CHAPTER V

THE VARIATIONAL EQUATIONS

THE VARIATIONAL PRINCIPLES

Lloyd and Best (1975) have shown that, in calculating the approximate density of states, the variational parameter(s) should be chosen so that the "pressure" function reaches its maximum value. Whereas the pressure function at a given energy is defined to be the integrated number of states from the lowest to the given energy states or, in notations,

$$P(E) \equiv \int_{-\infty}^{E} dE' \int_{-\infty}^{E'} dE'' \rho(E'') = \int_{-\infty}^{E} dE' (E - E') \rho(E'), \qquad (5.1)$$

where the last equation is the result of the integration by parts. Therefore, the variational equations for a two-parameter density of states can be derived as following

$$dP(E;\alpha,\beta) = \frac{\partial P(E;\alpha,\beta)}{\partial \alpha} d\alpha + \frac{\partial P(E;\alpha,\beta)}{\partial \beta} d\beta = 0. \qquad (5.2)$$

Two variational equations are obtained in this form,

$$\frac{\partial P(E;\alpha,\beta)}{\partial \alpha} = \int_{-\infty}^{E} dE' (E - E') \frac{\partial \rho(E';\alpha,\beta)}{\partial \alpha} = 0, \qquad (5.3)$$

provided that α and β are independent.

The asymptotic behaviour of this variational principle is to maximize the density of states, which is the so-called Halperin and Lax' variational ansatz (1966, 1967). That is

$$\frac{\partial \rho(E;\alpha,\beta)}{\partial \alpha}=0. \tag{5.4}$$

In Halperin and Lax' work, this equation has been reduced to a more simple one by thinking that the prefactor of the exponential slowly varies with respect to the exponential term. The left-hand side of (5.4) then becomes a partial derivative of the exponent of the density of states. In symbols,

$$\frac{\partial b(E;\alpha,\beta)}{\partial \alpha}=0. \tag{5.5}$$

DERIVING THE VARIATIONAL EQUATIONS

First, we shall introduce two new variables, $\tilde{y}=\sqrt{\frac{b}{\xi'}}$ and $y=\frac{b}{2\xi'}$, into the full ground-state and deep-tail cases, respectively. Each direct transformation gives us the variational equation (5.3) in the new forms,

$$\frac{\partial P_g(E)}{\partial \alpha}=\int_{\tilde{y}}^{\infty} d\tilde{y}'\,\frac{2(E'-E)}{\tilde{y}'}\left(\frac{1}{b'}\frac{\partial b'}{\partial E'}\right)^{-1}\frac{\partial \rho_g(E')}{\partial \alpha}=0 \tag{5.6}$$

and

$$\frac{\partial P_d(E)}{\partial \alpha} = \int_y^{\infty} dy' \frac{(E'-E)}{y'} \left(\frac{1}{b'} \frac{\partial b'}{\partial E'} \right)^{-1} \frac{\partial \rho_d(E')}{\partial \alpha} = 0 . \tag{5.7}$$

Also, it is useful to define the energy-independent variables, C_x and C_z, as

$$C_x \equiv \sqrt{\frac{b}{\xi'_L}} \left(\frac{3}{2}x - \frac{3}{4}x' + \eta \right)^{-1} = \frac{1}{\sqrt{\xi'_L}} \left(1 + \frac{4}{x'} \right)^{\frac{3}{4}} \tag{5.8}$$

and

$$C_z \equiv \sqrt{\frac{b}{\xi'_Q}} \left(3z^{-2} - \frac{3}{2} z^{-4} z'^2 + \eta \right)^{-1} = \frac{1}{\sqrt{\xi'_Q}} \left\{ \frac{\sqrt{\pi}}{2\sqrt{2}} \exp\left[-\frac{z'^2}{4} \right] D_{-3}^{-1}(z') \right\}^{\frac{1}{2}} . \tag{5.9}$$

In terms of C_x and C_z, the terms $\left(\frac{1}{b} \frac{\partial b}{\partial E} \right)^{-1}$ and $E' - E$ for both Gaussian and screened Coulomb cases, respectively, are

$$\left(\frac{1}{b} \frac{\partial b}{\partial E} \right)^{-1} = \frac{E_L}{2C_x} \sqrt{\frac{b}{\xi'_L}} , \tag{5.10}$$

$$\left(\frac{1}{b} \frac{\partial b}{\partial E} \right)^{-1} = \frac{E_Q}{2C_z} \sqrt{\frac{b}{\xi'_Q}} , \tag{5.11}$$

$$E' - E = \frac{1}{C_x} \left(\sqrt{\frac{b'}{\xi'_L}} - \sqrt{\frac{b}{\xi'_L}} \right), \tag{5.12}$$

and

$$E' - E = \frac{1}{C_z}\left(\sqrt{\frac{b'}{\xi'_Q}} - \sqrt{\frac{b}{\xi'_Q}}\right). \qquad (5.13)$$

