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PATH INTEGRAL DERIVATION OF MAGNUS FORCE IN SUPERCONDUCTIVITY

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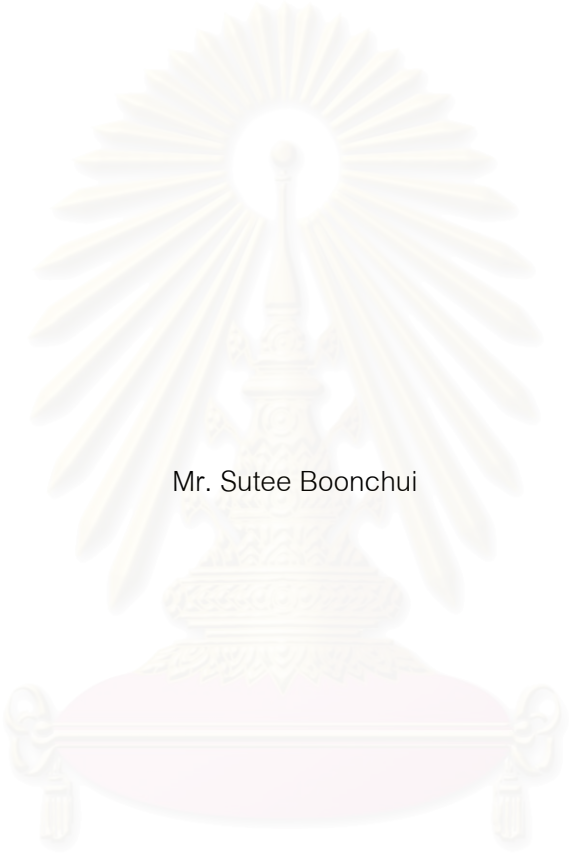

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ในวิทยานิพนธ์นี้ แสดงการกำเนิดของแรงแมกนัต จากแบบจำลองที่ประกอบด้วยวอร์เท็กซ์หนึ่งตัว มีอันตรกิริยากับประจุโบซอนซึ่งทั้งคู่ถูกบรรจุในพื้นหลังที่เป็นศักดิ์บวกคงที่ กิริยายังผลของวอร์เท็กซ์หนึ่งตัวได้มาจากการกำจัด พิกัดของประจุโบซอน ผลสรุปที่ได้จาก กิริยายังผลคือ แรง ฟายน์แมน-เฮลมานน์ จากการใช้ฟังก์ชันคลื่นที่สถานะฟื้น จะได้ว่านั้นคือ แรงแมกนัต และแสดงการมีอยู่จริงของแรงแมกนัต ว่าเป็นคุณสมบัติทั่วไปที่เกิดจากผลของการโคจรเฉพาะที่รอบวอร์เท็กซ์. ท้ายสุดเป็นการแสดงการวิเคราะห์ผลของสิ่งแวดล้อมต่อซึ่งเกิดขึ้นกับแรงแมกนัต ที่มีมาแต่เดิม

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SUTEE BOONCHUI : PATH INTEGRAL DERIVATION OF MAGNUS FORCE IN SUPERCONDUCTIVITY. THESIS ADVISOR : PROF. VIRULH SA-YAKANIT. F.D. ,60 pp. ISBN 974-346-533-2.

In this thesis we derive the Magnus force from a model system consisting of a single vortex imbedded in uniform positive background that is coupled with a mutual interaction charged boson. The effective action of a single vortex is obtained by eliminating the charged boson degree of freedom. It is shown that the resulting effective action can be used to derive the Feynman-Hellmann force. From the ground state contribution we obtain the Magnus force. We also show that the existence of the Magnus force is an effect of the transition amplitude of the suprecurrent and is independent on the effect of environment.

Department Physics	Student's signature .....
Field of study Physics	Advisor's signature .....
Academic year 2000	Co-advisor's signature .....

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Chapter I

Introduction

An understanding of superconductivity is based on the Bardeen-Cooper-Schriffer theory (BCS) [1], which involves the Bose condensation of pairs of electron which their binding energy is much less than the Fermi energy. Before the BCS theory was developed, Pippard [2] had discovered that the length scale ζ, which is related to the size of the Cooper pairs at low temperatures, is much longer than the penetration depth λ in type I superconductor. Afterwards, the type II superconductor had been discovered and they found that the penetration depth λ in this type is much longer than the length scale ζ. In such a superconductor, when magnetic field is applied to it with field strength exceeds a value referred to the lower critical field, B_{c1}, a magnetic flux is able to penetrate into the superconductor as a quantized unit of magnetic flux and forming a cylindrically symmetric domain called “vortex”. For an applied field slightly above B_{c1}, the magnetic field inside a type II superconductor is strongest in the cores of the vortex, and decreases with distance away from the core, and becomes very small at large distance. For much higher applied magnetic field there are many vortices occur and they are overlap. Hence the field inside the superconductor becomes strong everywhere.

proposed by Friedel, de Gennes and Matrican [4] and later developed and extended by Nozieres and Vinen [5] by including pinning and friction but they are phenomenological theories. More over, the first microscopic theory that tries to explain the Magnus force is due to Ao and Thouless [6]. They derived the force for an adiabatic motion of the vortex. They also found that the existence of the Magnus force is a general property of the vortex line and is not influenced by the presence of the disorder and the magnetic field. Since then there have been several attempts to derive the Magnus force from different fundamental approaches such as by the Chern-Simons theory[7], Feynman-Hellmann theorem [8], path integral derivation of Magnus force [9],etc.

1.1 The Magnus Force in Classical Hydrodynamics

Fluid dynamics concerns itself with the study of motion of fluids. Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. This means that any small volume element in the fluid is always supposed to be so large that it still contains a great number of molecules. Consider the volume V_0. The mass of fluid in this volume is $\int dV\ \rho$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $d\vec{A}$ of the surface bounding this volume is

$\rho \vec{V} \cdot d\vec{A}$, $\vec{V}$ is the velocity field, the magnitude of the vector $d\vec{A}$ is equal to the area of the surface element, and its direction is along the normal. We take $d\vec{A}$ along the outward normal. Then $\rho \vec{V} \cdot d\vec{A}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint d\vec{A} \cdot \rho \vec{V} \tag{1-1}$$

where the integration is taken over the whole of the closed surface surrounding the volume in question. Next, the decreasing the mass of the fluids per unit time in the volume can be written

$$-\frac{\partial}{\partial t} \int dV \ \rho \ . \tag{1-2}$$

Equating the two expressions, we have

$$\oint d\vec{A} \cdot \rho \vec{V} = -\frac{\partial}{\partial t} \int dV \rho \ . \tag{1-3}$$

The surface integral can be transformed by Divergence theorem to a volume integral

$$\oint d\vec{A} \cdot \rho \vec{V} = \int dV \ \vec{\nabla} \cdot \left(\rho \vec{V}\right). \tag{1-4}$$

Thus

$$\int dV \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) \right] = 0 \tag{1-5}$$

since this equation must hold for any volume, the integrand must vanish,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0 \quad . \tag{1-6}$$

The vector

$$\vec{j} = \rho \vec{V} \tag{1-7}$$

is called the mass flux density. Its direction is that of the motion of the fluid, while its magnitude is equal to the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

For a better understanding of the origin of the Magnus force it is worth recalling how the Magnus force arises in classical hydrodynamics. Let us consider an isolated straight vortex line in an incompressible inviscid liquid (see Fig. 1-2). The vortex line along the axis **z** induces the velocity field

$$\vec{V}_V (\vec{r}) = \frac{\vec{k} \times \vec{r}}{2 \pi r^2} \tag{1-8}$$

Here $\vec{r}$ is position vector in the moving frame, $\vec{V}_L$, and $\vec{k}$ is the circulation vector directed along the axis **z**. This circulation given by

$$k = \oint d\vec{l} \cdot \vec{V}_V \tag{1-9}$$

and may have arbitrary values in classical hydrodynamics. In addition, there is the fluid flow past the vortex line with a transport velocity $\vec{V}_{tr}$. Then the net velocity field around the line is

$$\vec{V} (\vec{r}) = \vec{V}_V (\vec{r}) + \vec{V}_{tr} \tag{1-10}$$

Fig. 1-2

Sketch of the vortex move in uniform background.

