CHAPTER VI

WAVELET THEORY IN STATISTICAL FIELD THEORY

Contents

6.1 Gaussian model

The Gaussian model's Hamiltonian is similar to the Landau-Ginzburg model's but without the quartic term (u = 0).

6.1.1 Formulation

The Gaussian model (free field theory) is given by the Hamiltonian

$$H = \frac{1}{2}\sum_{x\in\Gamma}\sum_{\mu}^{D}(a(x+e_\mu)-a(x))^2+\frac{m^2}{2}\sum_{x\in\Gamma}(a(x)^2) \quad (6.1)$$

To obtain the Hamiltonian in term of wavelet transform field, we must express the difference of two fields in wavelet coefficients. This is done by using the characteristic function of the link

$$L(x,\mu)(y) = \delta_{y,x+e_\mu}-\delta_{y,x} \quad (6.2)$$

and the scalar product in the space of lattice field, then

$$a(x+e_\mu)-a(x) = <a, L(x,\mu)>$$

$$= \sum_{y\in\Gamma}a(y)L(x,\mu)(y) \quad (6.3)$$

As the wavelet transform is orthogonal, it conserves the scalar product.

$$a(x+e_\mu)-a(x) = <\hat{a}, \hat{L}(x,\mu)>$$

$$= \sum_{n}\sum_{l}\sum_{x'}\hat{a}_l^{(n)}(x')\hat{L}_l^{(n)}(x,\mu)(x') \quad (6.4)$$

where $\hat{L}_l^{(n)}(x,\mu)(x')$ is the wavelet transform of $L(x,\mu)(y)$. Then the expectation value of the link Hamiltonian is

$$<\frac{1}{2}(a(x+e_\mu)-a(x))^2>_0 = \frac{1}{2}\sum_{n=1}^{N}\sum_{l}^{n_l}\left|\hat{L}_l^{(n)}(x,\mu)\right|^2 A_l^{(n)} \quad (6.5)$$

with the symbolic notation

$$\left|\hat{L}_i^{(n)}(x,\mu)\right|^2 = \sum_{x'\in\Gamma^n}(\hat{L}_i^{(n)}(x,\mu)(x')) \qquad (6.6)$$

This gives the power spectrum of the link (x,μ) on scale n, in the sense that the original norm of the function $L(x,\mu)$ is distributed over different scales. Accord to conservation of the norm

$$\sum_{n=1}^{N}\sum_{i}^{n_n}\left|\hat{L}_i^{(n)}(x,\mu)\right|^2 = \sum_{y}(L(x,\mu)(y))^2 \qquad (6.7)$$

The notation can be extended to

$$\left|\hat{L}_i^{(n)}\right|^2 = \sum_{x\in\Gamma}\sum_{\mu=1}^{D}\left|\hat{L}_i^{(n)}(x,\mu)\right|^2 \qquad (6.8)$$

This gives the power spectrum of the whole lattice.

Similarly, the site Hamiltonian gives

$$\frac{m^2}{2}S(x)^2 = \frac{m^2}{2}\sum_{n=1}^{N}\sum_{i}^{n_n}\left|\hat{S}_i^{(n)}(x)\right|^2 A_i^{(n)} \qquad (6.9)$$

with the characteristic function $S(x)$ of a site

$$S(x)(y) = \delta_{y,x} \qquad (6.10)$$

In this case, there is further analytical simplification as the sum over $x\in\Gamma$ can be performed

$$\sum_{x\in\Gamma}(S_i^{(n)}(x)(x'))^2 = \sum_{x\in\Gamma}(\psi_i^{(n)}(x')(x))^2$$

$$= \quad 1 \tag{6.11}$$

due to the normalization of wavelets.

The internal energy is therefore

$$U \quad = \quad <H>_0 \quad = \quad \frac{1}{2}\sum_{n=1}^{N}\sum_{t}^{n_t}\left|\hat{L}_t^{(n)}\right|^2 A_t^{(n)} + \frac{m^2}{2}\sum_{n=1}^{N}\sum_{t}^{n_t} N_n A_t^{(n)} \tag{6.12}$$

It is interesting when we use the characteristic function link .; both internal energy and entropy are linear in the variational parameters. The variation can then be performed analytically and leads to the result

$$A_t^{(n)} \quad = \quad \frac{1}{\beta}\frac{1}{m^2 + \left|\hat{L}_t^{(n)}\right|^2 / N_n} \tag{6.13}$$

With the notation (6.13), the quantity $\left|\hat{L}_t^{(n)}\right|^2 / N_n$ has taken place of the momentum square k^2 in the corresponding Fourier-space expression. The wavelet coefficient n=N, t= 0 gives the average of the function over all sites.

In the case of a characteristic function of a link (6.13), the average vanishes and thus

$$A_0^{(N)} \quad = \quad \frac{1}{\beta m^2} \tag{6.14}$$

is divergent for $m^2 \to 0$. The same divergence appears in the Fourier Transform associated with the k= 0 mode and is caused by the invariance under adding a constant to a massless field. As it appears only in a single wavelet coefficient, it can be easily subtracted in the wavelet case as with the Fourier Transform.

6.1.2 Correlator

According to C. Best and A.Schaferwork, I find something mistake in this part. To calculate the correlator of the Gaussian model, we first derive the scaling property of the fluctuation strengths $A_l^{(n)}$ with respect to n. Using the factorization property (5.4) of the wavelets, we find for (6.8)

$$\begin{aligned}\hat{L}_l^{(n)}(x,\mu)(x') &= \psi_l^{(n)}(x+e_\mu-x')-\psi_l^{(n)}(x-x') \\ &= \left(\prod_{k\neq\mu}^{D}\psi_l^{(1D,n)}(x_k-x_k')\right) \\ &\quad *\left(\psi_{l_\mu}^{(1D,n)}(x_\mu+1-x_\mu')-\psi_l^{(1D,n)}(x_\mu-x_\mu')\right) \qquad (6.15)\end{aligned}$$

Squaring and summing over x, x' and μ yields

$$\left|\hat{L}_l^{(n)}\right|^2 = \sum_{x,x',\mu}\left(\prod_{k\neq\mu}^{D}\psi_l^{(1D,n)}(x_k-x_k')\right)^2\left(\psi_{l_\mu}^{(1D,n)}(x_\mu+1-x_\mu')-\psi_l^{(1D,n)}(x_\mu-x_\mu')\right)^2 \qquad (6.16)$$

6.1.3 Correlation Function

Given the fluctuation strengths $A_l^{(n)}$, the correlation function can be calculated by applying the wavelet transform

$$<a(x)a(y)> = C(d) = \sum_n\sum_l C_l^{(n)}(x,y)A_l^{(n)} \qquad (6.17)$$

with the wavelet autocorrelation

$$C_i^{(n)}(x,y) = \sum_{x' \in \Gamma^n} \Psi_i^{(n)}(x')(x)\Psi_i^{(n)}(x')(y) \qquad (6.18)$$

These functions are not translation invariant as the translation group, has complicated representation in wavelet space.

The scaling relations between wavelet (which hold approximately for discrete wavelets of sufficiently high scale n imply that

$$C_i^{(n)}(x,y) \approx C_i^{(n)}(x-y)$$

$$\approx 2^{(n'-n)/2} C_i^{(n')}(2^{n'-n}(x-y)) \qquad (6.19)$$

These wavelet correlators have finite but nonzero extent (as the wavelets have). Thus, long range correlations can be obtained by (in wavelet space uncorrelated) fluctuations at sufficiently large scales. In this way, the wavelet theory can map a critical position-space theory with long -range correlations to less critical wavelet-space theory dominated by short-range correlations.