Substituting (5.10)-(5.13) into (5.6) and (5.7), the variational equations become

$$\int_{\tilde{y}}^{\infty} d\tilde{y}'(\tilde{y} - \tilde{y}')\frac{\partial \rho_g}{\partial \alpha} = 0 \qquad (5.14)$$

and

$$\int_{y}^{\infty} dy'\left(y^{\frac{1}{2}}y'^{\frac{1}{4}} - y'^{\frac{3}{4}}\right)\frac{\partial \rho_d}{\partial \alpha} = 0. \qquad (5.15)$$

Recalling that the density of states (4.45) and (4.48) have the same form:

$$\rho_g = const \times \frac{a}{b^{3/4}} \exp\left[-\frac{b}{4\xi'}\right] D_{\frac{1}{2}}\left(\sqrt{\frac{b}{\xi'}}\right), \qquad (5.16)$$

we then have

$$\frac{\partial \rho_g}{\partial \alpha} = \rho_g\left[\frac{1}{a}\frac{\partial a}{\partial \alpha} - \frac{3}{4}\left(\frac{1}{b}\frac{\partial b}{\partial \alpha}\right) - \frac{b}{2\xi'}\left(\frac{1}{b}\frac{\partial b}{\partial \alpha}\right) + \frac{3}{4}\sqrt{\frac{b}{\xi'}}\left(\frac{1}{b}\frac{\partial b}{\partial \alpha}\right)\frac{D_{3/2}\left(\sqrt{b/\xi'}\right)}{D_{1/2}\left(\sqrt{b/\xi'}\right)}\right]. \qquad (5.17)$$

While the density of states (4.52) and (4.53) and their derivatives can be written as

$$\rho_d = const \times a \exp\left[-\frac{b}{2\xi'}\right] \qquad (5.18)$$

and

$$\frac{\partial \rho_d}{\partial \alpha} = \rho_d\left[\frac{1}{a}\frac{\partial a}{\partial \alpha} - \frac{b}{2\xi'}\left(\frac{1}{b}\frac{\partial b}{\partial \alpha}\right)\right]. \tag{5.19}$$

As you see in (5.17) and (5.19), the derivatives of the density of states are in terms of $\frac{1}{a}\frac{\partial a}{\partial \alpha}$ and $\frac{1}{b}\frac{\partial b}{\partial \alpha}$ which make us easily obtain all wanted derivatives by calculating entirely four pairs of those terms. For the Gaussian case, we get

$$\frac{\partial \rho_g}{\partial x} = 3\rho_g\left[\frac{1}{x} - \frac{C_x}{2}\tilde{y} + \frac{3C_x}{4}\frac{D_{-5/2}(\tilde{y})}{D_{-3/2}(\tilde{y})}\right], \tag{5.20}$$

$$\frac{\partial \rho_g}{\partial x'} = 3\rho_g\left[-\frac{5\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{2x'^2} + \frac{C_x}{4}\tilde{y} + \frac{\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{x'^2}\tilde{y}^2 - \left(\frac{3C_x}{8} + \frac{3\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{2x'^2}\tilde{y}\right)\frac{D_{-5/2}(\tilde{y})}{D_{-3/2}(\tilde{y})}\right], \tag{5.21}$$

$$\frac{\partial \rho_d}{\partial x} = 3\rho_d\left[\frac{3C_x}{4\sqrt{2y}} + \frac{1}{x} - \frac{C_x}{\sqrt{2y}}y\right], \tag{5.22}$$

and

$$\frac{\partial \rho_d}{\partial x'} = 3\rho_d\left[-\frac{3C_x}{8\sqrt{2y}} - \frac{4\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{x'^2} + \left(\frac{C_x}{2\sqrt{2y}} + \frac{2\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{x'^2}\right)y\right]. \tag{5.23}$$

Similarly, for the screened Coulomb case,

$$\frac{\partial \rho_s}{\partial z} = 3\rho_s\left[-\frac{2}{z} + \frac{2C_z\left(1 - z^{-2}z'^2\right)}{z^3}\tilde{y} - \frac{3C_z\left(1 - z^{-2}z'^2\right)}{z^3}\frac{D_{-5/2}(\tilde{y})}{D_{-3/2}(\tilde{y})}\right], \tag{5.24}$$

$$\frac{\partial \rho_g}{\partial z'} = 3\rho_g\left[\frac{5}{4}\frac{D_{-4}(z')}{D_{-3}(z')} + C_z z^{-4} z' \tilde{y} - \frac{1}{2}\frac{D_{-4}(z')}{D_{-3}(z')}\tilde{y}^2 - \left(\frac{3C_z z^{-4} z'}{2} - \frac{3}{4}\frac{D_{-4}(z')}{D_{-3}(z')}\tilde{y}\right)\frac{D_{1/2}(\tilde{y})}{D_{1/2}(\tilde{y})}\right], \tag{5.25}$$

$$\frac{\partial \rho_d}{\partial z} = 3\rho_d\left[-\frac{3C_z}{\sqrt{2y}}\frac{1 - z^{-2}z'^2}{z^3} - \frac{2}{z} + \frac{C_z}{\sqrt{2y}}\frac{1 - z^{-2}z'^2}{z^3}y\right], \tag{5.26}$$

and

$$\frac{\partial \rho_d}{\partial z'} = 3\rho_d\left[-\frac{3C_z z^{-4} z'}{2\sqrt{2y}} + \frac{2D_{-4}(z')}{D_{-3}(z')} + \left(\frac{2C_z z^{-4} z'}{\sqrt{2y}} - \frac{D_{-4}(z')}{D_{-3}(z')}\right)y\right]. \tag{5.27}$$