The Euler equation for the liquid is

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V} = -\frac{1}{\rho}\left[\vec{\nabla} P - \vec{F}\,\delta\left(\vec{r}\right)\right] \qquad (1\text{-}11)$$

Here ρ is the liquid density and P is the pressure. This equation implies that an external force $\vec{F}$ is applied to the vortex line.

Assuming that the vortex line move with the constant velocity $\vec{V}_L$ and replacing the center point of vortex by

$$\vec{X} = \vec{X}_0 - \vec{V}_L t\,. \qquad (1\text{-}12)$$

Here $\vec{x}_0$ is the center point of vortex in the fixed frame at t = 0. Then the Euler equation (1-11) yields the Bernoulli law for the pressure:

$$\begin{aligned} P &= P_0 - \frac{1}{2}\rho\left[\vec{V}(\vec{r}) - \vec{V}_L\right]^2 \\ &= P_0' - \frac{1}{2}\rho\left[\vec{V}_V^{\,2}(\vec{r}) + 2\vec{V}_V \cdot \left(\vec{V}_{tr} - \vec{V}_L\right)\right]. \end{aligned} \tag{1-13}$$

Here P_0 and $P_0' = P_0 - \frac{1}{2}\rho\left[\vec{V}_{tr} - \vec{V}_L\right]^2$ are constant which are of no importance for the following derivation. Next one should consider the momentum balance for a cylindrical region of radius r_0 around the vortex lines by r_0 is the radius of vortex.

The momentum - flux tensor is given by

$$\Pi_{ij} = P\,\delta_{ij} + \rho\, V_j(\vec{r})V_i(\vec{r}) \tag{1-14}$$

or in the reference frame moving with the vortex velocity $\vec{V}_L$:

$$\Pi'_{ij} = P\,\delta_{ij} + \rho\left(V_j(\vec{r}) - V_{Lj}\right)\left(V_i(\vec{r}) - V_{Li}\right) \tag{1-15}$$

The momentum conservation law requires that the external force $\vec{F}$ on the vortex line must be equal to the momentum-flux through the entire cylindrical boundary in the reference frame moving with the vortex velocity $\vec{V}_L$. The latter is given by the integral $\int dA_i \Pi'_{ij}$ where dA_i are the components of the vector $d\vec{A}$ directed along the outer normal to the boundary of the cylindrical region and is equal to the elementary area of the boundary in magnitude.

Then using Equation (1-8), (1-10) and (1-13), the momentum balance yields the following relation:

$$\vec{F}_{Magnus} = \rho\left[\left(\vec{V}_L - \vec{V}_{tr}\right)\times \vec{k}\right] \qquad (1\text{-}16)$$

Thus a force is exerted on the vortex when it moves relative to the fluid density. This force is proportional and perpendicular to the vortex velocity, and proportional to the fluid density. It is the Magnus force. The Magnus force makes the vortex dynamics similar to that of charged particles in a magnetic field, which the role of the magnetic field played by the fluid density.

1.2 Magnus Force and Chern-Simons Vortices

In type II superconductors, we derive the Magnus force in a two-dimensional superconductor film at zero temperature. We start by used the conclusions of I. V. Barashenkov and A.O Harin [10]. They formulated a (2+1) dimensioned Chern-Simons theory in which the matter density is finite at infinity and found that the Euler-Lagrange equation gives to the vortex solutions. The Lagrangian is written solely in terms of the matter field $\varphi\left(\vec{q};\vec{X}\right)$

$$L = \int d^2\vec{q}\left[\varphi^+\left(\vec{q};\vec{X}\right)i\hbar\partial_t\varphi\left(\vec{q};\vec{X}\right) - H\left(\varphi\left(\vec{q};\vec{X}\right)\right)\right] \qquad (1\text{-}17)$$

where $H\left(\varphi\left(\vec{q};\vec{X}\right)\right)$ is the energy density of system and has not explicit on time.

The one-vortex solution centered at $\vec{X}$ has the form

$$\varphi\left(\vec{q};\vec{X}\right)=\rho^{\frac{1}{2}}\left(\vec{q}-\vec{X}\right)e^{i\Theta} \tag{1-18}$$

with $\vec{q}=(q_X,q_Y)$ and $\vec{X}=(X,Y)$. Here Θ is function of $\vec{q}-\vec{X}$ where the latter is given by $\Theta=\tan^{-1}\left[\frac{q_Y-Y}{q_X-X}\right]$ and $\rho\left(\vec{q}-\vec{X}\right)$ is the superfluid density

$$\rho\left(\vec{q}-\vec{X}\right)=\rho_0+\delta\rho\left(\vec{q};\vec{X}\right). \tag{1-19}$$

The superfluid density is the sum of the background density in the absence of the vortex plus the modification due to the presence of the vortex. The density $\rho\left(\vec{q}-\vec{X}\right)$ vanishes continuously at $\vec{q}=\vec{X}$ and approaches the background density ρ_0 as $\left|\vec{q}-\vec{X}\right|\to\infty$. This is done by regarding $\vec{X}$ as time dependent function and substituting equation (1-18) into (1-17) then we obtained effective Lagrangian describing the dynamics of the vortex center.

We consider the two terms in equation (1-17) separately. The second term, when integrated, gives the rest energy of the vortex and is irrelevant to our discussion of the Magnus force. The first term leads to

$$L^1_{eff}=\int d^2\vec{q}\left[\frac{i\hbar}{2}\partial_t\rho\left(\vec{q}-\vec{X}\right)-\rho\left(\vec{q}-\vec{X}\right)\hbar\partial_t\Theta\left(\vec{q}-\vec{X}\right)\right] \tag{1-20}$$

The number of particles is constant in time since $\partial_t\int d^2\vec{q}\rho\left(\vec{q}-\vec{X}\right)=0$, equation (1-20) can be simplified to

$$\begin{aligned} L^1_{\text{eff}} &= -\int d^2\vec{q}\rho(\vec{q}-\vec{X})\hbar\partial_t\Theta(\vec{q}-\vec{X}) \\ &= \dot{\vec{X}}\cdot\int d^2\vec{q}\rho(\vec{q}-\vec{X})\hbar\vec{\nabla}_q\Theta(\vec{q}-\vec{X}) \end{aligned} \tag{1-21}$$

where we have used the fact that Θ only depends on $\vec{q}-\vec{X}$ and $\vec{\nabla}_q$ is differentiation with respect to $\vec{q}$. Recalling that $\dot{\vec{X}}$ is the velocity of the vortex, we observe that L^1_{eff} describes the interaction of the vortex with the vector potential. The force $\vec{F}$ experienced by the vortex due to this interaction can then be obtained by varying (1-21) with respect to $\vec{X}$:

$$\vec{F}_{\text{Magnus}} = \dot{\vec{X}}\times\vec{B} \tag{1-22}$$

The field strength $\vec{B}$ is given by

$$\vec{B} = \vec{\nabla}_X\times\int d^2\vec{q}\,\rho(\vec{q}-\vec{X})\hbar\vec{\nabla}_q\Theta(\vec{q}-\vec{X}). \tag{1-22}$$

Since the integrand only depends on $\vec{q}-\vec{X}$, we write

$$\vec{B} = -\hbar\int d^2\vec{q}\,\vec{\nabla}_q\times\left[\rho(\vec{q}-\vec{X})\vec{\nabla}_q\Theta(\vec{q}-\vec{X})\right]. \tag{1-23}$$

Using the Stokes theorem $\int d^2\vec{r}\vec{\nabla}\times\vec{D} = \oint d\vec{l}\cdot\vec{D}$ and relation of

$$\vec{\nabla}\Theta(\vec{q}-\vec{X}) = \frac{\hat{k}\times(\vec{q}-\vec{X})}{|\vec{q}-\vec{X}|^2} \tag{1-24}$$

equation (1-23) can be simplified to

$$\vec{B} = -h\rho_0\hat{k}. \tag{1-25}$$

We can write the Magnus force as

$$\vec{F}_{\text{Magnus}} = -h\rho_0\dot{\vec{X}}\times\hat{k}. \tag{1-26}$$

The Magnus force was defined as the force between a vortex and superconductor at zero temperature. It corresponds to the Magnus force acting on a vortex in classical hydrodynamics, equation (1-16). This force appears if the vortex moves with respect to the liquid. This force is normal to the relative vortex velocity and does not produce work.