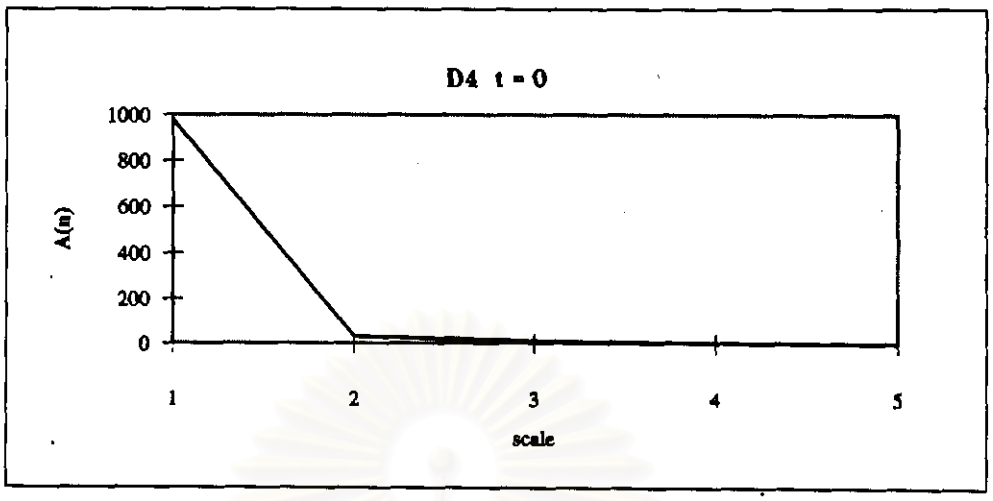
The correlation function can be solved exactly by a Fourier Transform (Itzykson Drouffe, 1992 and Zinn-Justin, 1989).

$$C(d) = \frac{1}{\beta}\frac{1}{N_0^{\,2}} \sum_{k \in F\Gamma} \frac{\cos(k.d)}{m^2 + \sum_{\mu}(2 - 2\cos(k.e_\mu))} \qquad (6.20)$$

with k on the dual lattice $F\Gamma$ with lattice spacing $\frac{2\pi}{2^N}$

Figure 6.1-6.3 show the wavelet coefficient fluctuations in two dimensional Gaussian model on 64*64, m = 0.032 for different t and sum all t for different wavelet types.

(a)



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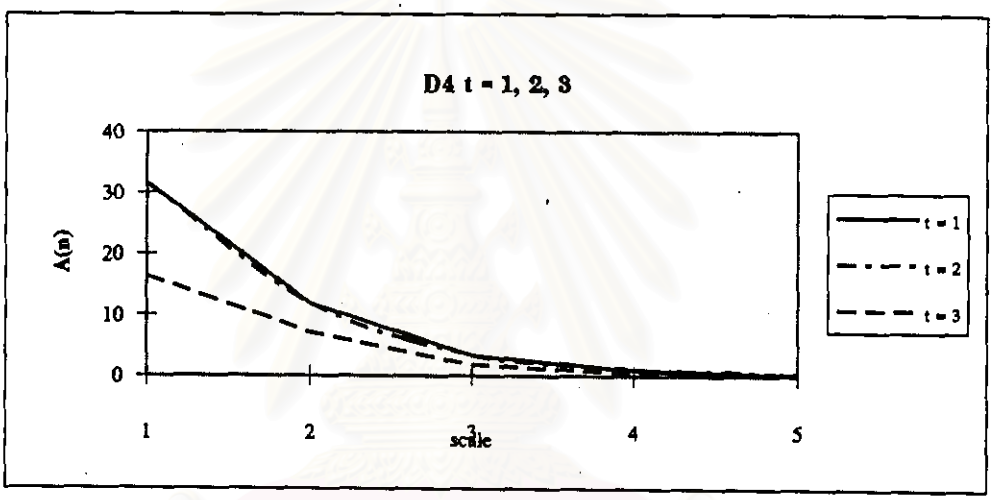

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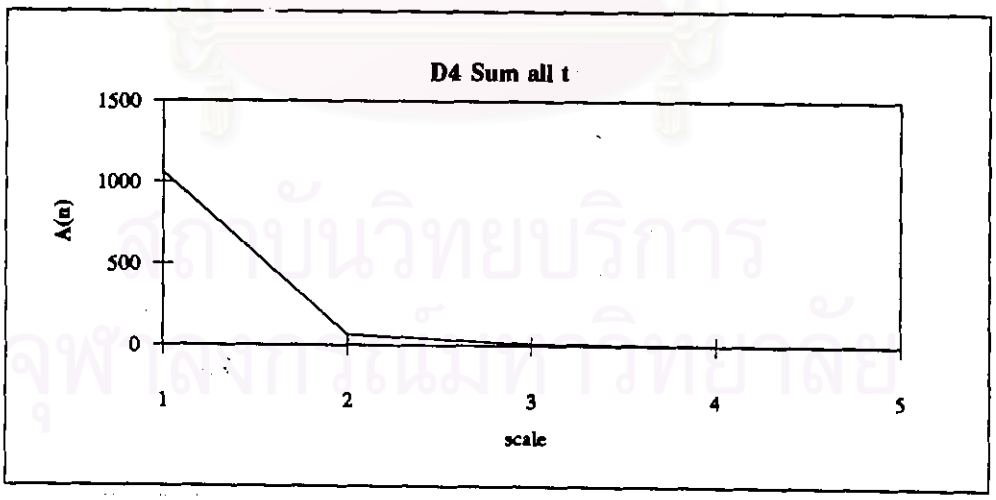


Figure 6.1 Wavelet coefficient fluctuation in two dimensional Gaussian model on 64*64, $m = 0.032$, $\beta = 1$, D4 (a) t = 0 (b) t = 1, 2, 3 (c) Sum all t

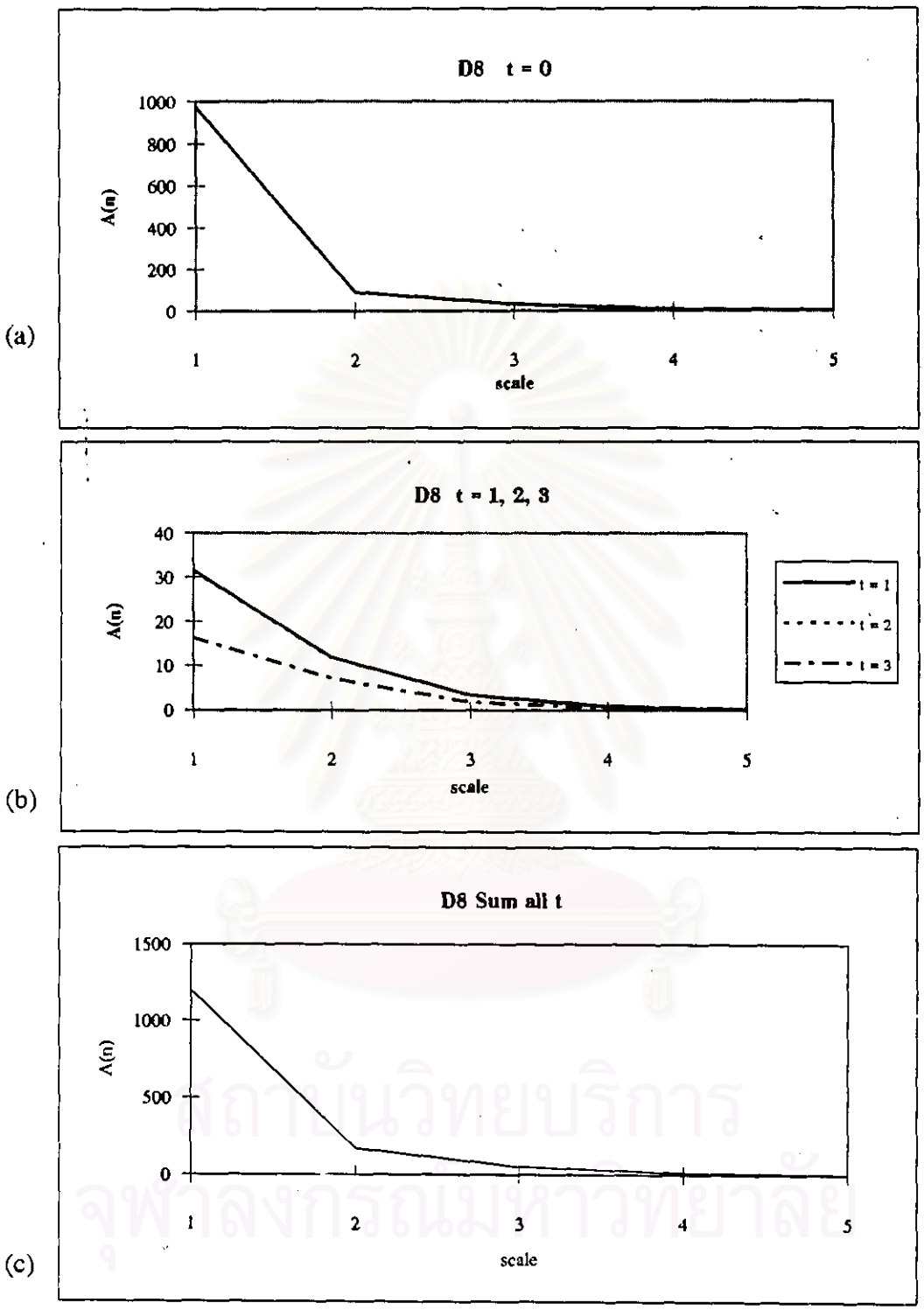


Figure 6.2 Wavelet coefficient fluctuation in two dimensional Gaussian model on 64*64, m = 0.032, β = 1, D8 (a) t = 0 (b) t = 1, 2, 3 (c) Sum all t