In the full-ground-state approximation, we can then write down the variational equations with the help of some properties of the parabolic cylinder function,

$$\int_{\tilde{y}}^{\infty} d\tilde{y}' \exp\left[-\frac{\tilde{y}'^2}{4}\right] D_{\frac{1}{2}}(\tilde{y}')\left[\left(\frac{1}{x} + \frac{C_x}{2\tilde{y}}\right)\tilde{y} - \frac{1}{x}\tilde{y}'\right] = 0, \tag{5.28}$$

$$\int_{\tilde{y}}^{\infty} d\tilde{y}' \exp\left[-\frac{\tilde{y}'^2}{4}\right] D_{\frac{1}{2}}(\tilde{y}')\left[\left(\frac{2}{x'^2}\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}} + \frac{C_x}{4\tilde{y}}\right)\tilde{y} - \frac{1}{2x'^2}\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}\tilde{y}'\right] = 0, \tag{5.29}$$

$$\int_{\tilde{y}}^{\infty} d\tilde{y}' \exp\left[-\frac{\tilde{y}'^2}{4}\right] D_{\frac{1}{2}}(\tilde{y}')\left[\left(-\frac{2}{z} - \frac{2C_z\sqrt{\xi_Q'}}{\tilde{y}}\frac{1 - z^{-2}z'^2}{z^3}\right)\tilde{y} + \frac{2}{z}\tilde{y}'\right] = 0, \tag{5.30}$$

and

$$\int_{\tilde{y}}^{\infty} d\tilde{y}' \exp\left[-\frac{\tilde{y}'^2}{4}\right] D_{\frac{1}{2}}(\tilde{y}') \left[\left(\frac{3}{4}\frac{D_{-4}(z')}{D_{-3}(z')} - \frac{C_z\sqrt{\xi_Q'}}{\tilde{y}} z^{-4} z'\right)\tilde{y} - \frac{3}{4}\frac{D_{-4}(z')}{D_{-3}(z')}\tilde{y}'\right] = 0, \tag{5.31}$$

for the Gaussian and screened Coulomb cases, respectively. By using the definition of the incomplete gamma function,

$$\int_{t}^{\infty} dt'\, t'^{a-1} \exp[-t'] = \Gamma(a,t), \tag{5.32}$$

and its recursion formulae, we obtain the variational equations in the deep-tail approximation for the Gaussian and screened Coulomb cases, respectively,

$$\left(\frac{1}{x} + \frac{1}{2} C_x\sqrt{\xi_L'}\, y^{-\frac{1}{2}}\right) y^{\frac{1}{2}}\, \Gamma\left(\frac{5}{4}, y\right) - \frac{1}{x}\Gamma\left(\frac{7}{4}, y\right) = 0, \tag{5.33}$$

$$\left(-\frac{3\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{2x'^2} - \frac{1}{4} C_x\sqrt{\xi_L'}\, y^{-\frac{1}{2}}\right) y^{\frac{1}{2}}\, \Gamma\left(\frac{5}{4}, y\right) - \frac{\left(C_x\sqrt{\xi_L'}\right)^{-\frac{4}{3}}}{2x'^2}\Gamma\left(\frac{7}{4}, y\right) = 0, \tag{5.34}$$

$$\left(-\frac{2}{z} - 2C_z\sqrt{\xi_Q'}\,\frac{1 - z^{-2}z'^2}{z^3}\, y^{-\frac{1}{2}}\right) y^{\frac{1}{2}}\, \Gamma\left(\frac{5}{4}, y\right) + \frac{2}{z}\Gamma\left(\frac{7}{4}, y\right) = 0, \tag{5.35}$$

and

$$\left(\frac{3}{4}\frac{D_{-4}(z')}{D_{-3}(z')} - C_z\sqrt{\xi_Q'}\, z^{-4} z'\, y^{-\frac{1}{2}}\right) y^{\frac{1}{2}}\, \Gamma\left(\frac{5}{4}, y\right) - \frac{1}{4}\frac{D_{-4}(z')}{D_{-3}(z')}\Gamma\left(\frac{7}{4}, y\right) = 0. \tag{5.36}$$

According to the Halperin and Lax' variational ansatz, (5.4), It is easy to get the variational equations by setting the equations (5.20)-(5.27) equal to zero. In the very low-lying energy tail, the variational equation (5.5) is obtained easily by taking the coefficient in front of y, in (5.22) and (5.23) for the Gaussian case and in (5.26) and (5.27) for the screened Coulomb case, equal to zero. It should be noted that the variational equation in this limit for the full-ground-state and deep-tail approximations are not different because they have the same exponent, except the factor 2.