1.3 Magnus Force and Feynman–Hellmann Theorem

In this section we consider the derivation of the Magnus force using Feynman-Hellmann theorem, following the review paper of E. $\breve{\text{S}}$ima′nek [8]. The derivation starts from the time-dependent Schrödinger equation for the superconductor film at zero temperature which contain a single vortex located at point $\vec{X}$. The Schrödinger equation for the time evolution of the superfluid wave function is

$$i\hbar \frac{\partial}{\partial t}\left|\varphi\left(\vec{X}(t)\right)\right\rangle = H\left(\vec{X}(t)\right)\left|\varphi\left(\vec{X}(t)\right)\right\rangle \qquad (1\text{-}27)$$

At any instant, for $\vec{X} = \vec{X}(t)$, the instantaneous eigenstates satisfy

$$H\left(\vec{X}(t)\right)\left|\varphi_n\left(\vec{X}(t)\right)\right\rangle = E_n\left(\vec{X}(t)\right)\left|\varphi_n\left(\vec{X}(t)\right)\right\rangle \qquad (1\text{-}28)$$

Differentiating the ground–state energy $E_0(X(t))$ with respect to the vortex center coordinate X_i, we obtain

$$\frac{\partial E_0\left(\vec{X}(t)\right)}{\partial X_i} = \left\langle \partial \varphi_0 / \partial X_i \left| H\left(\vec{X}(t)\right) \right| \varphi_0 \right\rangle + \left\langle \varphi_0 \left| H\left(\vec{X}(t)\right) \right| \partial \varphi_0 / \partial X_i \right\rangle + \left\langle \varphi_0 \left| \frac{\partial H\left(\vec{X}(t)\right)}{\partial X_i} \right| \varphi_0 \right\rangle. \tag{1-29}$$

Using equation (1-29) the first two term of this equation can be expressed in term of the derivative of $\left|\varphi_0\left(\vec{X}(t)\right)\right\rangle$. We have

$$\frac{\partial}{\partial t}\left|\varphi_0\left(\vec{X}(t)\right)\right\rangle = \sum_k \left|\frac{\partial \varphi_0}{\partial X_k}\right\rangle \dot{X}_k \tag{1-30}$$

equation (1-29) can be simplified to

$$-\left\langle \varphi_0 \left| \frac{\partial H}{\partial X_i} \right| \varphi_0 \right\rangle = -\frac{\partial E\left(\vec{X}\right)}{\partial X_i} + \sum_k \dot{X}_k \omega_{ki} \tag{1-31}$$

and

$$\omega_{ki} = i\hbar\left[\left\langle \frac{\partial \varphi_0}{\partial X_k} \middle| \frac{\partial \varphi_0}{\partial X_i} \right\rangle - \left\langle \frac{\partial \varphi_0}{\partial X_i} \middle| \frac{\partial \varphi_0}{\partial X_k} \right\rangle\right] \tag{1-32}$$

where ω_{ki} is the adiabatic curvature tensor.

The right–hand size of equation (1-31) is known as a time–dependent version of the **Feynman–Hellmann theorem**.

The force on the liquid at the vortex line is

$$F^k{}_{Magnus} = -\frac{\partial}{\partial x_k} E\left(\vec{X}\right) + \sum_k \dot{X}_k \omega_{ki} \,. \tag{1-33}$$

The second term makes the vortex dynamics similar to that of charged particles in a magnetic field, with the role of the magnetic filed played by the adiabatic curvature tensor. From equation (1-32) and (1-33), the components of Magnus force in x and y direction are

$$
\begin{aligned}
F_{X\,\mathrm{Magnus}} &= i\hbar\dot{X}_Y\left[\left\langle\frac{\partial\varphi_0}{\partial Y}\middle|\frac{\partial\varphi_0}{\partial X}\right\rangle-\left\langle\frac{\partial\varphi_0}{\partial X}\middle|\frac{\partial\varphi_0}{\partial Y}\right\rangle\right] \\
F_{Y\,\mathrm{Magnus}} &= i\hbar\dot{X}_X\left[\left\langle\frac{\partial\varphi_0}{\partial Y}\middle|\frac{\partial\varphi_0}{\partial X}\right\rangle-\left\langle\frac{\partial\varphi_0}{\partial X}\middle|\frac{\partial\varphi_0}{\partial Y}\right\rangle\right]
\end{aligned}
\qquad (1\text{-}34)
$$

respectively.

Scope of This Thesis

In this chapter, we show the derivation of Magnus force in classical hydrodynamics by following Sonin [11] and the was derived Magnus force occur in type II superconductor was derived reviewing the paper of I.V. Barashenkev and A.O. Harin [10] and E. Šima′nek [8].

In chapter II, we will give a brief review of the B–O approximation in molecular physics by follow textbook Geometric Phase in Physics, [12].

In chapter III, the mathematical method of path integration is reviewed [13] and its application to the geometrical phase will be given [14].

In chapter IV, we will present an attempt to clarify the origin of the Magnus force and the effect of the environment on the Magnus force will be discussed.

In chapter V, the discussion of the results and conclusion will be given.

Chapter II

Born–Oppenheimer Hamiltonian

In molecular physics, it is useful to treat the electronic and nuclear degrees of freedom as fast and slow variables, respectively. This is because the gap between nuclear levels, by a factor of order $\left(\frac{M}{m}\right)^{\frac{1}{4}}$. In the Born–Oppenheimer approximation, one solves for the electronic states in a fixed nuclear background. By the adiabatic theorem, one expects these electronic states to be approximately stationary with respect to the relatively slow motions of the nuclei. We can thus obtain an effective description for the nuclear motion, relative to a fixed electronic orbital, by integrating over electronic coordinates. We shall find that the effective nuclear Hamiltonian obtained in this way involves both an ordinary potential terms due to electronic energy levels and background gauge potential that couples to the nuclear current. This gauge potential takes into account the Berry phase accumulated by the electronic wave functions when the nuclear coordinates change adiabatically.

The Born–Oppenheimer effective Hamiltonion begins with the full Hamiltonion of system

$$H = \frac{\vec{P}^{\,2}}{2M} + \frac{\vec{p}^{\,2}}{2m} + V\left(\vec{R}, \vec{r}\right) \qquad (2\text{-}1)$$

and full Schrödinger equation can be written as

$$\left(\frac{\hat{\vec{P}}^{\,2}}{2M}+\frac{\hat{\vec{p}}^{\,2}}{2m}+V\left(\hat{\vec{R}},\hat{\vec{r}}\right)\right)\varphi(\vec{r},\vec{R})=E\,\varphi(\vec{r},\vec{R}), \qquad (2\text{-}2)$$

where $\frac{\vec{P}^{\,2}}{2M}$ is the kinetic energy term of the nuclei,

$\frac{\vec{p}^{\,2}}{2m}$ is the kinetic energy term of the electronic,

$V(\vec{R},\vec{r})$ contains the interaction energies of the nuclei and electronic, and $\vec{r}$ and $\vec{R}$ are the electronic and the nuclear coordinates respectively. We spit therefor the full Hamiltonian into the fast and slow part

$$\begin{aligned} H &= \frac{\vec{P}^{\,2}}{2M}+h(\vec{p},\vec{r},\vec{R}) \\ h(\vec{p},\vec{r},\vec{R}) &= \frac{\vec{p}^{\,2}}{2m}+V(\vec{r},\vec{R}) \quad . \end{aligned} \qquad (2\text{-}3)$$

Where the fast Hamiltonian $h(\vec{p},\vec{r},\vec{R})$ depends parametrically on the slow variable $\vec{R}$, the snapshot Hamiltonian (for fixed $\vec{R}$) lead to the Schrödinger equation

$$\hat{h}\left(\hat{\vec{p}},\hat{\vec{r}},\vec{R}\right)\phi_n(\vec{r},\vec{R})=\varepsilon_n(\vec{R})\phi_n(\vec{r},\vec{R}) \qquad (2\text{-}4)$$

where ε_n is the energy of the fast system. The wave function for the whole system $\varphi(\vec{r},\vec{R})$ is separated into nuclear and electronic components $\Phi_n(\vec{R})$ and $\phi_n(\vec{r},\vec{R})$ as

$$\varphi(\vec{r},\vec{R})=\sum_n \Phi_n(\vec{R})\phi_n(\vec{r},\vec{R}). \qquad (2\text{-}5)$$

Substituting the wave function $\varphi(\vec{r},\vec{R})$ into the full Schrödinger equation and using the equation for the fast variables we get

$$\sum_n \left[\frac{\hat{\vec{P}}^2}{2M}+\varepsilon_n(\vec{R})\right]\Phi_n(\vec{R})\phi_n(\vec{r},\vec{R})=E_n\sum_n \Phi_n(\vec{R})\phi_n(\vec{r},\vec{R})\cdot \quad (2\text{-}6)$$

Where E is the energy of the whole system.