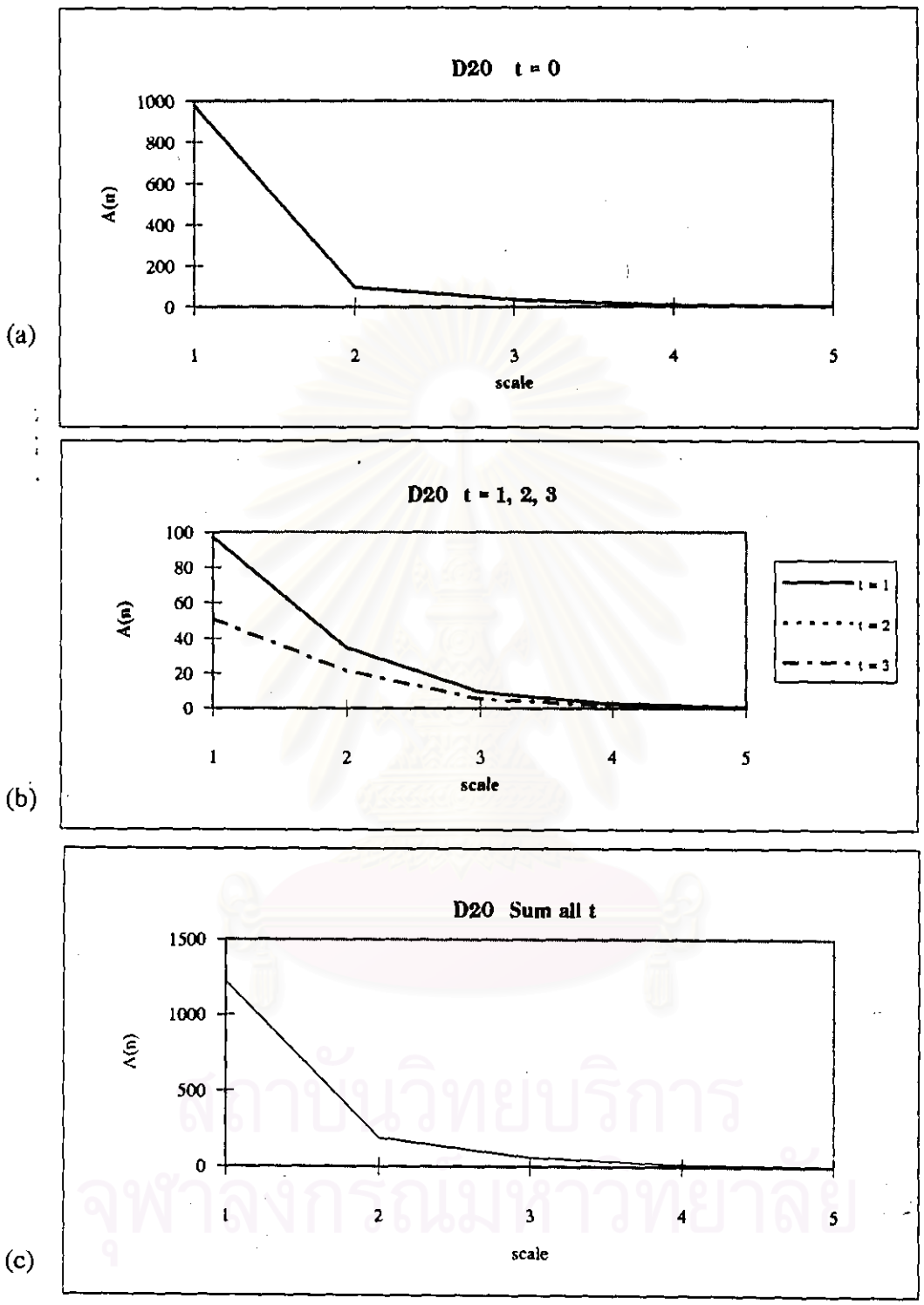
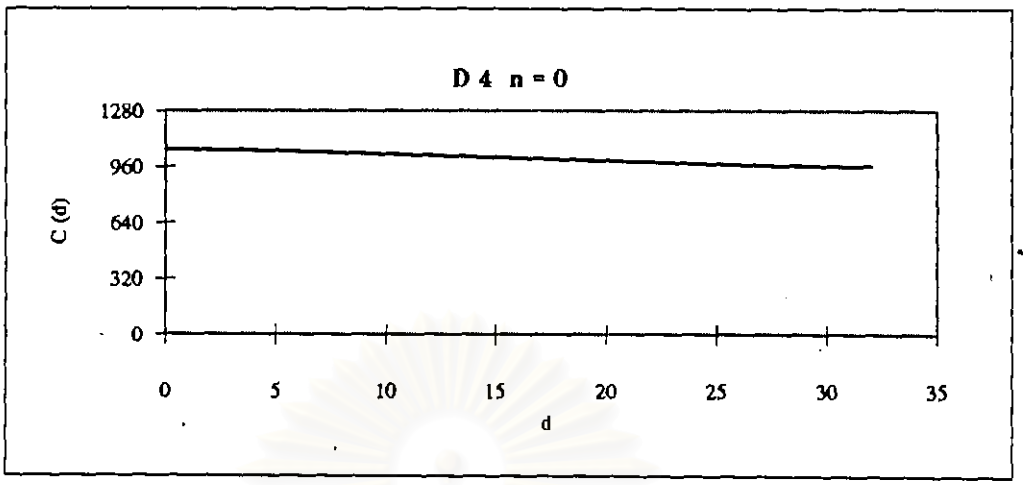


Figure 6.3 Wavelet coefficient fluctuation in two dimensional Gaussian model on 64*64, m = 0.032, β = 1, D20 (a) t = 0 (b) t = 1, 2, 3 (c) Sum all t

Figure 6.4-6.6 show the Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (n), for different wavelet types.

Figure 6.7 shows the prediction for correlation function from different wavelets. The higher order of the wavelet is closer to the exact result. Since the higher wavelets approach form of the free wave that diagonalizes the Hamiltonian.

(a)



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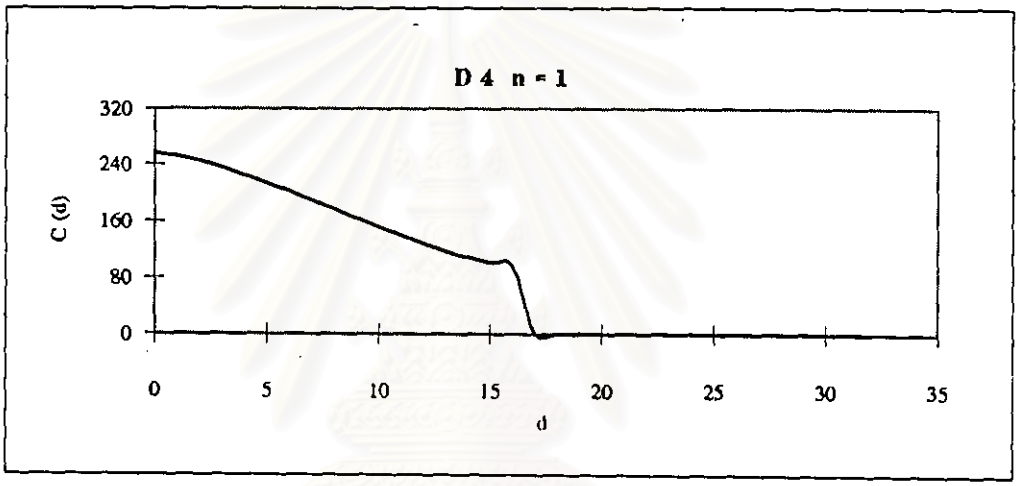