NUMERICAL RESULTS

In order to compare with other methods, the numerical results are important and necessary. Moreover, these are also useful in practice for calculating any quantity. Since we have derived many cases of the density of states, it is not easy to calculate and show all of them numerically. Hence, only the density of states for the screened Coulomb potential by maximizing themselves shall be performed and tabulated here. The expression for the density of states in the full-ground-state approximation is given by (4.48) and its variational equations are

$$-\frac{2}{z}+\frac{2C_z\left(1-z^{-2}z'^2\right)}{z^3}\tilde{y}-\frac{3C_z\left(1-z^{-2}z'^2\right)}{z^3}\frac{D_{1/2}(\tilde{y})}{D_{3/2}(\tilde{y})}=0 \tag{5.37}$$

and

$$\frac{5}{4}\frac{D_{-4}(z')}{D_{-3}(z')}+C_z z^{-4}z'\,\tilde{y}-\frac{1}{2}\frac{D_{-4}(z')}{D_{-3}(z')}\tilde{y}^2-\left(\frac{3C_z z^{-4}z'}{2}-\frac{3}{4}\frac{D_{-4}(z')}{D_{-3}(z')}\tilde{y}\right)\frac{D_{1/2}(\tilde{y})}{D_{3/2}(\tilde{y})}=0. \tag{5.38}$$

For the deep-tail approximation, it is given by (4.53) with the following constraints

$$-\frac{3C_z}{\sqrt{2y}}\frac{1-z^{-2}z'^2}{z^3}-\frac{2}{z}+\frac{C_z}{\sqrt{2y}}\frac{1-z^{-2}z'^2}{z^3}y=0 \tag{5.39}$$

and

$$-\frac{3C_z z^{-4}z'}{2\sqrt{2y}}+\frac{2D_{-4}(z')}{D_{-3}(z')}+\left(\frac{2C_z z^{-4}z'}{\sqrt{2y}}-\frac{D_{-4}(z')}{D_{-3}(z')}\right)y=0. \tag{5.40}$$

It is also useful to show the numerical results of the one-parameter theory. The density of states in both approximations are similar to those in the two-parameter theory except for the dimensionless functions. The two functions $a(\eta;z,z')$ and $b(\eta;z,z')$ must be replaced by

$$a(\eta\ ;z)=\frac{\sqrt{\pi}}{8\sqrt{2}}z^{-6}\left(\frac{3}{2}z^{-2}+\eta\right)^{\frac{1}{2}}\exp\left[\frac{-z^2}{2}\right]D_{-3}^{-2}(z) \tag{5.41}$$

and

$$b(\eta\ ;z)=\frac{\sqrt{\pi}}{2\sqrt{2}}\left(\frac{3}{2}z^{-2}+\eta\right)^{2}\exp\left[\frac{-z^2}{4}\right]D_{-3}^{-1}(z), \tag{5.42}$$

where η and z are defined as before. The variational equations for the full-ground-state and deep-tail approximations, respectively, are given below :

$$-\frac{2}{z}+\frac{5}{4}\frac{D_{-4}(z)}{D_{-3}(z)}+C_z z^{-3}\tilde{y}-\frac{1}{2}\frac{D_{-4}(z)}{D_{-3}(z)}\tilde{y}^2-\left(\frac{3C_z z^{-3}}{2}-\frac{3}{4}\frac{D_{-4}(z)}{D_{-3}(z)}\tilde{y}\right)\frac{D_{-\frac{1}{2}}(\tilde{y})}{D_{\frac{1}{2}}(\tilde{y})}=0,$$

(5.43)

and

$$-\frac{2}{z}-\frac{3C_z z^{-3}}{2\sqrt{2y}}+\frac{2D_{-4}(z)}{D_{-3}(z)}+\left(\frac{2C_z z^{-3}}{\sqrt{2y}}-\frac{D_{-4}(z)}{D_{-3}(z)}\right)y=0. \qquad (5.44)$$

When we have used $\tilde{y}=\sqrt{\frac{b(\eta;z)}{\xi_Q'}}$, $y=\frac{b(\eta;z)}{2\xi_Q'}$ and $C_z=\sqrt{\frac{b(\eta;z)}{\xi_Q'}}\left(\frac{3}{2}z^{-2}+\eta\right)^{-1}$ in (5.43) and (5.44). In addition, the density of states by maximizing the exponent, which satisfies the variational equation

$$\frac{2C_z z^{-3}}{\sqrt{2y}}=\frac{D_{-4}(z)}{D_{-3}(z)}, \qquad (5.45)$$

are also shown. Note that all numerical calculations are performed by using Mathematica.