We may now integrate out the electronic degrees of freedom to leave a system of equations for the nuclear wave function $\Phi_n(\vec{R})$ alone. Using bracket notation for the normalized electronic eigenstates, we get

$$\sum_m \langle\phi_m|\ \frac{\hat{\vec{P}}^2}{2M}\Phi_n(\vec{R})|\phi_n\rangle+\varepsilon_n\Phi_n(\vec{R})=E\Phi_n(\vec{R})\ . \quad (2\text{-}7)$$

The nuclear kinetic energy operator $T_{nuclear}=-\frac{\hbar^2}{2M}\vec{\nabla}_R{}^2$ operator on both the nuclear and electronic wave function $\Phi_n(\vec{R})$ and $\phi_n(\vec{r},\vec{R})$.

Thus the kinetic energy term in (2-7) are proportional

$$\langle\phi_m|-\tfrac{\hbar^2}{2M}\vec{\nabla}_R{}^2\Phi_n(\vec{R})|\phi_n\rangle=\frac{1}{2M}\sum_k\left(\delta_{mk}\vec{\nabla}_R-i\hbar\langle\phi_m|\vec{\nabla}_R|\phi_k\rangle\right)\left(\delta_{kn}\vec{\nabla}_R-i\hbar\langle\phi_k|\vec{\nabla}_R|\phi_n\rangle\right)\Phi_n(\vec{R})\ . \quad (2\text{-}8)$$

We can write a complete matrix-valued Schrödinger operator for the nuclear wave function

$$\sum_m H^{eff}_{mn}\Phi_m(\vec{R})=E\Phi_n(\vec{R}) \quad (2\text{-}9)$$

which act on the nuclear wave $\Phi_n(\vec{R})$.

We denote the matrix–valued Hamiltonian as

$$H_{mn}^{eff} = \frac{1}{2M}\sum_{k}\vec{\Pi}_{nk}\vec{\Pi}_{km} + \varepsilon_n\left(\vec{R}\right)\delta_{nm} \tag{2-10}$$

when

$$\begin{aligned}\vec{\Pi}_{nm} &= \delta_{mn}\hat{\vec{P}} - i\hbar\left\langle\phi_n\left(\vec{R}\right)\middle|\vec{\nabla}_R\middle|\phi_m\left(\vec{R}\right)\right\rangle\cdot\dot{\vec{R}} \\ &\equiv \delta_{mn}\hat{\vec{P}} - \vec{A}_{nm}\cdot\dot{\vec{R}}\end{aligned} \tag{2-11}$$

and $$\vec{A}_{nm} \equiv i\hbar\left\langle\phi_m\middle|\vec{\nabla}_R\middle|\phi_n\right\rangle.$$

2.1 Born-Oppenheimer Approximation

In the Born-Oppenheimer approximation, the effect of the off–diagonal matrix elements $\vec{A}_{mn}$ which mix different energy levels is ignored. Then for a non degenerate electronic level, the effective nuclear Schrödinger operator in the Born-Oppenheimer approximation is then simply

$$H_n^{BO} = -\frac{\hbar^2}{2M}\left(\vec{\nabla}_R - i\vec{A}_n\left(\vec{R}\right)\right)^2 + \varepsilon_n\left(\vec{R}\right). \tag{2-12}$$

Equation (2-12) looks like the Schrödinger operator of a charged particle in the presence of a background magnetic potential. To further strengthen this analogy, the vector field $\vec{A}_n$ even transforms likes a U(1) gauge potential. The phase each of the wave function $\phi_n\left(\vec{R}\right)$ is arbitrary, and our description of the dynamics of the nuclei must always

respect this arbitrariness. The use of a vector potential brings out the fact that our description possesses the freedom of performing gauge transformation in analogy with electromagnetism [Appendix A]. The effect of a redefinition of phases of electronic wave function

$$\phi_n(\vec{R}) \rightarrow e^{i\Lambda_n(\vec{R})}\phi_n(\vec{R}) \tag{2-13}$$

is to rotate the nuclear wave function oppositely

$$\Phi_n(\vec{R}) \rightarrow e^{-i\Lambda_n(\vec{R})}\Phi_n(\vec{R}) \tag{2-14}$$

so that the full wave function $\varphi(\vec{r},\vec{R})$ is preserved.

From Eq. (2-14), we see that the gauge potential transforms just as it should

$$\vec{A}_n(\vec{R}) \rightarrow \vec{A}_n(\vec{R}) + \vec{\nabla}_R \Lambda_n(\vec{R}) \tag{2-15}$$

and it is easy that the overall effect of the phase redefinition is to leave the Schrödinger equation invariant.

We conclude that the nuclei behave like charged particles in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}_n$ semiclassicall speaking, when the nuclei go around a closed path, the wave function will accumulate a geometrical phase proportional to the enclosed magnetic flux. This phase is the Berry phase.

Chapter III

Feynman Path Integral in Quantum Mechanics

In proceeding to evaluate the geometrical phase by path integral method, the basic ideas of constructing the Feynman path integral (Feynman 1961) will be presented in this chapter. We present in this chapter the mathematical formulation of the quantum–mechanical transformation of the propagator in the form of a path integral.

3.1 The Sum over all Paths

In classical mechanics, If we consider a particle at an initial time t_a start from the point x_a and goes to a final point x_b at time t_b , there will be one specific and particular trajectory which goes from a to b. This particular trajectory is called the " classical trajectory ", which satisfy the classical Lagrangian equation of motion

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \qquad (3\text{-}1)$$

where L is the Lagrangian for the system.

The classical path x is that for which S is extreme.

$$S = \int_{t_a}^{t_b} dtL \qquad (3\text{-}2)$$

where S is action for the system.

In quantum mechanics, the systems deal with probability amplitude so that if we consider a particle moves from a one point to an other point, there are many possible paths which the particle can take. By this reason, our function x(t) will have the property that $x(t_a) = x_a$ and $x(t_b) = x_b$. The probability amplitude K(b,a) to go from the point x_a at the time t_a to the point x_b at t_b is the sum of contribution from each path. The contribution of a path has a phase proportional to the action S :

$$K(b,a) = \sum_{\substack{\text{over all} \\ \text{paths from a to b}}} (\text{const}) \exp\left[\frac{i}{\hbar} S\{x(t)\}\right] \qquad (3\text{-}3)$$

If we need to find the probability amplitude of the particle going from a to b, we have to carry out the sum in Eq. (3-3). But the number or path from a to b is infinite, so Eq. (3-3) is very difficult to work with. Another method and more efficient method of computing the sum over all paths will now be described.

We choose a subset of all path by first separating the independent time into small interval, ε. This gives us a set of successive time t_1, t_2,… between the values t_a and t_b , where $t_{i+1} = t_i + \varepsilon$. At each time t_i we select some special point x_i and constructing a path by connecting all the points so selected of from a line. This process is shown in Fig. 3-1. It is possible to define a sum over all paths constructed in this manner by taking a

multiple integral over all values of x_i for i between 1 and N-1, where $N = t_b - t_a / \varepsilon$. By using this method, Eq. (3-3) then becomes

$$K(b,a) = \lim_{N\to\infty} \iint \ldots \int e^{\frac{i}{\hbar}S\{x(t)\}} dx_1 \ldots dx_N \ . \qquad (3\text{-}4)$$

This is called a path integral and the amplitude K(b,a) is known as the Feynman propagator.

