CHAPTER II

METHOD OF ANALYSIS

2.1 Introduction

The approach proposed here hinges on the shear lag phenomenon (1) and the energy approach presented by Chan, et al. (7) modified so that moment equilibrium is better satisfied. By assuming the shear lag effect and considering the moment equilibrium the axial forces and hence the strain energy due to axial deformation can be determined. This energy is added to that due to bending and shearing deformation in the members in the side panels. Applying the principle of minimum total potential energy and the Ritz technique the lateral displacement can be solved. Shear forces in the spandrel beams are obtained from static consideration.

The detail of the method will now be pursued.

2.2 Assumptions

The analysis for the responses of a frame - tube structure under lateral load (Fig. 1) is based on the following assumptions:

a) The material is homogeneous and linear elastic.

b) The points of contraflexure occur at midspans of columns and spandrel beams.

c) The floor slabs are assumed to be rigid in-plane, so that the axial deformation of the connecting beams can be neglected.

d) The out-of-plane bending of the frames is neglected.

e) The external shear is resisted by the side panels only.

f) Shearing deformation in the columns is neglected.

2.3 Formulation of the Method

2.3.1 Determination of Column Axial Forces Assuming Axial Deformation Distribution

In the proposed method we assume the distribution of axial deformation in the columns to approximately account for the shear lag effects. Extending the work of Chan, et. al. (7) and Moffat, et. al. (8) the following axial displacements in the columns are assumed:

for side panels

$$u_{cj}(y,z) = \left[\frac{\sinh\left(\frac{2y_j}{D}\right)^{m_1}}{\sinh\,(1)}\right] u_{cc}\,(z) \qquad (1)$$

for normal panels

$$u_{ci}(x,z) = \left[\left(\frac{2x_i}{c}\right)^{m_2} + m_3\left(1-\left(\frac{2x_i}{c}\right)^{m_2}\right)\right] u_{cc}\,(z) \qquad (2)$$

where x,y,z are the Cartesian coordinates shown in Fig. 2; i,j denote the positions of the spandrel beams and columns along x and y axes respectively; u_{cc}, u_{ci}, u_{cj} are the axial displacements of the corner, i^{th} and j^{th} columns respectively;

C,D are, respectively, lengths of normal and side panels; and m_1, m_2, m_3 are the coefficients in the axial deformation function.

The coefficients m_1, m_2 and m_3 depend primarily on the height-to-width ratio of the structure, the stiffness factor and the aspect ratio. The stiffness factor is defined as the ratio of bending stiffness $(12EI_b/a^3)$ of the spandrel beam to axial stiffness (A_cE/h) of the column. The ratio of the length of normal panel to that of the side panel is referred to as the aspect ratio. The relative bending stiffness of the girders and columns also affects, to a less extent, the values of m_1, m_2 and m_3 and thus will not be considered.

Based on the influence curves presented by Khan and Amin (1) for the case of frame-tubes having the same column sections at each floor level and a linearly varying stiffness ratio, these coefficients are determined for different aspect ratios and stiffness factors and given in the form of charts in Figs.14-18.

In practice the corner columns are sometimes made stiffer than the interior ones to improve the performance of the frame-tube. In this study, we also include the frame-tubes analysed by Schwaighofer and Ast (5). Table 1 shows the variation of axial deformation in the side panel columns from the results of Schwaighofer and Ast (5) together with the average values of m_1. For the normal panels we will adopt the values of m_2 and m_3 from Figs. 14 - 18.

With the distribution of axial displacements assumed, we can proceed to determine the axial forces as follows.

From the stress-strain relationship and the assumed distribution of axial deformation we can express the axial forces in the columns in terms of those in the corner ones, thus

$$P_{ci} = f(x_i)P_{cc} \frac{A_{ci}}{A_{cc}} \tag{3a}$$

$$P_{cj} = f(y_j)P_{cc} \frac{A_{cj}}{A_{cc}} \tag{3b}$$

in which P_{cc}, P_{ci}, P_{cj} are the axial forces in the corner, i^{th} and j^{th} columns, respectively; A_{cc}, A_{ci}, A_{cj} are cross-sectional areas of the corner, i^{th} and j^{th} columns, respectively, and

$$f(x_i) = \left(\frac{2x_i}{C}\right)^{m_2} + m_3 \left[1 - \left(\frac{2x_i}{C}\right)^{m_2}\right] \tag{4}$$