Table 5.1 The density of states for the screened coulomb potential using the one-parameter theory by maximizing the density of states and by maximizing the exponent when ξ_Q'=50

ξ_Q'	η	Maximizing Exponent		Maximizing Density of States			
				Deep-tail		Full-ground-state	
		z	$\rho(z)$	z	$\rho_d(z)$	z	$\rho_g(z)$
50	50	0.416109	$2.35783*10^{-24}$	0.394504	$2.71563*10^{-24}$	0.394503	$1.44006*10^{-25}$
	45	0.430792	$6.62348*10^{-20}$	0.404932	$7.80024*10^{-20}$	0.404931	$4.13402*10^{-21}$
	40	0.447815	$7.73402*10^{-16}$	0.146254	$9.37517*10^{-16}$	0.416252	$4.96493*10^{-17}$
	35	0.46792	$3.66123*10^{-12}$	0.428461	$4.6136*10^{-12}$	0.428455	$2.44072*10^{-13}$
	30	0.492245	$6.80422*10^{-9}$	0.441368	$9.05184*10^{-9}$	0.441356	$4.78139*10^{-10}$
	25	0.522637	$4.75274*10^{-6}$	0.454348	$6.85017*10^{-6}$	0.454321	$3.61005*10^{-7}$
	20	0.562374	$1.17141*10^{-3}$	0.465633	$1.91866*10^{-3}$	0.465556	$1.00732*10^{-4}$
	15	0.61805	$9.20534*10^{-2}$	0.470557	$1.89406*10^{-1}$	0.470319	$9.6778*10^{-3}$
	10	0.705918	$1.90207*10^{0}$	0.458505	$6.32563*10^{0}$	0.457671	$3.25777*10^{-1}$
	5	0.885785	$6.20127*10^{0}$	0.417754	$7.42246*10^{1}$	0.415444	$3.73995*10^{0}$
	4	0.952892	$5.51637*10^{0}$	0.406758	$1.08909*10^{2}$	0.404115	$5.45894*10^{0}$
	3	1.04696	$3.99752*10^{0}$	0.395358	$1.54997*10^{2}$	0.392412	$7.72613*10^{0}$
	2	1.19559	$2.09635*10^{0}$	0.383816	$2.14557*10^{2}$	0.38061	$1.06362*10^{1}$
	1	1.50091	$5.59447*10^{-1}$	0.372357	$2.89672*10^{2}$	0.368944	$1.42813*10^{1}$
	0	-	-	0.361158	$3.81419*10^{2}$	0.357591	$1.87520*10^{1}$

Table 5.2 The density of states for the screened coulomb potential using the two-parameter theory by maximizing the density of states when ξ'_Q=50

		Maximizing Density of States					
		Deep-tail			Full-ground-state		
ξ'_Q	η	z	z'	$\rho_d(z,z')$	z	z'	$\rho_g(z,z')$
50	50	0.410101	0.393622	$2.41328 \cdot 10^{-24}$	0.410101	0.393621	$1.27972 \cdot 10^{-25}$
	45	0.423514	0.403695	$6.80184 \cdot 10^{-20}$	0.423513	0.403693	$3.60485 \cdot 10^{-21}$
	40	0.438792	0.414448	$7.97635 \cdot 10^{-16}$	0.438791	0.414444	$4.22406 \cdot 10^{-17}$
	35	0.456399	0.425686	$3.79753 \cdot 10^{-12}$	0.456397	0.425679	$2.00894 \cdot 10^{-13}$
	30	0.476941	0.436815	$7.11385 \cdot 10^{-9}$	0.476937	0.436801	$3.75748 \cdot 10^{-10}$
	25	0.501149	0.446173	$5.02831 \cdot 10^{-6}$	0.501138	0.446136	$2.64956 \cdot 10^{-7}$
	20	0.529573	0.448931	$1.26398 \cdot 10^{-3}$	0.529536	0.448819	$6.63362 \cdot 10^{-5}$
	15	0.560451	0.429343	$1.03352 \cdot 10^{-1}$	0.560291	0.428886	$5.36337 \cdot 10^{-3}$
	10	0.575404	0.332319	$2.4123 \cdot 10^{0}$	0.574269	0.329951	$1.23666 \cdot 10^{-1}$
	5	0.521299	0.133934	$1.62762 \cdot 10^{1}$	0.517675	0.130752	$8.13836 \cdot 10^{-1}$
	4	0.505991	0.103546	$2.15075 \cdot 10^{1}$	0.50215	0.100952	$1.0705 \cdot 10^{0}$
	3	0.491165	0.0783978	$2.78129 \cdot 10^{1}$	0.487214	0.0764044	$1.37638 \cdot 10^{0}$
	2	0.477133	0.0577744	$3.53246 \cdot 10^{1}$	0.473089	0.056344	$1.74348 \cdot 10^{0}$
	1	0.463911	0.040905	$4.41847 \cdot 10^{1}$	0.45963	0.0399773	$2.17223 \cdot 10^{0}$
	0	0.451512	0.0270958	$5.45432 \cdot 10^{1}$	0.447418	0.0266069	$2.67137 \cdot 10^{0}$