Fig. 3-1

Diagram showing the path integration can be constructed. [13]

3.2 The Feynman Propagator

We shall be interested in the path integral framework in topological structure by the recent discovery of quantum adiabatic theorem. The problem this is stated as follows. Kuratsji and Iida find this way of formulating the Born–Oppenheimer idea much more appropriate than the usual formulation in terms of fast and slow variables. In the derivation of effective Lagrangian, we should expect, and will find, that geometrical phases occur. This is particularly clear if we think in term of path integral. Then along any particular path the slow degrees of freedom can be considered as external parameters governing the state of the fast ones.

Consider the trace of the evolution operator $K(T) = Tr[\exp(\frac{-i}{\hbar}\hat{H}T)]$, which is the evolution operator of two interacting systems. We adopt a Hamiltonian

$$\hat{H} = \hat{h}(\hat{q}, \hat{p}, \hat{X}) + \hat{H}_0(\hat{X}, \hat{P}) \qquad (3\text{-}5)$$

which are described by the variable conventionally called “internal” and “collective” coordinate $\hat{q}$ and $\hat{X}$ respectively. Where the internal Hamiltonian $\hat{h}(\hat{q}, \hat{p}, \hat{X})$ is assumed to dependent on coordinates $\hat{q}$, $\hat{X}$, conjugate momentum $\hat{p}$ and not explicitly on conjugate momentum $\hat{P}$ and the collective Hamiltonian $\hat{H}$ dependent on $\hat{X}$ and $\hat{P}$.

The trace of the evolution operator which is written as

$$K(T) = \sum_{n} \int \langle n(X_0), X_0 \,|\exp[\frac{-i}{\hbar}\hat{H}T]|\, n(X_0), X_0 \rangle dX_0 \,. \qquad (3\text{-}6)$$

In Equation (3-6) one naturally picks up the transition amplitude for the quantum process starting from the initial state of product form $|n(X_0),X_0\rangle \equiv |n(X_0)\rangle \otimes |X_0\rangle$ and returning via closed loops C to the same state, where $|\, X_0 \,\rangle$ denotes the eigenstate of $\hat{X}$ and $|n(X_0)\rangle$ is the eigenstate of $\hat{h}(\hat{q},\hat{p},\hat{X})$ at $X = X_0$ with eigenvalue $E_n(X_0)$. Then with the aid of the time–discretization together with the completeness relation holding for X, we get

$$\langle n(X_0), X_0 \,|e^{\frac{-i}{\hbar}\hat{H}T}\,| n(X_0), X_0 \rangle = \langle n(X_0), X_0 \,|e^{\frac{-i}{\hbar}\hat{H}\varepsilon} \ldots e^{\frac{-i}{\hbar}\hat{H}\varepsilon}\,| n(X_0), X_0 \rangle \qquad (3\text{-}7)$$

with $\varepsilon = T/N$. Further noting the relation for $\varepsilon \approx 0$,

$$\langle n(X_0), X_0 |e^{\frac{-i}{\hbar}\hat{H}T}| n(X_0), X_0 \rangle = \lim_{N\to\infty} \int\int \ldots \int \prod_{k=1}^{N-1} dX_k \big[\, \langle n(X_0), X_0 |e^{\frac{-i}{\hbar}\hat{H}\varepsilon}| X_{N-1}\rangle\langle X_{N-1}|e^{\frac{-i}{\hbar}\hat{H}\varepsilon}| X_{N-2}\rangle \ldots$$
$$\ldots \langle X_k |e^{\frac{-i}{\hbar}\hat{H}\varepsilon}| X_{k-1}\rangle \ldots \langle X_1 |e^{\frac{-i}{\hbar}\hat{H}\varepsilon}| n(X_0), X_0\rangle \,\big] \qquad (3\text{-}8)$$

and

$$\begin{aligned}
\langle X_k |e^{\frac{-i}{\hbar}\hat{H}\varepsilon}| X_{k-1}\rangle &\approx \langle X_k |e^{\frac{-i}{\hbar}\hat{H}_0\varepsilon}| X_{k-1}\rangle e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon} \\
&= \int dP_k \langle X_k |P_k\rangle\langle P_k |e^{\frac{-i}{\hbar}\hat{H}_0\varepsilon}| X_{k-1}\rangle e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon} \\
&= \int dP_k \langle X_k |P_k\rangle\langle P_k |X_{k-1}\rangle e^{\frac{-i}{\hbar}\hat{H}_0(P_k,X_k)\varepsilon} \cdot e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon} \\
&= \int dP_k \exp\left[iP_k(X_k - X_{k-1}) - \frac{i}{\hbar}\hat{H}_0(X_k,P_k)\right] \cdot e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon} \,.
\end{aligned} \qquad (3\text{-}9)$$

From quantum mechanics, the momentum eigenfunction $\langle X|P\rangle$ are

$$\langle X|P\rangle = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \exp[\frac{-i}{\hbar} P.X]. \qquad (3\text{-}10)$$

Eq. (3-6) can be expressed as

$$K(T) = \sum_n \int \prod_k dX_k dP_k \quad T_{nn}(C) e^{\frac{-i}{\hbar} S_0(C)} \qquad (3\text{-}11)$$

where $S_0(C) = \int_0^T dt\left[P(t)\cdot \dot{X}(t) - H_0(X(t),P(t))\right]$ is the action for the collective motion along closed loops C. $T_{nn}(C)$ is just the internal transition amplitude and give by

$$T_{nn}(C) = \langle n(X_0)| e^{\frac{-i}{\hbar}\hat{h}(0)\varepsilon} ... e^{\frac{-i}{\hbar}\hat{h}(k)\varepsilon} ... e^{\frac{-i}{\hbar}\hat{h}(T)} |n(X_0)\rangle \qquad (3\text{-}12)$$

where $\hat{h}(k)$ denotes the internal Hamiltonian at the point X = X_k on the loop C . If we denote $|\Phi_n(T)\rangle$ as a solution of the time–dependent Schrödinger equation $\left[i\hbar\frac{\partial}{\partial t} - \hat{h}(\hat{p},\hat{q},X)\right]|\Phi_n(t)\rangle = 0$ with the boundary condition $|\Phi_n(0)\rangle = |n(X_0)\rangle$, $T_{nn}(C)$ is written $T_{nn}(C) = \langle n(X_0)|\Phi_n(T)\rangle$.

Under the above prescription we turn to the case of the adiabatic motion where the period T is large. By inserting the completeness holding for the internal state on each point of external variables X_k; $\sum_{m_k} |m_k\rangle\langle m_k| = 1$.

Eq. (3-12) is written as

$$T_{nn}(C)=\sum_{m_N}\cdots\sum_{m_k}\cdots\sum_{m_1}\langle n(X_0)|e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_0)\varepsilon}|m_N\rangle\cdots\langle m_k|e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon}|m_{k-1}\rangle\cdots\langle m_1|e^{\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_1)}|n(X_0)\rangle \cdot \tag{3-13}$$

In the adiabatic approximation, we pick up the quantum transition only between the states with the same quantum number n, $\langle n_k|\exp\left[\frac{-i}{\hbar}\hat{h}(\hat{p},\hat{q},X_k)\varepsilon\right]|n_{k-1}\rangle$. Then using the relation $\hat{h}(\hat{p},\hat{q},X_k)|n_k\rangle = E_n(X_k)|n_k\rangle$ where $E_n(X_k)$ is an energy of an adiabatic level n at $X = X_k$. We obtain

$$T_{nn}(C)=\langle n(X_0)|n(X_T)\rangle_C \exp\left[\frac{-i}{\hbar}\int_0^T dt E_n(X(t))\right]. \tag{3-14}$$

Here the overlap function

$$\langle n(X_0)|n(X_T)\rangle_C = \lim_{N\to\infty}\prod_k^N\langle n(X_k)|n(X_{k-1})\rangle. \tag{3-15}$$

Thus, by using the approximate relation

$$\begin{aligned}\langle n(X_k)|n(X_{k-1})\rangle &\approx 1-\langle n(X_k)|\vec{\nabla}_X n(X_k)\rangle\cdot\Delta X\\ &\approx \exp[i\Delta\omega]\end{aligned} \tag{3-16}$$

where $\Delta X = X_k - X_{k-1}$ and $\Delta\omega = i\langle n(X_k)|\vec{\nabla}_X n(X_k)\rangle\cdot\Delta X$.