$$f(y_j) = \frac{\sinh\left(\frac{2y_j}{D}\right)^{m_1}}{\sinh(1)} \tag{5}$$

The forces P_{ci} and P_{cj} acting at the points of contraflexures in the columns are shown qualitatively in Fig. 2. From symmetry and the moment equilibrium considerations about the horizontal axis lying in the plane of symmetry of the side panels and passing through the inflection points we obtain;

$$P_{cc} = \frac{(N-1)(P_0 + \frac{2}{3}P_H + \frac{z}{3H} P_H)(H - z)^2}{2\left[4\left(\sum_{j=1}^{M1} f(y_j)y_j \frac{A_{cj}}{A_{cc}} + \sum_{i=2}^{N1} f(x_i)\frac{D}{2}\frac{A_{ci}}{A_{cc}}\right) + 2\alpha\right]} \tag{6}$$

where N,M are the number of columns in normal and side panel, respectively; H is the total height of the structure; P_0 is the lateral load intensity (per column) at the base of frame-tube; P_H is the difference of lateral load intensities (per column) at the top and the base of frame-tube; and M1, N1 and α are numbers determined as follows:

for even or odd numbers of columns in side panels,

$M1 = \frac{M}{2}$ and the summation is j = 1 to integer

for even number of columns in normal panels,

$$N1 = \frac{N}{2}$$

$$\alpha = 0$$

for odd numbers of columns in normal panels

$$N1 = \frac{N-1}{2}$$

$$\alpha = f\left(x_{\frac{(N+1)}{2}}\right)\frac{D}{2}\frac{A_c}{A_{cc}}\frac{(N+1)}{2}$$

Substituting the stress-strain relation $\frac{du_{cc}}{dz} = \frac{P_{cc}}{A_{cc}E}$ into Eq. (6) leads to

$$\frac{du_{cc}}{dz} = \frac{(N-1)(P_0 + \frac{2}{3}P_H + \frac{z}{3H} P_H)(H-z)^2}{2A_{cc}E\left[4\left(\sum_{j=1}^{M1} f(y_j)y_j \frac{A_{cj}}{A_{cc}} + \sum_{i=2}^{N1} f(x_i)\frac{D}{2}\frac{A_{ci}}{A_{cc}}\right) + 2\alpha\right]}$$

$$= \frac{(N-1)}{2\phi}(P_0 + \frac{2}{3}P_H + \frac{z}{3H}P_H)(H-z)^2 \tag{7a}$$

$$\text{where} \quad \theta = A_{cc}E\left[4\left(\sum_{j=1}^{M1} f(y_j)\,y_j\frac{A_{cj}}{A_{cc}} + \sum_{i=2}^{N1} f(x_i)\,\frac{D}{2}\,\frac{A_{ci}}{A_{cc}}\right) + 2\alpha\right] \tag{7b}$$

and E is the modulus of elasticity

Integrating Eq. (7a), yields

$$\begin{aligned} u_{cc}(z) &= \int_0^z \frac{(N-1)}{2\,\theta}\left(P_o + \frac{2}{3}P_H + \frac{z}{3H}P_H\right)(H-z)^2\,dz \\ &= \frac{(N-1)}{2\,\theta}P_o\left(H^2z - Hz^2 + \frac{z^3}{3}\right) \\ &\quad + \frac{(N-1)}{2\,\theta}\left(\frac{2}{3}P_H\right)\left(H^2z - Hz^2 + \frac{z^3}{3}\right) \\ &\quad + \frac{(N-1)}{2\,\theta}\left(\frac{P_H}{3}\right)\left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right) + C_1 \end{aligned} \tag{8}$$

For a frame - tube which is fixed at the base,

$$u_{cc}(0) = 0 \tag{9}$$

In view of Eq.(9), the constant C_1 in Eq.(8) is found to be zero, thus

$$\begin{aligned} u_{cc}(z) &= \frac{(N-1)}{2\,\theta}P_o\left(H^2z - Hz^2 + \frac{z^3}{3}\right) \\ &\quad + \frac{(N-1)}{2\,\theta}\left(\frac{2}{3}P_H\right)\left(H^2z - Hz^2 + \frac{z^3}{3}\right) \\ &\quad + \frac{(N-1)}{2\,\theta}\left(\frac{P_H}{3}\right)\left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right) \end{aligned} \tag{10}$$

The lateral displacement due to axial deformation only can be approximately determined by considering the kinematics of a typical unit shown in Fig. 4.