Table 5.3 The density of states for the screened coulomb potential using the one-parameter theory by maximizing the density of states and by maximizing the exponent when $\xi_Q'=5.0$

		Maximizing Exponent		Maximizing Density of States			
				Deep-tail		Full-ground-state	
ξ_Q'	η	z	$\rho(z)$	z	$\rho_d(z)$	z	$\rho_g(z)$
5.0	5.0	0.885785	$1.82596 \cdot 10^{-6}$	0.788311	$2.40053 \cdot 10^{-6}$	0.788251	$7.10047 \cdot 10^{-7}$
	4.5	0.916862	$1.46019 \cdot 10^{-5}$	0.80768	$1.99231 \cdot 10^{-5}$	0.802586	$5.88277 \cdot 10^{-6}$
	4.0	0.952692	$9.9122 \cdot 10^{-5}$	0.816979	$1.41798 \cdot 10^{-4}$	0.816827	$4.17768 \cdot 10^{-5}$
	3.5	0.995455	$5.64688 \cdot 10^{-4}$	0.830514	$8.59578 \cdot 10^{-4}$	0.830258	$2.52525 \cdot 10^{-4}$
	3.0	1.04696	$2.65836 \cdot 10^{-3}$	0.842034	$4.40394 \cdot 10^{-3}$	0.841582	$1.28887 \cdot 10^{-3}$
	2.5	1.11134	$1.01158 \cdot 10^{-2}$	0.849362	$1.89116 \cdot 10^{-2}$	0.84853	$5.50845 \cdot 10^{-3}$
	2.0	1.19559	$3.00779 \cdot 10^{-2}$	0.849037	$6.75515 \cdot 10^{-2}$	0.847471	$1.95325 \cdot 10^{-2}$
	1.5	1.3136	$6.59382 \cdot 10^{-2}$	0.836847	$1.99946 \cdot 10^{-1}$	0.833777	$5.7262 \cdot 10^{-2}$
	1.0	1.50091	$9.47005 \cdot 10^{-2}$	0.809316	$4.93163 \cdot 10^{-1}$	0.804585	$1.39671 \cdot 10^{-1}$
	0.5	1.88698	$6.36509 \cdot 10^{-2}$	0.769501	$1.03356 \cdot 10^{0}$	0.762909	$2.89014 \cdot 10^{-1}$
	0.4	2.03237	$4.90072 \cdot 10^{-2}$	0.760641	$1.17825 \cdot 10^{0}$	0.753741	$3.26625 \cdot 10^{-1}$
	0.3	2.23769	$3.33808 \cdot 10^{-2}$	0.751631	$1.33642 \cdot 10^{0}$	0.744453	$3.71791 \cdot 10^{-1}$
	0.2	2.56619	$1.64594 \cdot 10^{-2}$	0.742526	$1.5086 \cdot 10^{0}$	0.735104	$4.18631 \cdot 10^{-1}$
	0.1	3.25976	$6.37982 \cdot 10^{-3}$	0.733385	$1.69526 \cdot 10^{0}$	0.725746	$4.69258 \cdot 10^{-1}$
	0.0	-	-	0.724248	$1.89666 \cdot 10^{0}$	0.716427	$5.23762 \cdot 10^{-1}$

Table 5.4 The density of states for the screened coulomb potential using the two-parameter theory by maximizing the density of states when ξ_Q'=5.0

		Maximizing Density of States					
		Deep-tail			Full-ground-state		
ξ_Q'	η	z	z'	$\rho_d(z,z')$	z	z'	$\rho_g(z,z')$
5.0	5.0	0.854346	0.778818	$1.86615*10^{-6}$	0.854323	0.778742	$5.51889*10^{-7}$
	4.5	0.879019	0.789609	$1.49514*10^{-5}$	0.87898	0.789484	$4.4136*10^{-6}$
	4.0	0.906227	0.798255	$1.01732*10^{-4}$	0.906159	0.798042	$2.99598*10^{-5}$
	3.5	0.936052	0.602319	$5.81334*10^{-4}$	0.935922	0.801924	$1.70662*10^{-4}$
	3.0	0.967973	0.796749	$2.74885*10^{-3}$	0.967697	0.795953	$8.03463*10^{-4}$
	2.5	0.999472	0.770358	$1.05391*10^{-2}$	0.998802	0.766558	$3.06117*10^{-3}$
	2.0	1.02135	0.89738	$3.18688*10^{-2}$	1.01943	0.892842	$9.16967*10^{-3}$
	1.5	1.00728	0.535243	$7.38913*10^{-2}$	1.00186	0.525479	$2.09779*10^{-2}$
	1.0	0.943456	0.325976	$1.34202*10^{-1}$	0.93519	0.318825	$3.75945*10^{-2}$
	0.5	0.875202	0.179102	$2.098*10^{-1}$	0.866948	0.174334	$5.82125*10^{-2}$
	0.4	0.863194	0.157561	$2.2693*10^{-1}$	0.855039	0.15353	$6.28663*10^{-2}$
	0.3	0.85175	0.138032	$2.44792*10^{-1}$	0.843702	0.134675	$6.77134*10^{-2}$
	0.2	0.840842	0.120285	$2.63416*10^{-1}$	0.832901	0.117543	$7.27609*10^{-2}$
	0.1	0.830435	0.104119	$2.82826*10^{-1}$	0.822601	0.101937	$7.80161*10^{-2}$
	0.0	0.820497	0.0893546	$3.03057*10^{-1}$	0.812766	0.0876823	$8.34862*10^{-2}$