Eq. (3-12) is written as

$$\langle n(X_0)|n(X_T)\rangle_C = \exp[i\Gamma_n(C)] \tag{3-17}$$

and

$$\Gamma_n(C)=\oint\langle n(X(t))|\vec{\nabla}_X n(X(t))\rangle\cdot dl. \tag{3-18}$$

Equation (3-18) is essentially the same as the phase obtained by Berry. Thus we arrive at the effective path integral associated with the adiabatic change of the external dynamical variable X,

$$K^{eff}(T) = \sum_n \int D[X(t)]D[P(t)]\exp\left[\frac{-i}{\hbar}(S_n^{ad} + \hbar\Gamma_n(C))\right] \qquad (3\text{-}19)$$

where $S_n^{ad} \equiv S_0 - \int_0^T dt E_n(X(t))$ is the adiabatic action function. From equation (3-19) we get a natural explanation that the phase $\Gamma_n(C)$ appears as topological action function which is to be added to as the usual dynamical action.

Chapter IV

Path Integral Derivation of the Magnus Force

In this chapter, firstly, we will describe the exact derivation of the transverse force acting on the quantized single vortex moving in a uniform background. The derivation is based on the model of the charged boson embedded in a constant positive back ground represented the superconductor. The basic of mathematical method in this derivation of the Magnus force is due to Kuratsuji and Iide [14] and we use it to calculate all the quantum transition exactly.

Secondly, we will show that the conditions of the occurrence of the Magnus force in superconductors are that a vortex has a local circulation around it and the velocity of vortex must be finite. Therefore the net velocity of the fluid that flows past the vortex in the reference frame which moving with the vortex is $\vec{V} = \vec{v}_v - \dot{\vec{R}}$.

Thirdly, we will analyze the effect of environment on the original Magnus force by using model Hamiltonian of Ao and Thouless [15]. In their work they considered tunneling of a vortex influenced by the pinning potential and the dissipation which represented by A set of N harmonic oscillator.

4.1 The Original Magnus Force

In this section, we shall show the mathematical approach to the problem and assumption. We derive the Magnus force from a model system consisting of a single vortex imbedded in uniform positive background and coupled to a mutual interaction charged bosons by following the model of Michael R. Geller, Carlos Wexler and David J.Thouless [16] but we shall neglect the current-current interaction between the charged bosons.

Fig. 4-1

This is model of supercurrent, it is 2-dimensional charged boson gas under the influence of positive background.

The full Hamiltonian for a quantized vortex coupled to the interacting charged boson is given as

$$\hat{H} = \hat{H}_V + \hat{h}_c + \hat{h} . \tag{4-1}$$

Where $\hat{H}_V(\hat{\vec{P}}, \hat{\vec{R}})$ is the Hamitonian for a quantized vortex, the second term

$$\hat{h}_c \equiv -\frac{e^2}{2m^2c} \sum_{n \neq n'} p_n^i T^{ij}(\vec{x}_n - \vec{x}_{n'}) p_{n'}^j \tag{4-2}$$

with $T^{ij}(\vec{x}) \equiv (\delta^{ij}|\vec{x}|^{-1} + x^i x^j |\vec{x}|^{-3})/2c$, is the current-current interaction, lowest-order relativistic effects. This interaction form was first obtained by Darwin in 1920 [17].

$$\hat{h} = \sum_i \frac{(\hat{\vec{p}}_i - \frac{2e}{c}\vec{a}(\hat{\vec{x}}_i - \hat{\vec{R}}))^2}{2m} + \frac{1}{2}\sum_{i<j} V(\hat{\vec{x}}_i - \hat{\vec{x}}_j) + \hat{h}_b \tag{4-3}$$

is the Hamiltonian representing N bosons with negative charge -2*e*, interacting with the vector potential $\vec{a}(\hat{\vec{x}}_n - \hat{\vec{R}})$ satisfying the equation $\oint \vec{a}(\vec{x}_n - \vec{R}) \cdot d\vec{l} = \phi_0 = hc/2e$. The V term represents the Coulomb interaction. Finally $\hat{h}_b$ is a uniform positive background and $\hat{h}_b \equiv -\sum_i \int d^3\vec{x}'\, e^2 \bar{n}(\vec{x}')|\vec{x}_i - \vec{x}'|^{-1}$, $\bar{n}(\vec{x}')$ is the charge distribution of lattice. This problem is simplified by using the symmetry of the lattice.

Next, the full Hamiltonian $\hat{H}$ can be separate into the internal and collective part. The internal part is $\hat{h}_{in} \equiv \hat{h} + \hat{h}_c$ and it depends on the center point of the vortex, $\vec{R}$ and not explicitly on the conjugate momentum of the

vortex, $\vec{P}$. The collective Hamiltonian, $\hat{H}_V$ is the Hamiltonian for a quantized vortex.

4.1.1 Propagator as a Transition Amplitude

We consider the Propagator as the probability amplitude for the system at an initial time t_a starting from the position $\vec{x}_{1a},...,\vec{x}_{Na}$; $\vec{R}_a$ to the final position $\vec{x}_{1b},...,\vec{x}_{Nb}$; $\vec{R}_b$ at time t_b. The Hamiltonian operator is independent of time. Thus the time–evolution operator is

$$U(t_b, t_a) \equiv e^{-\frac{i}{\hbar}\hat{H}(t_b - t_a)}. \tag{4-4}$$

The Propagator can be written as

$$\begin{aligned} K\left(\{\vec{x}\}_b, \{\vec{x}\}_a, \vec{R}_b, \vec{R}_a; t_b, t_a\right) &= \left\langle \vec{x}_{1b},...,\vec{x}_{Nb} \middle| \left\langle \vec{R}_a \middle| U(t_b, t_a) \right| \vec{R}_b \right\rangle \middle| \vec{x}_{1a},...,\vec{x}_{Na} \right\rangle \\ &= \left\langle \vec{x}_{1b},...,\vec{x}_{Nb} \middle| \left\langle \vec{R}_a \middle| \exp\left[-\frac{i}{\hbar}\hat{H}(t_b - t_a)\right] \right| \vec{R}_b \right\rangle \middle| \vec{x}_{1a},...,\vec{x}_{Na} \right\rangle \\ &= \sum_m \sum_n \left\langle \vec{x}_{1b},...,\vec{x}_{Nb}; \vec{R}_b \middle| m; \vec{R}_b \right\rangle \left\langle m; \vec{R}_b \middle| \left\langle \vec{R}_b \middle| \exp\left[-\frac{i}{\hbar}\hat{H}(t_b - t_a)\right] \right| \vec{R}_a \right\rangle \\ &\quad \times \left| n; \vec{R}_a \right\rangle \left\langle n; \vec{R}_a \middle| \vec{x}_{1a},...,\vec{x}_{Na}; \vec{R}_a \right\rangle \end{aligned} \tag{4-5}$$

where $\left| n; \vec{R} \right\rangle, \left| m; \vec{R} \right\rangle$ denote the eigenvector of the internal Hamiltonian $\hat{h}_{in}$ at $\vec{R}$ with eigenvalue $E_n(\vec{R}), E_m(\vec{R})$ respectively.