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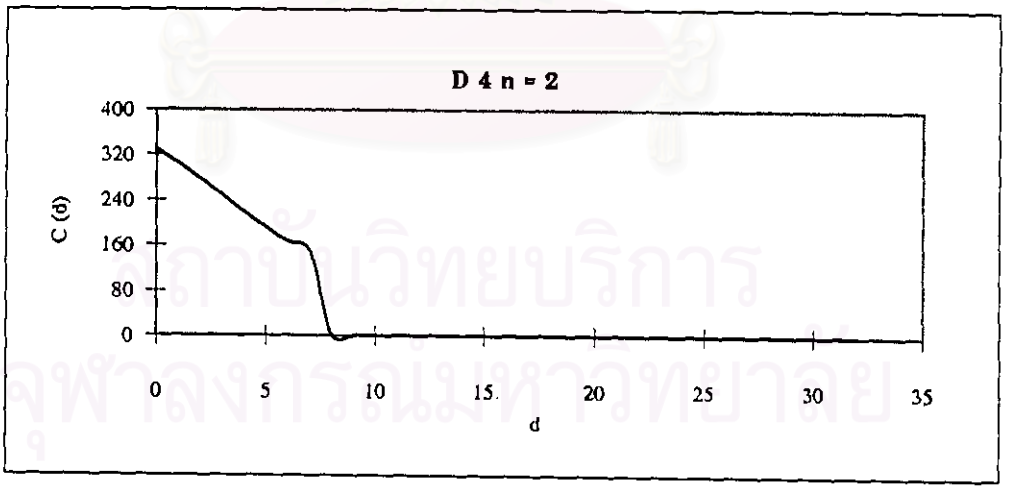
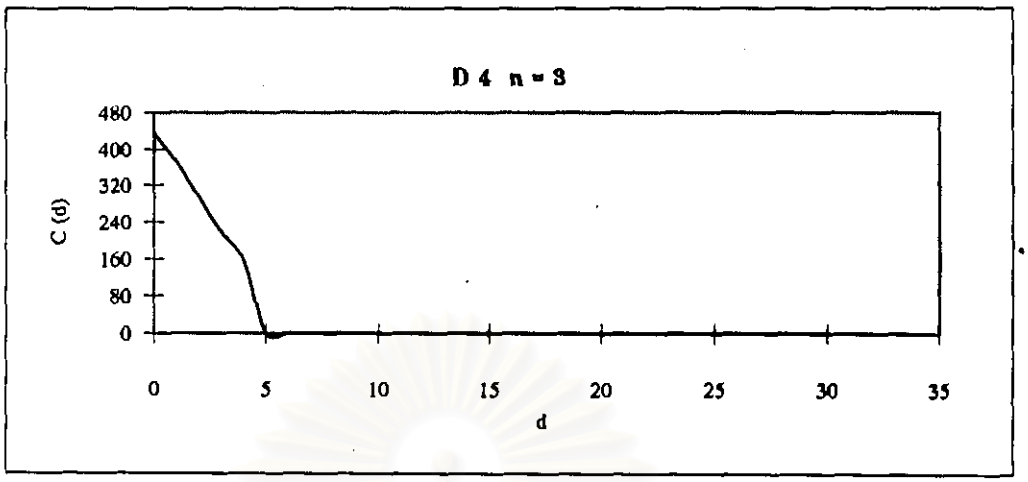
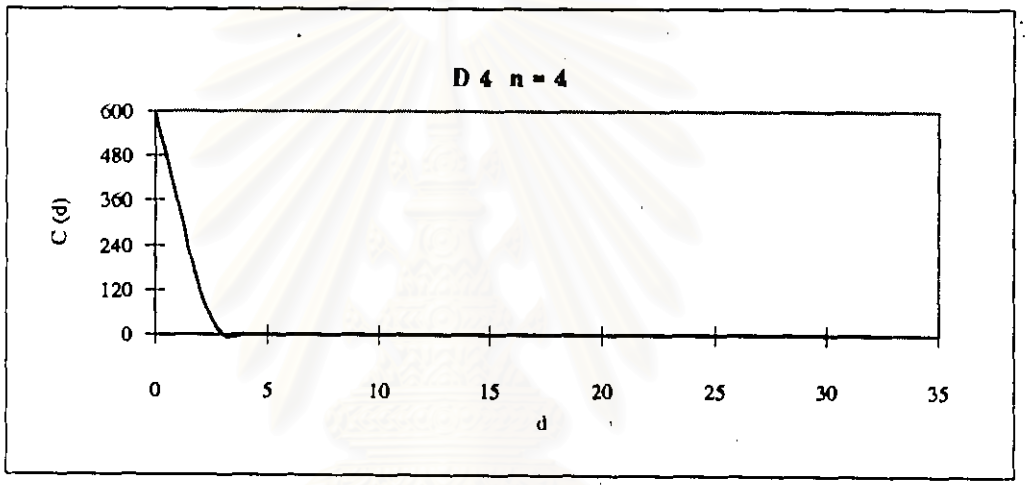


Figurre 6.4 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D4

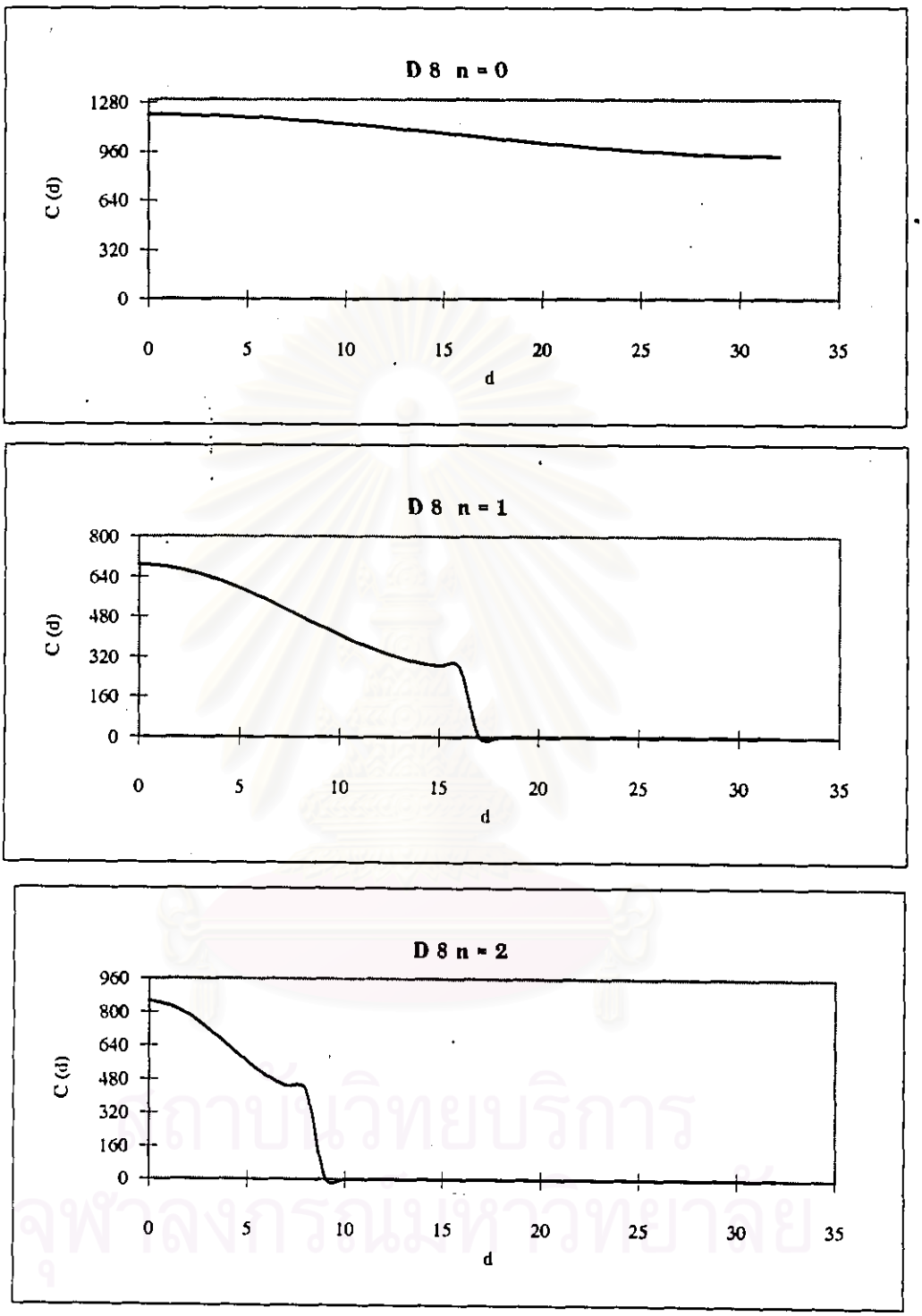
(d)



(e)