Table 5.5 The density of states for the screened coulomb potential using the one-parameter theory by maximizing the density of states and by maximizing the exponent when ξ_Q'=0.5

		Maximizing Exponent		Maximizing Density of States			
				Deep-tail		Full-ground-state	
ξ_Q'	η	z	$\rho(z)$	z	$\rho_d(z)$	z	$\rho_g(z)$
0.5	0.5	1.88698	$2.04724*10^{-5}$	1.61939	$2.73514*10^{-5}$	161882	$4.50763*10^{-5}$
	0.45	1.9542	$4.30791*10^{-5}$	1.64761	$5.93753*10^{-5}$	1.6468	$9.76339*10^{-5}$
	0.4	2.03237	$8.66023*10^{-5}$	1.67624	$1.24068*10^{-4}$	1.67508	$2.03464*10^{-4}$
	0.35	2.12504	$1.65359*10^{-4}$	1.70447	$2.48862*10^{-4}$	1.70276	$4.06791*10^{-4}$
	0.3	2.23769	$2.97456*10^{-4}$	1.73089	$4.77673*10^{-4}$	1.72831	$7.77702*10^{-4}$
	0.25	2.37933	$4.98031*10^{-4}$	1.75316	$8.7263*10^{-4}$	1.74917	$1.41644*10^{-3}$
	0.2	2.58619	$7.61199*10^{-4}$	1.76768	$1.52019*10^{-3}$	1.76148	$2.44822*10^{-3}$
	0.15	2.83152	$1.02565*10^{-3}$	1.76979	$2.50329*10^{-3}$	1.78032	$4.0026*10^{-3}$
	0.1	3.25978	$1.13006*10^{-3}$	1.75512	$3.8976*10^{-3}$	1.74145	$6.18089*10^{-3}$
	0.05	4.17952	$8.07729*10^{-4}$	1.72243	$5.74584*10^{-3}$	1.70439	$9.03185*10^{-3}$
	0.04	4.53949	$6.70531*10^{-4}$	1.71397	$6.1713*10^{-3}$	1.69513	$9.88342*10^{-3}$
	0.03	5.06093	$5.09554*10^{-4}$	1.70497	$6.61535*10^{-3}$	1.68537	$1.03619*10^{-2}$
	0.02	5.92697	$3.29881*10^{-4}$	1.69548	$7.07793*10^{-3}$	1.67517	$1.1067*10^{-2}$
	0.01	7.87469	$1.44095*10^{-4}$	1.68556	$7.55896*10^{-3}$	1.6646	$1.17986*10^{-2}$
	0	-	-	1.67526	$8.05637*10^{-3}$	1.6537	$1.25566*10^{-2}$

Table 5.6 The density of states for the screened coulomb potential using the two-parameter theory by maximizing the density of states when ξ_Q'=0.5

		Maximizing Density of States					
		Deep-tail			Full-ground-state		
ξ_Q'	η	z	z'	$\rho_d(z,z')$	z	z'	$\rho_g(z,z')$
0.5	0.5	1.7797	1.58408	$2.01384*10^{-5}$	1.7794	1.58325	$3.31637*10^{-5}$
	0.45	1.82675	1.60054	$4.22342*10^{-5}$	1.62628	1.5993	$6.93742*10^{-5}$
	0.4	1.87737	1.61117	$8.45415*10^{-5}$	1.87661	1.60922	$1.38431*10^{-4}$
	0.35	1.93065	1.61021	$1.60531*10^{-4}$	1.92933	1.60695	$2.61805*10^{-4}$
	0.3	1.98327	1.5856	$2.8666*10^{-4}$	1.98074	1.57989	$4.65046*10^{-4}$
	0.25	2.02481	1.51162	$4.7534*10^{-4}$	2.01931	1.49971	$7.65631*10^{-4}$
	0.2	2.02361	1.329	$7.18666*10^{-4}$	2.01006	1.30346	$1.1461*10^{-3}$
	0.15	1.92248	0.987152	$9.73862*10^{-4}$	1.89746	0.951163	$1.53515*10^{-3}$
	0.1	1.77519	0.665294	$1.20519*10^{-3}$	1.75131	0.641721	$1.88425*10^{-3}$
	0.05	1.66227	0.461772	$1.42104*10^{-3}$	1.84231	0.448698	$2.21081*10^{-3}$
	0.04	1.84374	0.430817	$1.46354*10^{-3}$	1.62442	0.419284	$2.27518*10^{-3}$
	0.03	1.62629	0.402369	$1.50598*10^{-3}$	1.60756	0.392103	$2.33948*10^{-3}$
	0.02	1.60961	0.375968	$1.54841*10^{-3}$	1.59164	0.366893	$2.40377*10^{-3}$
	0.01	1.59422	0.351436	$1.59088*10^{-3}$	1.57655	0.343435	$2.46812*10^{-3}$
	0	1.57944	0.328567	$1.63342*10^{-3}$	1.56224	0.321539	$2.53259*10^{-3}$