We can insert complete set of coordinate states and complete set of momentum states at $t = t_k$

$$\int d\vec{R}_k \left| \vec{R}_k \right\rangle\!\left\langle \vec{R}_k \right| = 1 \quad , \quad \int d\vec{P}_k \left| \vec{P}_k \right\rangle\!\left\langle \vec{P}_k \right| = 1. \tag{4-6}$$

We get

$$\begin{aligned} &\left\langle m;\vec{R}_b \left| \left\langle \vec{R}_b \right| \exp\left[-\frac{i}{\hbar}\hat{H}(t_b - t_a)\right] \right| \vec{R}_a \right\rangle \left| n;\vec{R}_a \right\rangle \\ &\qquad = \int \prod_{k=1}^{L-1} d\vec{R}_k \left\langle m;\vec{R}_b \left| \left\langle \vec{R}_b \right| \exp\left[-\frac{i}{\hbar}\hat{H}\varepsilon\right] \right| \vec{R}_{L-1} \right\rangle \cdots \left\langle \vec{R}_1 \left| \exp\left[-\frac{i}{\hbar}\hat{H}\varepsilon\right] \right| \vec{R}_a \right\rangle \left| n;\vec{R}_a \right\rangle \end{aligned} \tag{4-7}$$

with $\varepsilon = \dfrac{t_b - t_a}{L}$. Further noting the relation as $\varepsilon \to 0$,

$$\begin{aligned} \left\langle \vec{R}_k \left| \exp\left[-\frac{i}{\hbar}\hat{H}\varepsilon\right] \right| \vec{R}_{k-1} \right\rangle &\approx \left\langle \vec{R}_k \left| \exp\left[-\frac{i}{\hbar}\hat{H}_V\varepsilon\right] \right| \vec{R}_{k-1} \right\rangle \exp\left[-\frac{i}{\hbar}\hat{h}_{in}\left(\vec{R}_k\right)\varepsilon\right] \\ &= \int d\vec{P}_k \exp\left[\frac{i}{\hbar}\left[\vec{P}_k\left(\frac{\vec{R}_k - \vec{R}_{k-1}}{\varepsilon}\right)\right] - \hat{H}_V\left(\vec{R}_k, \vec{P}_k\right)\varepsilon\right] e^{-\frac{i}{\hbar}\hat{h}_{in}\left(\vec{R}_k\right)\varepsilon} \end{aligned} \tag{4-8}$$

equation (4-5) can be expressed as

$$K(t_b, t_a) = \sum_m \sum_n \psi_m\left(\vec{x}_{1b},\ldots,\vec{x}_{Nb};\vec{R}_b\right)\psi^*_n\left(\vec{x}_{1a},\ldots,\vec{x}_{Na};\vec{R}_a\right)\int D[\vec{P}]D[\vec{R}]\, T_{mn} e^{-\frac{i}{\hbar}S_{Vortex}\left[\vec{R}(t),\vec{P}(t)\right]} \tag{4-9}$$

with $S_{Vortex}\left[\vec{R}(t),\vec{P}(t)\right] \equiv \int_{t_a}^{t_b}\left[\vec{P}\cdot\vec{R} - H_V\right]dt$ is the action for a single vortex motion along path between a to b, $\psi_m\left(\vec{x}_{1b},\ldots,\vec{x}_{Nb};\vec{R}_b\right)$ is the wave function of charge boson gas at $\vec{R} = \vec{R}_b$.

T_{mn} is just the transition amplitude between the quantum state from $\left| m;\vec{R}_b \right\rangle$ to $\left| n;\vec{R}_a \right\rangle$ and is given by

$$T_{mn} \equiv \left\langle m;\vec{R}_b \left| \exp\left[-\frac{i}{\hbar}\hat{h}_{in}\left(\vec{R}_b\right)\right] \ldots \exp\left[-\frac{i}{\hbar}\hat{h}_{in}\left(\vec{R}_a\right)\right] \right| n;\vec{R}_a \right\rangle . \qquad (4\text{-}10)$$

This expression of the transition amplitude T_{mn} can be obtained by integrating equation (4-11)

$$i\hbar\frac{d}{dt}\left|n,t;\vec{R}(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|n,t;\vec{R}(t)\right\rangle + i\hbar\dot{\vec{R}}\cdot\vec{\nabla}_R\left|n,t;\vec{R}(t)\right\rangle \qquad (4\text{-}11)$$

by using the Schrödinger equation, we obtain

$$\left|n,t;\vec{R}(t)\right\rangle = \left|n,t_0;\vec{R}(t_0)\right\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\left[\hat{h}_{in} + i\hbar\dot{\vec{R}}\cdot\vec{\nabla}_R\right]\left|n,t';\vec{R}(t')\right\rangle dt' . \qquad (4\text{-}12)$$

This equation (4-12) can be solved by iteration

$$\left|n,t;\vec{R}(t)\right\rangle = \left[1 + \sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{U}(t_1)\hat{U}(t_2)\cdots\hat{U}(t_n)\right]\left|n,t_0;\vec{R}(t_0)\right\rangle \qquad (4\text{-}13)$$

where $$\hat{U}(t) = \hat{h}_{in}(\vec{R}(t)) + i\hbar\dot{\vec{R}}\cdot\vec{\nabla}_R \qquad (4\text{-}14)$$

and using the relation $\left[\hat{h}_{in}(t'),\hat{h}_{in}(t)\right] = 0$, this equation(4-13)can be simplify to

$$\left|n,t;\vec{R}(t)\right\rangle = \left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\left[\hat{h}_{in}(\vec{R}(t')) + i\hbar\dot{\vec{R}}\cdot\vec{\nabla}_R\right]}\right]\left|n,t_0;\vec{R}(t_0)\right\rangle . \qquad (4\text{-}15)$$

We arrive at the effective path integral associated with exact external dynamical variable $\vec{R}$,

$$K(t_b, t_a) = \sum_m \sum_n \psi_m(\vec{x}_{1b}, ..., \vec{x}_{Nb}; \vec{R}_b)\psi^*_n(\vec{x}_{1a}, ..., \vec{x}_{Na}; \vec{R}_a)K_{mn} \quad (4\text{-}16)$$

here K_{mn} just gives the usual dynamical evolution of the wave function for the charged boson with an additional effect from the motion of the external variable $\vec{R}(t)$ over all possible paths. We can express the transition matrix K_{mn} as

$$K_{mn} = \int D[\vec{P}]D[\vec{R}]\left\langle m;\vec{R}_b\right| \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left(\left[\vec{P}\cdot\dot{\vec{R}} - H_V\right]\delta_{m,n} - \hat{h}_{in}(\vec{R}) + i\hbar\dot{\vec{R}}\cdot\vec{\nabla}_R\right)\right]\left| n;\vec{R}_a\right\rangle . \quad (4\text{-}17)$$

In the adiabatic approximation, we pick up the quantum transition only between the states with the same quantum number. Therefore, the Propagator reduce to the diagonal form

$$K(t_b, t_a) = \sum_n \psi_m(\vec{x}_{1b}, ..., \vec{x}_{Nb}; \vec{R}_b)\psi^*_n(\vec{x}_{1a}, ..., \vec{x}_{Na}; \vec{R}_a)K_{nn} \quad (4\text{-}18)$$

and $$K_{nn} = \int D[\vec{P}]D[\vec{R}]\exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} dt\left(\left[\vec{P}\cdot\dot{\vec{R}} - H_V\right] - E_n(\vec{R}) + i\hbar\dot{\vec{R}}\cdot\vec{A}_{n,n}\right)\right]. \quad (4\text{-}19)$$

We denote the effective Lagrangian as

$$L^{eff}_{n,n} = \vec{P}\cdot\dot{\vec{R}} - H_V - E_n(\vec{R}) + i\hbar\,\vec{A}_{n,n}\cdot\dot{\vec{R}}, \quad (4\text{-}20)$$

which corresponds to the effective Schrödinger equation, equation (2-9) for the molecular physics, in chapter two. It is a matrix-valued Schrödinger

operator for the nuclear wave function. Equation (4-18), is the effective Lagrangian which additive the effect of quantum transition is added which is similar to the vortex acting as a charge particle in a magnetic field $\vec{B}_{n,n}$. The magnetic field defined as

$$\vec{B}_{n,n} = i\hbar \vec{\nabla}_R \times \vec{A}_{n,n}(\vec{R}), \tag{4-21}$$

and the scalar potential $E_n(\vec{R})$.

Then the new force, which in addition to the original force can be written as

$$\vec{F} \equiv -\frac{\partial U_n}{\partial \vec{R}} + \frac{d}{dt}\left(\frac{\partial U_n}{\partial \dot{\vec{R}}}\right) \tag{4-22}$$

and $U_n \equiv -E_n(\vec{R}) + i\hbar \vec{A}_{n,n} \cdot \dot{\vec{R}}$ is the generalized potential or velocity-dependent potential. This equation (4-22) is a time dependent version of the Feynman-Hellmann theorem [18].

