let L be a locally cyclic subgroup of G containing M. If $L = \sum_{p \in \mathbb{P}} L_p$, then each L_p is indecomposable, by Lemma 3.5. Since L_p contains M_p and M_p is a maximal indecomposable subgroup of G_p, $L_p = M_p$ for each $p \in \mathbb{P}$. Hence M is a maximal locally cyclic subgroup of G.

Now the theorem is completely proved.

<u>Lemma 3.7.</u> No infinite cyclic group contains a non-zero element of finite order.

<u>Proof</u>. Let $[a]$ be an infinite cyclic group. Then the order of a, denoted by $O(a)$, is $+\infty$. If there exists a non-zero $x \in [a]$ such that $O(x) < +\infty$, then

$$na = x,$$

for some integer n. Now

$$O(x)na = O(x)x = 0,$$

so that $O(a)$ divides $O(x)n$ and, therefore, $O(a) < +\infty$, contradicting the assumption.

Hence the lemma is proved.

<u>Lemma 3.8.</u> No finite cyclic group contains a non-zero element of infinite order.

<u>Proof</u>. Let $[a]$ be a finite cyclic group. Then $O(a) < +\infty$. For any non-zero element $x \in [a]$, there exists a non-zero integer n such that

$$x = na,$$

and so

$$O(a)x = O(a)na = 0.$$

Hence $O(x)$ divides $O(a)$ and , therefore , $O(x) < +\infty$.

Thus the lemma is proved .

Lemma 3.9. Let G be a group and M a non-zero locally cyclic subgroup of G . Then M is either torsion or torsion-free .

Proof. Since $M \neq 0$, there exists a non-zero $g \in M$. For any element $x \in M$, x and g generate a cyclic subgroup $[c]$ of G .

If the order of g is finite , then $[c]$ is the finite cyclic subgroup of G , by Lemma 3.7 and , therefore , the order of x is finite , by Lemma 3.8 .

If the order of g is infinite , then $[c]$ is the infinite cyclic subgroup of G , by Lemma 3.8 and , therefore , the order of x is infinite , by Lemma 3.7 .

Hence M is either torsion or torsion-free .

Thus the lemma is proved .

Lemma 3.10. Let G be a torsion-free group and let g be in G . Then

$$\langle g \rangle = \left\{ x \in G \;/\; mx \in [g] \text{, for some non-zero integer } m \right\}$$

is isomorphic to a subgroup of the additive group $\mathcal{Q}$ of rational numbers and , therefore , $\langle g \rangle$ is locally cyclic .

Proof. Clearly , $\langle g \rangle$ is a torsion-free subgroup of G . The case when $g = 0$ is obvious , assume $g \neq 0$. For any non-zero $x \in \langle g \rangle$,

$$mx = ng ,$$

for some non-zero integers m , n . Define

$$\varphi(x) = n/m$$

and

$$\varphi(0) = 0 .$$

To show that φ is a well-defined map from $\langle g \rangle$ into

$\mathcal{Q}$, let x be any non-zero element in $\langle g \rangle$. Suppose that there exist non-zero integers m , n and p , q such that

$$mx = ng$$

and

$$px = qg .$$

Then

$$mpx = npg = mqg ,$$

so that

$$(np - mq)\, g = 0 .$$

Since $g \neq 0$ and $\langle g \rangle$ is torsion-free ,

$$np = mq ,$$

and hence

$$n/m = q/p .$$

Hence φ is well-defined and is then obviously a map from $\langle g \rangle$ into $\mathcal{Q}$.

For any non-zero elements x , y $\in \langle g \rangle$,

$$mx = ng ,$$

for some non-zero integers m , n , so that

$$\varphi(x) = n/m ;$$

and also

$$py = qg ,$$

for some non-zero integers p , q , so that

$$\varphi(y) = q/p .$$

Since $\langle g \rangle$ is commutative ,

$$(mp)(x + y) = (mp)x + (mp)y .$$

But

$$mpx = npg ,$$
$$mpy = mqg$$

so that

$$(mp)(x + y) = npg + mqg = (np + mq)g .$$

Then

$$\varphi(x + y) = (np + mq)/mp$$
$$= n/m + q/p$$
$$= \varphi(x) + \varphi(y) ,$$

and hence φ is homomorphism .