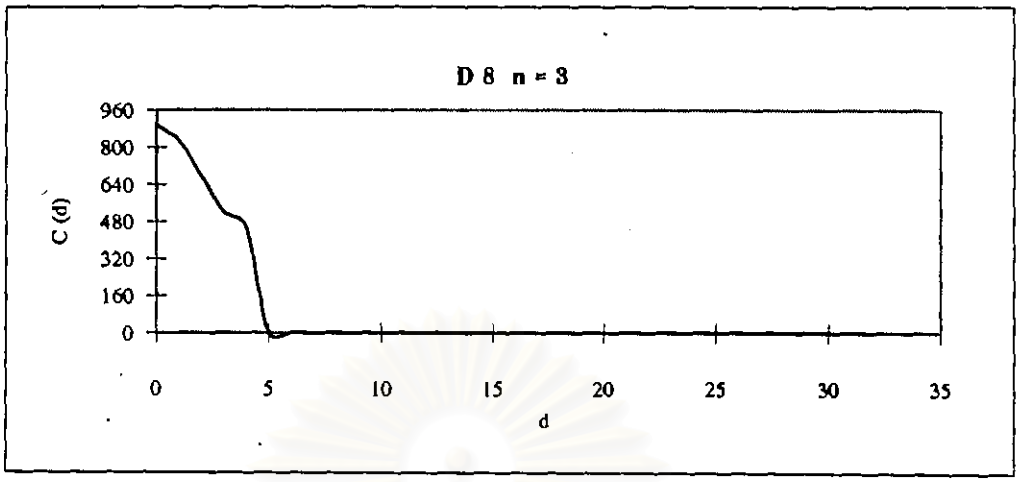


Figurre 6.4 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D4

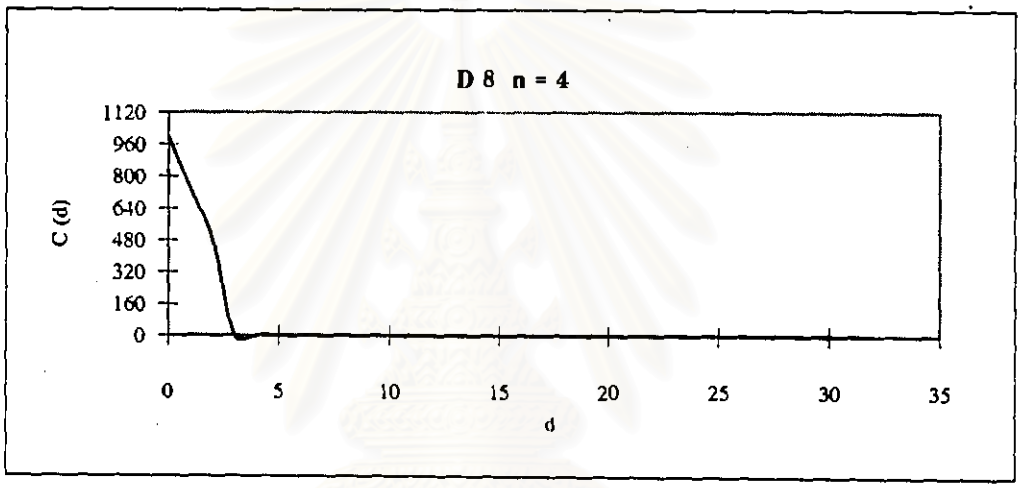


Figurre 6.5 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D8

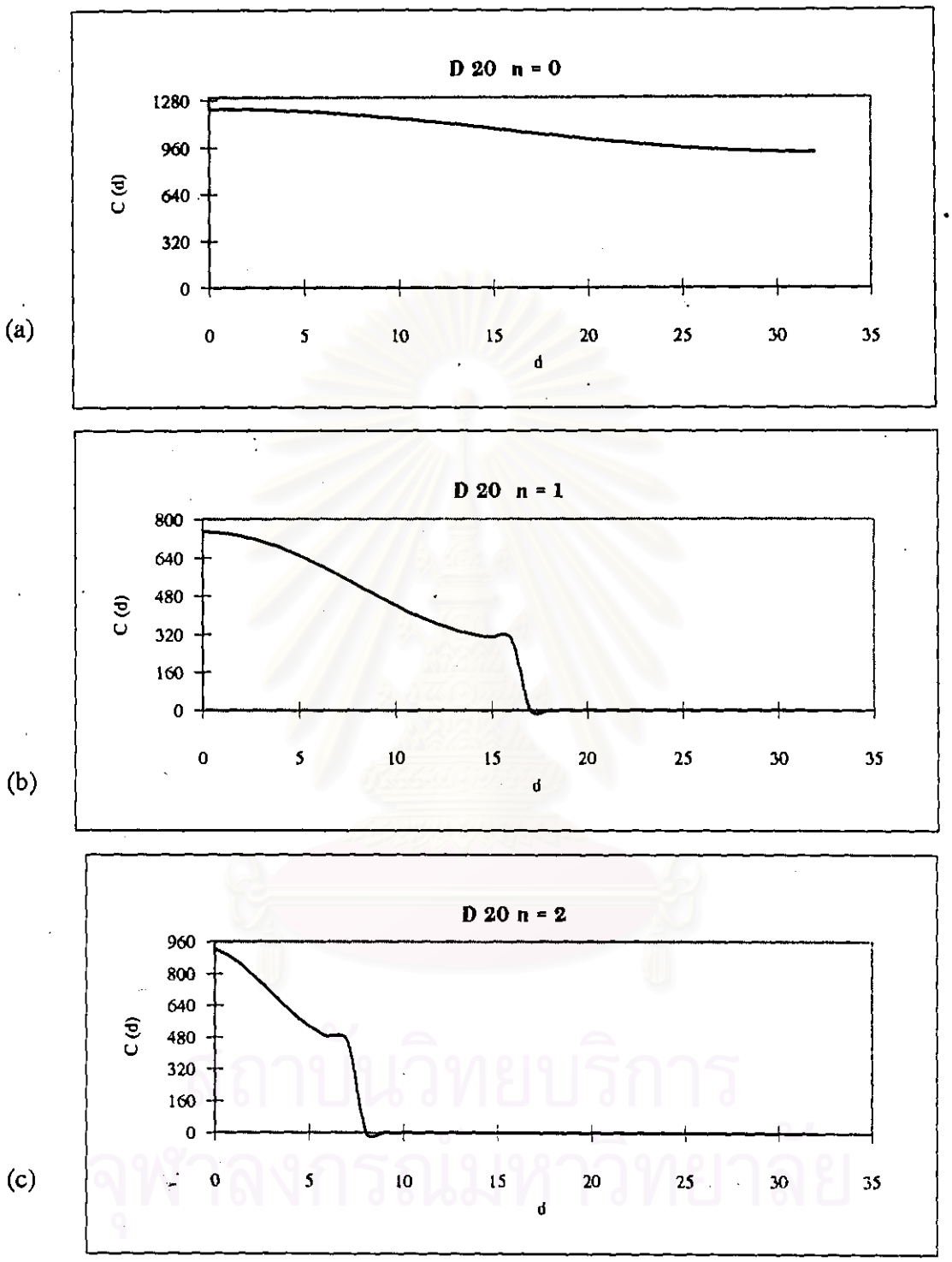
(d)



(e)

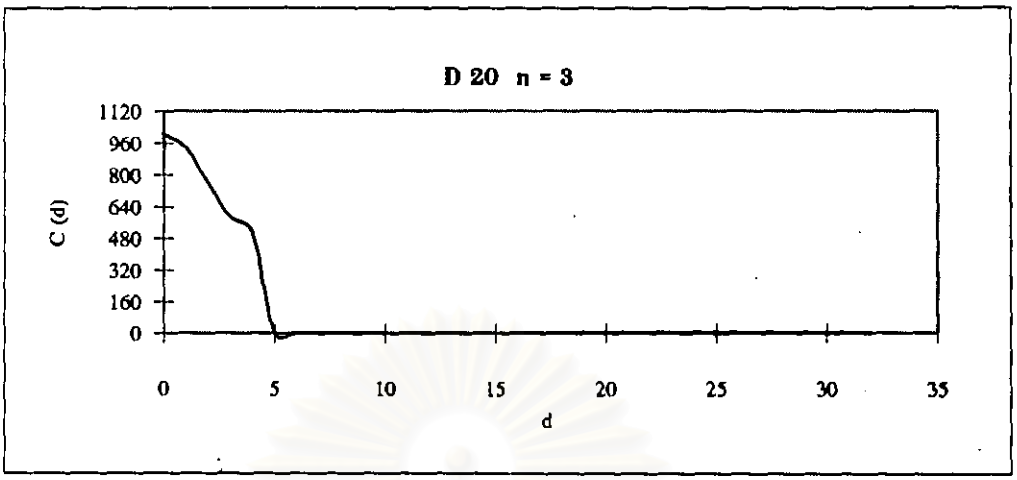


Figurre 6.5 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D8



Figurre 6.6 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D20

(d)