Let $u_{c(j-1)}$ and u_{cj} be the vertical displacements of the hinges immediately to the left and right of the joint under consideration, respectively. Due to these displacements the unit will undergo an angle of rotation, θ, given by.

$$\tan \theta = \frac{u_{c(j-1)} - u_{cj}}{\frac{a_{(j-1)}}{2} + \frac{a_j}{2}} \tag{11}$$

where $a_{(j-1)}$ and a_j are the width of bay $(j-1)$ and j respectively.

The lateral displacement Δ_{Aj} due to axial deformation is therefore

$$\Delta_{Aj} = h \tan \theta \tag{12}$$

From Eq. (1),

$$u_{c(j-1)} = \left[\frac{\sinh \left(\frac{y_j + \frac{a_{(j-1)}}{2}}{D/2} \right)^{m_1}}{\sinh (1)} \right] u_{cc}(z) \tag{13a}$$

$$\text{and } u_{cj} = \left[\frac{\sinh \left(\frac{y_j - \frac{a_j}{2}}{D/2} \right)^{m_1}}{\sinh (1)} \right] u_{cc}(z) \tag{13b}$$

Substituting Eqs. (13a) and (13b) into Eq. (11) and the result in Eq. (12) yields

$$\Delta_{Aj} = k_{cj} \left(\frac{4h}{D} \right) u_{cc}(z) \tag{14a'}$$

in which

$$k_{cj} = \frac{D}{2(a_{(j-1)}+a_j)} \left\{ \frac{\sinh\left(\frac{2y_j+a_{(j-1)}}{D}\right)^{m_1} - \sinh\left(\frac{2y_j-a_j}{D}\right)^{m_1}}{\sinh(1)} \right\} \qquad (14b)$$

$$k_{cc} = \frac{D}{2a_1} \left[1 - \frac{\sinh\left(1-\frac{a_1}{D}\right)^{m_1}}{\sinh(1)} \right] \qquad (14c)$$

The lateral displacement due to axial deformation, Δ_{Aj}, depends, to a considerable extent, on the assumed displacement field u_{cj}. In this study the same displacement shape is assumed throughout the height of the structure. Consequently $u_{cc}(z)$ will be overestimated at the upper stories since in a tall building the axial forces in the exterior columns may change sign at high levels. To account for the approximations involved, a correction factor, β is applied to Eq. (14a′), thus

$$\Delta_{Aj} = \beta k_{cj} \left(\frac{4h}{D} \right) u_{cc}(z) \qquad (14a)$$

2.3.2 Lateral Displacement due to Bending and Shearing Deformations

The strain energy, ΔU_{sj}, due to bending and shear deformations in each typical unit of the side panels shown in Fig. 3, is given by

$$\Delta U_{sj} = 2\left(\frac{1}{2}\int_0^{\frac{h-d_b}{2}} \frac{V_j^2 z^2}{EI_{cj}}\, dz\right) + \frac{1}{2}\int_0^{\frac{c_{(j-1)}}{2}} \frac{Q_{(j-1)}^2 y^2}{EI_{b(j-1)}}\, dy$$

$$+ \frac{1}{2} \int_{0}^{\frac{c}{2}(j-1)} \frac{Q^2_{(j-1)}}{G A^*_{b(j-1)}} dy + \frac{1}{2} \int_{0}^{\frac{c_j}{2}} \frac{Q_j^2 \; y^2}{EI_{bj}} dy$$

$$+ \frac{1}{2} \int_{0}^{\frac{c_j}{2}} \frac{Q_j^2}{GA^*_{bj}} dy$$

Carrying out the integration leads to,

$$\triangle U_{sj} = \frac{V_j^2 (h-d_b)^3}{24EI_{cj}} + \frac{Q^2_{(j-1)} \; c^3_{(j-1)}}{48EI^*_{b(j-1)}} + \frac{Q_j^2 \; c_j^3}{48EI^*_{bj}} \qquad (15a)$$

in which $$I^*_{bj} = \frac{I_{bj}}{1+ \frac{12EI_{bj}}{GA^*_{bj} \; c_j^2}}, \quad I^*_{b(j-1)} = \frac{I_{b(j-1)}}{1+ \frac{12EI_{b(j-1)}}{GA^*_{b(j-1)} \; c^2_{(j-1)}}} \qquad (15b)$$