For any non-zero elements x , $y \in \langle g \rangle$ such that

$$\varphi(x) = \varphi(y) = n/m ,$$

for some non-zero integers m , n . Then

$$mx = ng = my ,$$

so that

$$mx - my = 0 .$$

Since $\langle g \rangle$ is commutative ,

$$m(x - y) = mx - my = 0 .$$

Since $\langle g \rangle$ is torsion-free and $m \neq 0$,

$$x = y ,$$

and hence φ is one-to-one .

Thus $\langle g \rangle$ is isomorphic to an additive subgroup of $\mathbb{Q}$. Since subgroups of the additive group $\mathbb{Q}$ are locally cyclic , $\langle g \rangle$ is locally cyclic .

Hence the lemma is completely proved .

We are now ready to prove the main theorem of this chapter .

Theorem 3.11. Let G be a group and M a subgroup of G . Then M is a maximal locally cyclic subgroup if and only if either

(a) $M = \langle g \rangle$, for some $g \in G \setminus \{0\}$ of infinite order , or

(b) M is the direct sum of maximal indecomposable subgroups of the p-primary components of the torsion subgroup tG of G , one such subgroup from each component .

Proof. Suppose M is a maximal locally cyclic subgroup of G . If $M = 0$ and if there exists a non-zero $x \in G$, then

M is contained properly in $[x]$, contradicting the assumption . Thus $G = 0$ and , therefore , M satisfies (b) .

If $M \neq 0$, then M is either torsion or torsion-free , by Lemma 3.9.

If M is torsion , then M satisfies (b) , by Theorem 3.6.

If M is torsion-free and if we choose a non-zero $g \in M$, then for any non-zero $x \in M$, x and g generate a cyclic subgroup $[c]$ of G . Let

$$x = mc$$

and

$$g = nc$$

for some non-zero integers m , n . Then

$$nx = mnc = mg ,$$

and hence $x \in \langle g \rangle$; i.e. , $M \subseteq \langle g \rangle$. Since $g \in M$, $\langle g \rangle$ is a locally cyclic subgroup of M , by Lemma 3.10 and , therefore , $M = \langle g \rangle$.

Conversely , suppose that (a) or (b) holds .

If (a) holds , then $M = \langle g \rangle$, for some $g \in G \setminus \{0\}$ of infinite order . By Lemma 3.10 , $M = \langle g \rangle$ is locally cyclic . If M is not a maximal locally cyclic subgroup of G , then there exists $x \in G \setminus \langle g \rangle$ such that $M \cup \{x\}$ is contained in a maximal locally cyclic subgroup of G , and hence x and g generate a cyclic subgroup $[c]$ of G . Let

$$x = mc$$

and

$$g = nc$$

for some non-zero integers m , n . Then

$$nx = mnc = mg ,$$

so that $x \in \langle g \rangle$, contradicting the choice of x . Hence M is a maximal locally cyclic subgroup of G .

If (b) holds, then M is a maximal locally cyclic subgroup of G, by Theorem 3.6.

Hence the theorem is completely proved.

The above theorem gives a result, which will be used in Chapter V.

<u>Corollary 3.12</u>. Let G be a torsion-free group which is not locally cyclic. Then the intersection of all its maximal locally cyclic subgroups, denoted by $\overline{G}$, is the trivial subgroup 0.

<u>Proof</u>. By Theorem 3.11, M is a maximal locally cyclic subgroup of G if and only if $M = \langle g \rangle$, for some $g \in G \setminus \{0\}$. Then

$$\{ \langle g \rangle \;/\; g \in G \setminus \{0\} \}$$

coincides with the set of all maximal locally cyclic subgroups of G. It suffices to show that

$$\langle g \rangle \cap \langle h \rangle = 0$$

if $\langle g \rangle \neq \langle h \rangle$.

If there exists a non-zero $x \in \langle g \rangle \cap \langle h \rangle$, then

$$mx = ng$$

and

$$sx = th ,$$

for some non-zero integers m, n, s, t, so that

$$msx = nsg = mth .$$

Since $ns \neq 0$ and $mt \neq 0$, $g \in \langle h \rangle$ and $h \in \langle g \rangle$. Hence $\langle g \rangle \subseteq \langle h \rangle$ and $\langle h \rangle \subseteq \langle g \rangle$ and, therefore, $\langle g \rangle = \langle h \rangle$. Thus, if $\langle g \rangle \neq \langle h \rangle$, then $\langle g \rangle \cap \langle h \rangle = 0$.

Now $\overline{G} = 0$, and the corollary is proved.