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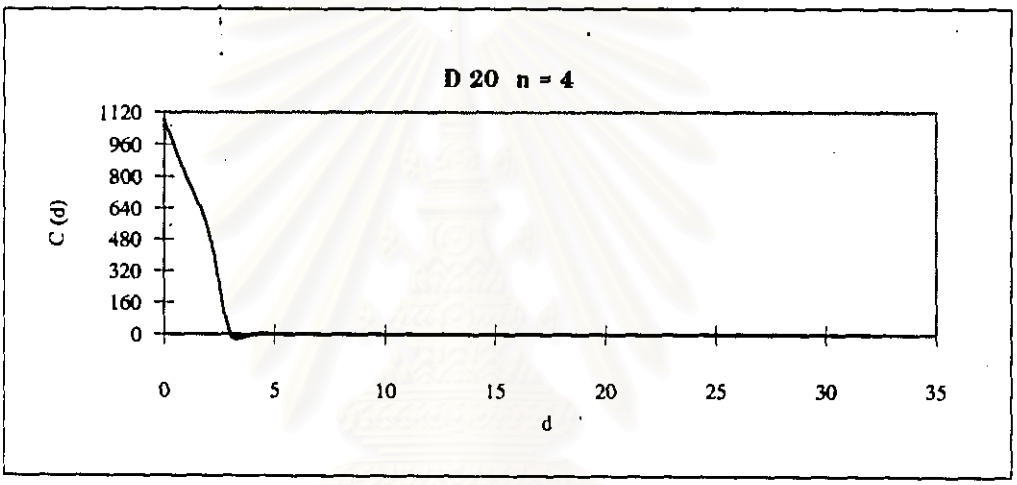

Figurre 6.6 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1 at different scales (a) n = 0 (b) n = 1 (c) n = 2 (d) n = 3 (e) n =4 for D20

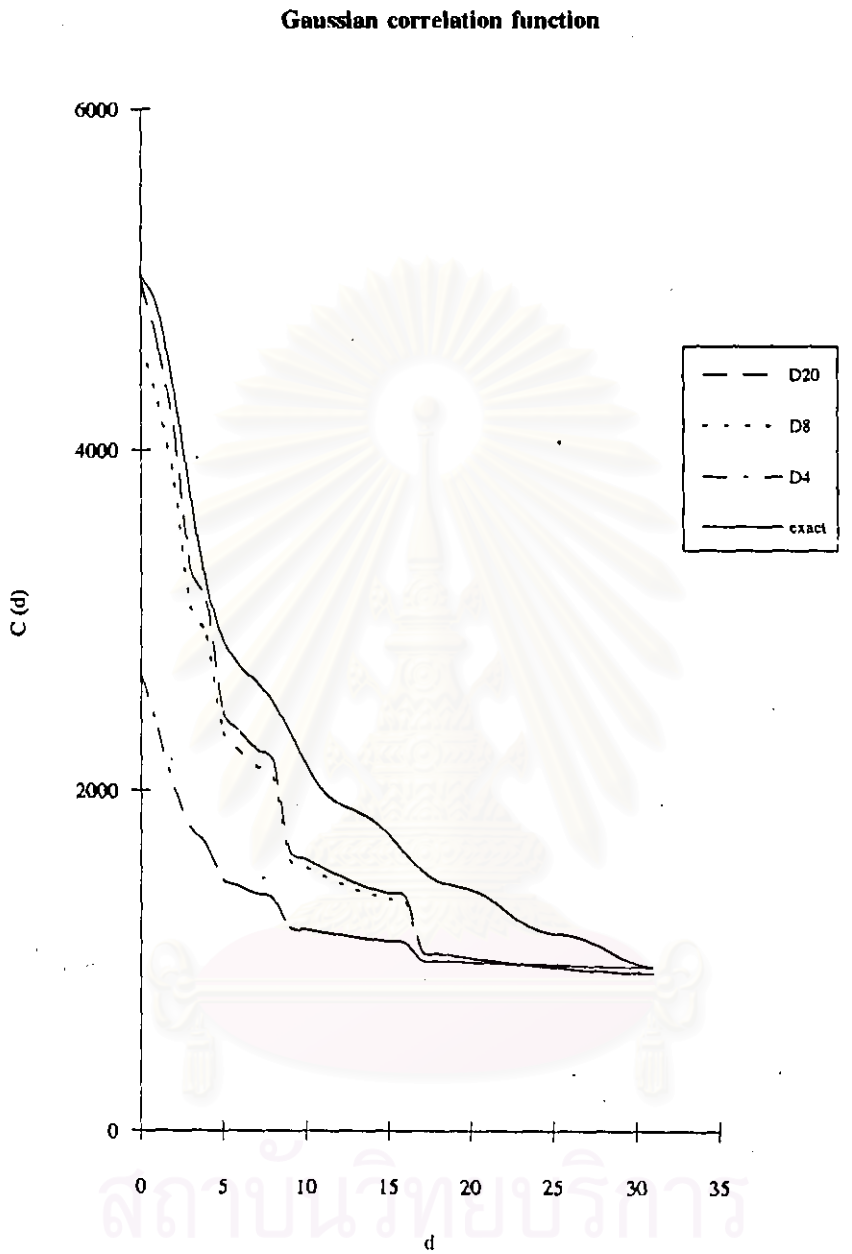

Figure 6.7 Gaussian correlation function m = 0.032, 64*64 lattice points, β =1, sum all t, n. The solid line is the exact result, the broken lines the result of the variational procedure with different Daubechies wavelets types

where $\Theta(d)$ is step function defined by

$$\Theta(d) \quad = \quad \begin{cases} 0 & where\ d < 0 \\ 1 & where\ d \geq 0 \end{cases}$$

The correlation function of the field theory is then

$$< a(x)a(x+d) > \quad = \quad \sum_{n=1}^{N} \sum_{i}^{n_{max}} (1-2^{1-n}d)\Theta(1-2^{1-n}d)\Theta(d) \qquad (6.22)$$

Let $2^{n_0} < d < 2^{n_0+1}$. Then only terms with $n \geq n_0$ contribute to the sum. If $A_i^{(n)} \approx \alpha^n A^{(0)}$ the sum can be performed, sending $N \to \infty$ so

$$C(d) \quad = \quad A^{(0)} n_\omega \Theta(d) \sum_{n=n_0}^{\infty} \alpha^n 2^{-Dn} (1-2^{1-n}d)$$

$$= \quad A^{(0)} n_\omega \Theta(d) \alpha^{n_0} 2^{-Dn_0} \left(\frac{2^D}{2^D - \alpha} - 2^{1-n_0} d \frac{2^{D+1}}{2^{D+1} - \alpha} \right) \qquad (6.23)$$

inserting $2^{n_0} \approx d$ and thus $\alpha^{n_0} \approx d^{\log_2 \alpha}$, the correlation function becomes

$$C(d) \quad \approx \quad d^{-(D-2)} \qquad (6.24)$$

This corresponds to the equation (3.2) which the critical exponent (η) equal to zero in Gaussian model.

6.2 Landau-Ginzburg Model

6.2.1 Formulation

To consider the Landau-Ginzburg theory in D dimensions with single component order parameter $S(x)$, the Hamiltonian is

$$H = \int d^D x \left[\frac{1}{2} (\nabla S(x))^2 + \frac{r_0}{2} S(x)^2 + \frac{u_0}{2} S(x)^4 \right] \quad (6.25)$$

with coupling constants r_0 and u_0.

The general expansion of the field $S(x)$ in a wavelet basis is given by

$$S(x) = \sum_n \sum_i^{n_i} \sum_{x' \in \Gamma^n} \hat{S}^{(n)}(x') \psi_i^{(n)}(x)(x') + \bar{S} \quad (6.26)$$

As wavelets are objects whose zero-th moments (their average over the real axis) vanish, they cannot represent an overall magnetization of the system. The over all magnetization is written separately as $\bar{S}$,

$$< S(x) > = \bar{S} \quad (6.27)$$

This constant, it implies for the wavelet coefficients

$$< \hat{a}_i^{(n)}(x') > = \delta_{n,N} \delta_{i,0} \bar{S} \quad (6.28)$$

i.e., the only coefficient with a nonzero expectation value is $\hat{a}_0^{(N)}$, associated with scaling function at the topmost level. An extra degree of freedom, $\bar{S}$, giving the

center of the probability distribution of $\hat{a}_0^{(N)}$, appears only in the internal energy and does not influence the entropy.

The expansion (6.26) is substituted into the Hamiltonian. Assuming $S=0$, the first term becomes

$$H_1 = \int d^D x[\nabla S]^2$$

$$= \sum_{n_1 n_2 t_1 t_2} \sum_{x_1' x_2'} \hat{S}_{t_1}^{n_1}(x_1') \hat{S}_{t_2}^{n_2}(x_2') \left(-\hat{\Delta}_{t_1 t_2}^{(n_1 n_2)}(x_1' x_2')\right) \tag{6.29}$$

with the representation of the Laplace operator in wavelet space.

$$\hat{\Delta}_{t_1 t_2}^{(n_1 n_2)}(x_1' x_2') = \int d^D x \Psi_{t_2}^{(n_1)}(x_1')(x) \Delta \Psi_{t_2}^{n_2}(x_2')(x) \tag{6.30}$$

Fortunately, this quantity has been investigated by [Beylkin et at., 1991], who gave an existence proof for some wavelet and scaling form. Remarkably, it does not exist for the Haar wavelet which can not differentiate twice.