and V_j are the shear forces at points of contraflexure in the column of the typical unit; $Q_{(j-1)}$, Q_j are the shear forces at points of contraflexure in the $(j-1)^{th}$ and j^{th} spandrel beams of the typical unit, respectively; $c_{(j-1)}$, c_j are clear lengths of the $(j-1)^{th}$ and j^{th} spandrel beams, respectively; h is the story height; d_b is the depth of spandrel beam; I_{cj}, $I_{b(j-1)}$, I_{bj} are the second moment of area of the column, $(j-1)^{th}$ and j^{th} spandrel beams, respectively; $A^*_{b(j-1)}$, A^*_{bj} are the effective cross-sectional area of the $(j-1)^{th}$ and j^{th} spandrel beams, respectively; G is the shear modulus.

From statics we can express the shear forces in the spandrel beams in terms of the column shear forces V_j as follows:

$$Q_{(j-1)} = \frac{2h\, V_j}{a_j + a_{(j-1)}} \qquad (16a)$$

$$Q_j = \frac{2h\, V_j}{a_j + a_{(j-1)}} \qquad (16b)$$

Substituting Eqs.(16a) and (16b) into Eq. (15a) yields, after simplifying:

$$\Delta U_{sj} = \frac{h\, V_j^2}{2K_{sj}} \qquad (17a)$$

$$\text{in which } K_{sj} = \frac{12EI_{cj}}{h^2\left(1-\frac{d_b}{h}\right)^3 (1+2\lambda_j)} \qquad (17b)$$

$$\text{and } \lambda_j = \frac{\left(\frac{I_{cj}}{h}\right)}{\left(1-\frac{d_b}{h}\right)^3}\left[\frac{\left(\frac{c_j}{a_j + a_{(j-1)}}\right)^3}{\left(\frac{I^*_{bj}}{a_j + a_{(j-1)}}\right)} + \frac{\left(\frac{c_{(j-1)}}{a_j + a_{(j-1)}}\right)^3}{\left(\frac{I^*_{b(j-1)}}{a_j + a_{(j-1)}}\right)}\right] \qquad (17c)$$

This strain energy is equal to the work done by the column shear forces, i.e.

$$\Delta U_{sj} = \frac{1}{2}(V_j)(\Delta_{sj}) \qquad (18)$$

$$\text{therefore } \Delta_{sj} = \frac{h}{K_{sj}}(V_j) \qquad (19)$$

where Δ_{sj} is the lateral displacement due to shearing action.

2.3.3 Total Potential Energy and Approximate Solution by Ritz Method

The total lateral displacement of each floor, Δ, is the sum of the lateral displacement due to shearing action of the lateral forces and axial deformation in the columns, i.e.

$$\Delta = \Delta_{sj} + \Delta_{Aj}$$

In view of Eq. (14a), Eq. (19) and Eq. (10) we obtain

$$\frac{\Delta}{h} = \frac{V_j}{K_{sj}} + \frac{2\beta k_{cj}}{D\ \phi} (N-1)P_o\left(H^2z - Hz^2 + \frac{z^3}{3}\right)$$

$$+ \frac{4\beta k_{cj}}{3\ D\phi} (N-1)P_H \left(H^2z - Hz^2 + \frac{z^3}{3}\right)$$

$$+ \frac{2\beta\ k_{cj}}{3\ D\phi} (N-1)\ P_H \left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right) \qquad (20)$$

$$\text{or } V_j = K_{sj}\left[\frac{\Delta}{h} - \frac{2\beta k_{cj}}{D\ \phi} (N-1)\ P_o \left(H^2z - Hz^2 + \frac{z^3}{3}\right)\right.$$

$$- \frac{4\beta k_{cj}}{3\ D\ \phi} (N-1)\ P_H\left(H^2z - Hz^2 + \frac{z^3}{3}\right)$$

$$\left.- \frac{2\beta k_{cj}}{3\ D\ \phi} (N-1)\ P_H \left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right)\right] \qquad (21)$$

Substituting Eq. (21) into Eq. (17a) results in

$$\frac{\Delta U_{sj}}{h} = \frac{K_{sj}}{2}\left[\frac{\Delta}{h} - \frac{2\beta k_{cj}}{D\,\phi}(N-1)P_o\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{4\beta k_{cj}}{3\,D\,\phi}(N-1)\,P_H\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{2\beta k_{cj}}{3\,D\,\phi}(N-1)P_H\left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right)\right]^2 \qquad (22)$$