Table 5.7 The density of states for the screened coulomb potential using the one-parameter theory by maximizing the density of states and by maximizing the exponent when ξ_Q'=0.05

ξ_Q'	η	Maximizing Exponent		Maximizing Density of States			
				Deep-tail		Full-ground-state	
		z	$\rho(z)$	z	$\rho_d(z)$	z	$\rho_g(z)$
0.05	0.05	4.17952	3.41942*10^{-8}	3.74299	4.00188*10^{-8}	3.74233	3.72277*10^{-7}
	0.045	4.34501	6.61925*10^{-8}	3.85182	7.84553*10^{-8}	3.85097	7.28907*10^{-7}
	0.04	4.53949	1.27137*10^{-7}	3.97454	1.53024*10^{-7}	3.97338	1.41957*10^{-6}
	0.035	4.77293	2.41915*10^{-7}	4.1144	2.96795*10^{-7}	4.1128	2.74837*10^{-6}
	0.03	5.06093	4.54905*10^{-7}	4.2758	5.71986*10^{-7}	4.27348	5.28505*10^{-6}
	0.025	5.42961	8.41889*10^{-7}	4.46469	1.09403*10^{-6}	4.46116	1.00808*10^{-5}
	0.02	5.92697	1.52185*10^{-6}	4.68904	2.07269*10^{-6}	4.6833	1.90302*10^{-5}
	0.015	6.65396	2.64504*10^{-6}	4.95873	3.87599*10^{-6}	4.94849	3.54145*10^{-5}
	0.01	7.67469	4.24691*10^{-6}	5.28137	7.10705*10^{-6}	5.26087	8.44869*10^{-5}
	0.005	-	-	5.63894	1.26067*10^{-5}	5.59275	1.13202*10^{-4}
	0.004	11.7863	5.26971*10^{-6}	5.70694	1.40533*10^{-5}	5.65227	1.25857*10^{-4}
	0.003	13.4542	4.63169*10^{-6}	5.7702	1.5625*10^{-5}	5.70564	1.39535*10^{-4}
	0.002	16.2736	3.69766*10^{-6}	5.82643	1.73222*10^{-5}	5.75069	1.5422*10^{-4}
	0.001	-	-	5.87329	1.91424*10^{-5}	5.78529	1.69876*10^{-4}
	0	-	-	5.90848	2.10805*10^{-5}	5.80762	1.86447*10^{-4}

Table 5.8 The density of states for the screened coulomb potential using the two-parameter theory by maximizing the density of states when ξ_Q'=0.05

		Maximizing Density of States					
		Deep-tail			Full-ground-state		
ξ_Q'	η	z	z'	$\rho_d(z,z')$	z	z'	$\rho_g(z,z')$
0.05	0.05	3.9569	3.71601	3.35716*10^-8	3.95652	3.71523	3.12213*10^-7
	0.045	4.08751	3.81633	6.49278*10^-8	4.08699	3.8173	6.03009*10^-7
	0.04	4.2364	3.93185	1.24592*10^-7	4.23586	3.93042	1.15528*10^-6
	0.035	4.40817	4.05814	2.36865*10^-7	4.40708	4.05808	2.19204*10^-6
	0.03	4.60891	4.19831	4.45096*10^-7	4.60722	4.19519	4.1091*10^-6
	0.025	4.84644	4.35131	8.23553*10^-7	4.84357	4.34622	7.57913*10^-6
	0.02	5.1289	4.5081	1.49024*10^-6	5.12345	4.49891	1.36563*10^-5
	0.015	5.45406	4.62905	2.60211*10^-6	5.44171	4.60963	2.36981*10^-5
	0.01	5.73775	4.54054	4.25488*10^-6	5.70081	4.48889	3.83662*10^-5
	0.005	5.44443	3.60603	6.08938*10^-6	5.33209	3.45216	5.40626*10^-5
	0.004	5.25213	3.29161	6.39548*10^-6	5.1317	3.16725	5.66059*10^-5
	0.003	5.04284	2.96202	6.67092*10^-6	4.92248	2.89586	5.88863*10^-5
	0.002	4.63757	2.69908	6.91853*10^-6	4.72296	2.59574	6.09359*10^-5
	0.001	4.64863	2.45171	7.14262*10^-6	4.54236	2.36214	6.27945*10^-5
	0	4.48032	2.23968	7.34754*10^-6	4.38287	2.16285	6.44989*10^-5