The quadratic term becomes

$$H_2 = \int d^D x S(x)^2 = \sum_{n} \sum_{t_1} \sum_{x'} \left[\hat{S}_{t_1}^{(n)}(x')\right]^2 \tag{6.31}$$

by the orthonormality properties of Daubechies wavelets.

The quartic term has a complicated four-point interaction

$$H_3 = \int d^D x S(x)^4 \tag{6.32}$$

$$= \sum_{n_1 \ldots n_4} \sum_{t_1 \ldots t_4} \sum_{x_1' \ldots x_4'} \hat{S}_{t_1}^{(n_1)}(x_1') \ldots \hat{S}_{t_4}^{(n_4)}(x_4') \overline{M}_{t_1 t_2 t_3 t_4}^{(n_1 n_2 n_3 n_4)}(x_1' x_2' x_3' x_4') \tag{6.33}$$

The matrix $\overline{M}$ is obtained by Cartesian composition

$$\overline{M}^{(n_1n_2n_3n_4)}_{l_1l_2l_3l_4}(x'_1x'_2x'_3x'_4) = \prod_{j=1}^{D} M^{(n_1n_2n_3n_4)}_{l_{1i}l_{2i}l_{3i}l_{4i}}(x'_{1i}, x'_{2i}, x'_{3i}, x'_{4i}) \qquad (6.34)$$

from the matrix element

$$M^{(n_1n_2n_3n_4)}_{l_1l_2l_3l_4}(x'_1, x'_2, x'_3, x'_4) = \int dx \Psi^{(n_1)}_{l_{1i}}(x'_1)(x)\ldots\Psi^{(n_4)}_{l_4}(x'_4)(x) \qquad (6.35)$$

This complexity can be simplified by using wavelet space instead of position space. The scaling relations of wavelets allow considerable simplifications, at least an approximate solution can be obtained analytically.

6.2.2 Scaling Form of the Matrix Elements

To calculate the internal energy, the scaling forms of the involved matrix elements are derived. For the derivative term (6.29), we need the matrix element (6.30) of the Laplace operator for $n_1 = n_2 = n$ and $x'_1 = x'_2 = 0$ (for simplicity, the system is assumed to have translational invariance)

By rescaling the integration variables,

$$\hat{\Delta}^{(n\,n)}_{l_1l_2}(0,0) = 2^{-nD}.2^{nD}2^{-2nD}\int d^D x \Psi_{l_1}(x)\Delta\Psi_{l_2}(x)$$

$$= 2^{-2nD}\hat{\Delta}^{(00)}_{l_1l_2}(0,0) \qquad (6.36)$$

the powers of 2 stem from the normalization of wavelets, from the rescaling of the integration variable and from the substitution under the double derivative,

respectively. The remaining coefficient $\hat{\Delta}^{(0,0)}_{l_1 l_2}(0,0) \equiv \Delta_{l_1 l_2}$ has been shown to exist and calculated by Beylkin et al., 1991, for Daubechies wavelet starting at DAUB6. The representation of Laplace operator does not exist for Haar wavelet or DAUB4.

For the four point matrix element, the situation is not simple as the expression is reduced to

$$M^{(n_1 n_1 n_2 n_2)}_{l_1 l_2 l_3 l_4}(0,0,x',x')$$

$$= \int dx 2^{-n_1/2}\Psi_{l_1}\left(x\right)2^{-n_1/2}\Psi_{l_2}(x)2^{-n_2/2}\psi_{l_3}(x')(x)2^{-n_2/2}\psi_{l_4}(x')(x)$$

$$= 2^{-n_2}\int dx\Psi_{l_1}\left(x\right)\Psi_{l_2}(x)$$

$$* \ \psi_{l_3}(2^{-n_2}(2^{n_1}(x-x'))(x)\psi_{l_4}(2^{-n_2}(2^{n_1}(x-x')) \qquad (6.37)$$

Assuming $n_2 \le n_1$, x' may be substituted by $x'_2 - x'_1$ (as Γ^{n_1} contains all points of Γ^{n_1}). When $n_2 << n_1$, assuming the third term and fourth wavelet under the integral change much faster than the other two and are thus essentially localized around $x = 2^{-n_1}x'$. Then the first two terms can be considered approximately constant under the integral and perform the remaining integration using the orthonormality of wavelet.

$$M^{(n_1 n_1 n_2 n_2)}_{l_1 l_2 l_3 l_4}(0,0,x',x') \approx 2^{-n_2}\psi_{l_3}(2^{-n_1}x')\psi_{l_4}(2^{-n_1}x') \qquad (6.38)$$

In order to evaluate the free energy, it is necessary to sum over x'_1, x'_2, then the scaling expression becomes

$$\sum_{x_1' \in \Gamma^{n_1}} \sum_{x_2' \in \Gamma^{n_2}} M^{(n_1 n_1 n_2 n_2)}_{i_1 i_2 i_3 i_4}(x_1', x_1', x_2', x_2') \approx N_0 2^{-(n_1+n_2)} \delta_{i_1,i_2} \delta_{i_3,i_4} \quad (6.39)$$

6.2.3 Effective Internal Energy

The complete expression of the internal energy is derived. From the derivative term, the contribution is

$$\frac{U_1}{N_0} = \frac{1}{2} \sum_n 2^{-3nD} \sum_{i_1 i_2} \left(-\Delta_{i_1 i_2}\right) A^{(n)}_{i_1 i_2} \quad (6.40)$$

from the quadratic term

$$\frac{U_2}{N_0} = \frac{r_0}{2} \left[\sum_n \sum_i 2^{-nD} A^{(n)}_{ii} + \bar{S}^2 \right] \quad (6.41)$$

from the quartic term, using Wick's theorem (Bellac 1991) and equation (6.39)

$$\frac{U_3}{N_0} = \frac{3}{2} u_0 \sum_{n_1 n_2} \sum_{i_1 i_2} 2^{-(n_1+n_2)D} A^{n_1}_{i_1 i_1} A^{n_2}_{i_2 i_2} + \frac{u_0}{2} S^4 + 3 u_0 \bar{S}^2 \sum_n \sum_i 2^{-nD} A^n_{ii} \quad (6.42)$$

6.2.4 Approximate Solution

For simplicity, the nondiagonal term in A is neglected and written

$$A^n_{i_1 i_2} = \delta_{i_1 i_2} A^n_{i_1} \quad (6.43)$$

Let $$A = \sum_n \sum_i 2^{-nD} A^{(n)}_i \quad (6.44)$$

Then the free energy can be written

$$\frac{U}{N_0} = \frac{1}{2}\sum_{n} 2^{-3nD}\left(-\Delta_{i,i_2}\right)A_i^{(n)} + \frac{r_0}{2}A + \frac{r_0}{2}\bar{S}^2$$

$$+3\frac{U_0}{2}A^2 + 3u_0\bar{S}^2A + \frac{U_0}{2}\bar{S}^4 \qquad (6.45)$$

The free energy is minimized with respect to $\bar{S}$,

$$\frac{\partial U}{\partial \bar{S}} = r_0\bar{S} + 2u_0\bar{S}^3 + 6u_0\bar{S}A = 0$$

which lead to the trivial solution $\bar{S} = 0$, the nontrivial solution is

$$\bar{S} = \frac{1}{2}\sqrt{\frac{-r_0}{2u_0} - 3A} \qquad (6.46)$$

The square root is positive whenever

$$A < \frac{-r_0}{6u_0} \qquad (6.47)$$

In the "Mexican hat potential" that induces the symmetry breaking, $r_0 < 0$ and $u_0 > 0$, thus the condition for the existence of a nontrivial minimum requires that the fluctuation are smaller than a certain value.