By approximating the term $\frac{\Delta U_{sj}}{h}$ as $\frac{d\,U_{sj}}{dz}$ and $\frac{\Delta}{h}$ as $\frac{d\Delta}{dz}$ we obtain:

$$\frac{dU_{sj}}{dz} = \frac{K_{sj}}{2}\left[\frac{d\Delta}{dz} - \frac{2\beta k_{cj}}{D\,\phi}(N-1)P_o\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{4\beta k_{cj}}{3\,D\,\phi}(N-1)P_H\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{2\beta k_{cj}}{3\,D\,\phi}(N-1)\,P_H\left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right)\right]^2 \qquad (23)$$

Summing the contribution from all units at the same level and integrating over the height of the structure yields the total strain energy, U_s, thus

$$U_s = \frac{1}{2}\sum_{j=1}^{M} 2\int_0^H K_{sj}\left[\frac{d\Delta}{dz} - \frac{2\beta k_{cj}}{D\,\phi}(N-1)P_o\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{4\beta k_{cj}}{3\,D\,\phi}(N-1)P_H\left(H^2 z - Hz^2 + \frac{z^3}{3}\right) - \frac{2\beta k_{cj}}{3\,D\,\phi}(N-1)\,P_H\left(\frac{Hz^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4H}\right)\right]^2 dz \qquad (24)$$

The potential energy of the external load, U_L is given by

$$U_L = -(N-1) \int_0^H P(z)\, \Delta(z)\, dz \qquad (25)$$

At equilibrium, the principle of minimum total potential energy demands that

$$\Pi = U_s + U_L = \text{minimum} \qquad (26)$$

in which Π = total potential energy.

In this study an approximate solution of this problem is obtained by means of the Ritz method with the lateral displacement assumed to be

$$\Delta(z) = Az^2 + Bz \qquad (27)$$

The result is, for K_{sj} and k_{cj} independent of z,

$$A = 3\,\frac{(N-1)}{K_s}\left(\frac{\beta K_s^* P_o H^2}{10\,D\,\phi} + \frac{7}{90}\,\frac{\beta\,K_s^* P_H H^2}{D\,\phi} - \frac{P_o}{6} - \frac{5P_H}{36}\right) \qquad (28a)$$

$$B = \frac{(N-1)}{K_s}\left(P_o H + \frac{31}{36} P_H H + \frac{\beta\,K_s^* P_o H^3}{5D\,\phi} + 2\,\frac{\beta\,K_s^* P_H H^3}{15\,D\,\phi}\right) \qquad (28b)$$

where

$$K_s = 2 \sum_{j=1}^{M} K_{sj} \quad \text{and} \quad K_s^* = 2 \sum_{j=1}^{M} K_{sj}\, k_{cj} \qquad (28c)$$

For the special case of a uniform lateral load, we have

$$A = \frac{3(N-1)}{K_s}\left(\frac{\beta K_s^* P_o H^2}{10\ D\ \phi} - \frac{P_o}{6}\right) \tag{29a}$$

$$B = \frac{(N-1)}{K_s}\left(P_o H + \frac{\beta K_s^* P_o H^3}{5\ D\ \phi}\right) \tag{29b}$$

and for a triangular lateral load,

$$A = \frac{(N-1)}{K_s}\left(\frac{7\beta\ K_s^* P_H H^2}{30\ D\ \phi} - \frac{5\ P_H}{12}\right) \tag{30a}$$

$$B = \frac{(N-1)}{K_s}\left(\frac{31}{36} P_H\ H + \frac{\beta\ 2 K_s^* P_H H^3}{15\ D\ \phi}\right) \tag{30b}$$

2.3.4 Shear Forces in Spandrel Beams and Columns

To calculate the shear forces in spandrel beams we first determine the shear force at the inflection point of the external spandrel beam from the difference of axial forces in normal panel columns at the floor under consideration. The shear forces in the inner spandrel beams and columns can then be calculated from the law of statics, thus: (refer to Fig. 3.)

$$Q_j = Q_{(j-1)} + P_{cj}(z_1) - P_{cj}(z_2) \tag{31}$$

$$V_j(z_1) = \frac{1}{2h}\left(Q_j a_j + Q_{(j-1)} a_{(j-1)}\right) \tag{32}$$

where z_1, z_2 are the height from the base to the point of contraflexure of the lower and upper columns, respectively.