To minimize free energy with respect to the fluctuation $A_i^{(n)}$,

$$\frac{1}{2^{-nD}N_0}\frac{\partial U}{\partial A_i^{(n)}} = \frac{1}{2}2^{-nD}(-\Delta tt) + \frac{r_0}{2}3u_0A + 3u_0\bar{S}^2 - \frac{1}{2\beta A_i^{(n)}}$$

$$= \quad 0 \tag{6.48}$$

This yields

$$A_i^n \quad = \quad \frac{1}{2\beta}\frac{1}{2^{-2nD}\left(-\Delta_u\right)+r_0+3u_0\left(A+\bar{S}^2\right)} \tag{6.49}$$

Note that A depends on the $A_i^{(n)}$. Putting $u_0 = 0$, we get the Gaussian model results

$$A_i^{(n)}\ (\text{Gaussian}) \quad = \quad \frac{1}{2\beta}\frac{1}{2^{-2nD}\left(-\Delta_u\right)+r_0} \tag{6.50}$$

The power law behavior $A_i^{(n)} \approx 2^{2n}$ corresponds to infinite correlation length. At the transition point, the correlation length is infinite and the fluctuation strengths have scaling form

$$A_i^{(n)} \quad = \quad 2^{2n}A_i^0 \tag{6.51}$$

Sum the equation (6.44) by using $A_i^{(n)}$ from (6.49)

$$A \quad = \quad \frac{\sum_i A_i^{(0)}}{2^{D-2}-1} \quad = \quad \sum_i \frac{1}{2\beta\left(-\Delta_u\right)\left(2^{(D-2)}-1\right)}$$

The fluctuation strengths are finite for $D > 2$. On the other hand, this result can be equated to the value from (6.47)

$$\beta \quad = \quad \frac{3u_0}{r_0}\sum_i \frac{1}{-\Delta_u}\frac{1}{2^{D-2}-1}$$

This approach can be extended to allow a small but non vanishing mass, equivalent to a small deviation from criticality. Expanding in this mass parameter, it reveals that the solution is unstable for $D \leq 4$ as the sum A diverges.

The upper critical dimensions can be obtained as $D > 4$, which the mean field result holds.

6.2.5 Renormalization Group Transformation

6.2.5.1 Formulation

By the principle of minimal free energy, the calculations of the partition function to a nonlinear minimization problem, have been reduced in a moderate number of variables.

In the concept of Wilson renormalization group, to solve this minimization successively with respect to each scale is attempted, thus eliminating the variable associated with scale and obtaining an effective internal energy function of the remaining variables.

Assuming $A_t^n = A^n$, this simplifies the renormalization group transformation considerably. As the solution calculation is derived, this will not change the quantitative behavior.

When $n < 0$ have been eliminated, the rescaled fields are introduced on the next-coarser lattice by the transformation

$$\tilde{A}^{(n)} \equiv 2^{-2} A^{(n+1)} \tag{6.52}$$

The scale factor is chosen in such a way that the kinetic term remains unchanged under the scale transformation.

Let us first assume that $A^{(0)}$ vanishes, i.e. the coarse lattice is considered exclusively. Then the Hamiltonian is

$$\frac{U_0'}{2^{-D}N_0} = \sum_n 2^{-nD}\left[\frac{1}{2}2^{-2nD}C_\Delta + 2^2\frac{r_0 n_t}{2}\right]\tilde{A}^{(n)} + \frac{3u_0 n_t^2}{2}2^{4-D}\sum_{n_1 n_2}2^{-(n_1+n_2)D}\tilde{A}^{(n_1)}\tilde{A}^{(n_2)} \quad (6.53)$$

where $C_\Delta = \sum_l\left(-\Delta_{ll}\right)$. The volume factor $2^{-D}N_0$ on the left hand side takes the reduced number of sites on the next-higher lattice into account.

This is identical with the original expression when the substitutions

$$r_0 \rightarrow r_0' = 2^2 r_0 \quad (6.54)$$

$$u_0 \rightarrow u_0' = 2^{4-D}u_0 \quad (6.55)$$

Thus, the canonical dimension of the coupling constant is obtained. The complete internal energy in form of $A^{(0)}$ is

$$\frac{U_0}{N_0} = \frac{U_0'}{N_0} + \left(\frac{1}{2}C_\Delta + \frac{r_0 n_t}{2}\right)A^{(0)} + \frac{3u_0 n_t^2}{2}\left[A^{(0)}\right]^2 + 3u_0 n_t^2\left(\sum_n 2^{-nD}A^{(n)}\right)\left(A^{(0)}\right) \quad (6.56)$$

The renormalization step is performed by

- minimizing the expression with respect to $A^{(0)}$

- minimizing the expression with respect to $A^{(0)}$
- calculating the new effective free energy by inserting the value of $A^{(0)}$ found, which will be a function of the $A^{(n)}$, $n>0$
- expanding it into powers of $A^{(n)}$, and
- absorbing the new terms into a renormalization of the coupling constants

To perform this calculation, the following shorthand notation is used .

Let

$$a = \left(\frac{1}{2}C_{\Delta} + n_t\frac{r_0}{2}\right) \tag{6.57}$$

$$\hat{A} = 3n_t^2\sum_{n>0}2^{-nD}A^{(n)} \tag{6.58}$$

$$b = \frac{3}{2}n_t^2 \tag{6.59}$$

Then the internal energy becomes

$$\frac{U}{N_0} = \frac{U'}{N_0} + \left(a + u_0\hat{A}\right)A^{(0)} + u_0 b\left(A^{(0)}\right)^2 \tag{6.60}$$

The corresponding entropy is

$$\frac{S}{N_0} = \frac{n_t}{2}\sum_n 2^{-2nD}\ln A^{(n)} \tag{6.61}$$

Minimizing the free energy with respect to $A^{(0)}$ leads to the equation

$$\left(a + u_0 A\right) + 2u_0 bA^{(0)} - \frac{n_t}{2\beta A^{(0)}} = 0 \tag{6.62}$$

In the spirit of the ε expansion, the Guassian model ($u_0 = 0$) is assumed a good approximation. Then $A^{(0)}$ is expanded in a series of the quartic coupling u_0. The internal energy is obtained to $O(u_0^3)$:

$$\frac{U_0}{N_0} = \frac{U_0'}{N_0} + \left(\frac{n_t}{2\beta} - \frac{3n_t^4 u_0}{8a^2\beta^2} + \frac{9n_t^7 u_0^2}{8a^4\beta^3} - \frac{135n_t^{10}u_0^3}{32a^6\beta^4} \right)$$

$$+ \left(\frac{9n_t^6 u_0^2}{4a^3\beta^2} - \frac{27n_t^9 u_0^3}{2a^5\beta^3} \right) \sum_{n>0} 2^{-nD} A^{(n)}$$

$$- \frac{81n_t^8 u_0^3}{8a^4\beta^2} \sum_{n_1>0} \sum_{n_2>0} 2^{-(n_1+n_2)D} A^{(n_1)} A^{(n_2)} \qquad (6.63)$$

The second and third can be absorbed into a redefinition of coupling constants. To lowest order in u_0

$$r_0^{ren} = r_0 + \frac{9n_t^6 u_0^2}{4a^3\beta^2} \qquad (6.64)$$

$$u^{ren} = u_0 - \frac{81n_t^8 u_0^3}{8a^4\beta^2} \qquad (6.65)$$

Thus, according to (6.30) and (6.31), the renormalization flow is

$$r_0' = 2^2 \left(r_0 + \frac{9n_t^6 u_0^2}{4a^3\beta^2} \right)$$

$$u_0' = 2^{4-D} u_0 \left(1 - \frac{81n_t^8 u_0^2}{8a^4\beta^2} \right)$$

Note that the variable a depend on r_0

6.2.5.2 Renormalization Flow and Fixed Point

At first, there is always the fixed point $r_0 = u_0 = 0$, corresponding to massless Gaussian model with infinite correlation length. The critical exponents are given by the linearized renormalization transformation around this point. The matrix M of transformation is derived

$$\frac{\partial(r_0', u_0')}{\partial(r_0, u_0)} = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^{4-D} \end{pmatrix} \tag{6.66}$$

This is the well-known result for the critical behavior around the Guassian fixed point. In particular, it is stable for $D > 4$ in the u direction.

For $D < 4$, there is another fixed point. Using the shorthand

$$r_0' = 2^2(r_0 + \eta u_0^2) \tag{6.67}$$

$$u_0' = 2^{4-D} u_0 (1 - \gamma u_0^2) \tag{6.68}$$

and solving for nontrivial fixed point in the second equation, one gets

$$u^* = \pm\sqrt{\frac{1-2^{D-4}}{8}} \tag{6.69}$$

(only the positive solution is physical). The corresponding r_0 is

$$r^* = -(1-2^{D-4})\frac{4\eta}{3\gamma} \tag{6.70}$$

Resubstituting and taking into account that a η and gamma involve r_0, one gets

$$r^* = -\left(1+\frac{4(1-2^{\alpha-4})}{27n_f}\right)^{-1} 4\frac{(1-2^{\alpha-4})}{27n_f^2}C_\Delta \qquad (6.71)$$

Besides numerical factor is dominated by the matrix element of the Laplace operator C_Δ. However, the stability of the fixed point is investigated by neglecting the influence of r_0 on the coefficient a. The two eigenvalues are calculated

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = \frac{32^D - 32}{2D} \qquad (6.72)$$

Both of them are ≥ 1 and give the Wilson-Fisher fixed point instable. It remains to be seen whether the instability is inherent to the variational approach or the perturbative expansion in implementing the remormalization group and in calculation of its fixed points.