The strong analog between the behavior of electron in the strong magnetic field and a single vortex in the superconductors is a transverse force, which proportional to the velocity. So that we can define the components of Magnus force as

$$\begin{aligned} F_{Y\,n,n} &= i\hbar \dot{R}_X \left[\left\langle \frac{\partial n}{\partial Y} \middle| \frac{\partial n}{\partial X} \right\rangle - \left\langle \frac{\partial n}{\partial Y} \middle| \frac{\partial n}{\partial X} \right\rangle \right] \\ F_{X\,n,n} &= -i\hbar \dot{R}_Y \left[\left\langle \frac{\partial n}{\partial Y} \middle| \frac{\partial n}{\partial X} \right\rangle - \left\langle \frac{\partial n}{\partial Y} \middle| \frac{\partial n}{\partial X} \right\rangle \right]. \end{aligned} \tag{4-23}$$

4.1.2 The Magnus Force at Ground State

We now return to the model system of an ideal charged boson gas. In thermal equilibrium at absolute zero temperature, all particles occupy in ground state according to Bose-Einstein distribution. We consider equation (4-21) at absolute zero temperature,

$$\begin{aligned} F^X &= i\hbar\, \dot{R}^X \left[\left\langle \frac{\partial \psi_0}{\partial Y} \middle| \frac{\partial \psi_0}{\partial X} \right\rangle - \left\langle \frac{\partial \psi_0}{\partial X} \middle| \frac{\partial \psi_0}{\partial Y} \right\rangle \right] \\ F^Y &= -i\hbar \dot{R}^Y \left[\left\langle \frac{\partial \psi_0}{\partial Y} \middle| \frac{\partial \psi_0}{\partial X} \right\rangle - \left\langle \frac{\partial \psi_0}{\partial X} \middle| \frac{\partial \psi_0}{\partial Y} \right\rangle \right]. \end{aligned} \tag{4-24}$$

In finding the wave function of charged boson system at the ground state energy we must understand about this equation (4-1). The full Hamiltonian for a quantized vortex corresponds to charged particles influence by an infinitely long straight solenoid that encloses a magnetic flux, as in the Aharanov-Bohm effect. The energy eigenvector $\left| m, \vec{R} \right\rangle$ of the Hamiltonian $\hat{h}_{in}$ satisfy

$$\begin{aligned} \hat{h}_{in} \left| m; \vec{R} \right\rangle &= \left[\sum_i^N \frac{1}{2m} \left[\vec{p}_i - \frac{2e}{c} \vec{a}\left(\vec{x}_i - \vec{R} \right) \right]^2 + \sum_{i<j} V(\vec{x}_i - \vec{x}_j) \right] \left| m; \vec{R} \right\rangle \\ &= E_m\left(\vec{R} \right) \left| m; \vec{R} \right\rangle. \end{aligned} \tag{4-25}$$

On the external variables $\vec{R}_k$ which is the center of vortex and the external magnetic $\vec{B} = \vec{\nabla} \times \vec{a}\left(\vec{x} - \vec{R} \right)$ must satisfy

$$\vec{B} = \phi_0 \delta^2 \left(\vec{x} - \vec{R} \right) \quad \hat{k}. \tag{4-26}$$

In this situation a possible choice of the vortex potential is

$$\vec{a}\left(\vec{x}_i-\vec{R}\right)=\frac{\phi_0}{2\pi}\vec{\nabla}_x\theta\left(\vec{x}_i-\vec{R}\right). \tag{4-27}$$

This is Cherm–Simons vector potential. $\theta\left(\vec{x}-\vec{R}\right)$ is function of $\vec{x}_i-\vec{R}$ which is given by $\theta\left(\vec{x}-\vec{R}\right)=\tan^{-1}\left[\frac{x_Y-R_Y}{x_X-R_X}\right]$. However, Under a gauge transformation, it can change a problem of charged particles subjected to vector potential to problem which the vector potential is absent, so that non-relativistic $\left|\, m,\vec{R}\,\right\rangle$ eigenvector would acquire a phase factor

$$\left|\, m;\vec{R}\,\right\rangle\rightarrow\left|\,\tilde{m};\vec{R}\,\right\rangle=\exp\left[i\sum_{i=1}^{N}\theta\left(\vec{x}_i-\vec{R}\right)\right]\left|\, m;\vec{R}\,\right\rangle \tag{4-28}$$

and

$$\begin{aligned}\tilde{\hat{h}}_{in} &= \exp\left[i\sum_{i=1}^{N}\theta\left(\vec{x}_i-\vec{R}\right)\right]\hat{h}_{in}\exp\left[-i\sum_{i=1}^{N}\theta\left(\vec{x}_i-\vec{R}\right)\right]\\ &= \sum\frac{\vec{p}_i^{\,2}}{2m}+\sum_{i<j}V\left(\vec{x}_i-\vec{x}_j\right).\end{aligned} \tag{4-29}$$

We see that the equation (4-23) reduces to

$$\begin{aligned}\tilde{\hat{h}}_{in}\left|\,\tilde{m};\vec{R}\,\right\rangle &= \left[\sum_i^N\frac{\vec{p}_i^{\,2}}{2m}+\sum_{i<j}V\left(\vec{x}_i-\vec{x}_j\right)\right]\left|\,\tilde{m};\vec{R}\,\right\rangle\\ &= E_m\left(\vec{R}\right)\left|\,\tilde{m};\vec{R}\,\right\rangle.\end{aligned} \tag{4-30}$$

Here the eigenvector $\left|\,\tilde{m},\vec{R}\,\right\rangle$ describe the system of boson gas in the absence of the vector potential $\vec{a}\left(\vec{x}-\vec{R}\right)$. If we consider system at absolute zero temperature, all boson particles occupy in the ground state. The many-body wave function has the form

$$\psi_0\left(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}\right)=\exp\left[i\sum_{i=1}^{N}\theta\left(\vec{x}-\vec{R}\right)\right]\tilde{\psi}_0\left(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}\right). \tag{4-31}$$

Here $\tilde{\psi}_0$ is a many-body wave function of superconductor in the absence of the external magnetic field. The many-body wave function ψ_0 has the properties

1. $\int d^2x_1 \ldots d^2x_N \left| \psi_0 \right|^2 = N$, number of charged boson

2. $\int d^2x_1 \ldots d^2x_{N-1} \left| \psi_0 \right|^2 = \rho(\vec{x}, \vec{R})$.

By $\rho(\vec{x}, \vec{R}) = \rho_0 + \delta\rho(\vec{x}, \vec{R})$ is the probability density of finding particle at $\vec{x}$. It is the sum of values ρ_0 in the absence the vortex plus modification due to the presence of the vortex which depends on the size of a single vortex. The probability density $\rho(\vec{x}, \vec{R})$ is a function of $\vec{x} - \vec{R}$ so that the probability density $\rho(\vec{x}, \vec{R})$ must satisfy the boundary condition,

1. $\lim_{|\vec{x} - \vec{R}| \to \infty} \rho(\vec{x}, \vec{R}) = \rho_0$

2. $\vec{\nabla}_R \rho(\vec{x}, \vec{R}) = -\vec{\nabla}_x \rho(\vec{x}, \vec{R})$.

By virtue of the properties of the many-body wave function and boundary conditions, the Magnus force becomes

$$\vec{F}_{\text{Magnus}} = 2\pi\rho_S \hbar \hat{k} \times \dot{\vec{R}} . \qquad (4\text{-}32)$$

This is the Magnus force at absolute zero temperature. Where $\rho_S = N \rho_0$ is the number charge boson density, $\dot{\vec{R}}$ is velocity of vortex, $\hat{k}$ is unit vector in **z** direction.

4.2 The Condition for the Existent of Magnus Force

In chapter 1, we show the existent of Magnus force in classical hydrodynamic. The condition of the occurrence of the Magnus force is that the circulation must exist. The local circulation breaks symmetry of pressure around the vortex structure. The difference of pressure creates a force on the vortex, which is the Magnus force.

In the case of superconductors at zero temperature, the behaviors of bosons are governed by a single wave function of coherent phase. We used a time independent wave function which both amplitude and phase can vary in space and time,

$$\psi_0\left(\vec{x}_1,...,\vec{x}_N;\vec{R}\right)=\tilde{\psi}_0\left(\vec{x}_1,...,\vec{x}_N;\vec{R}\right)\exp\left[\frac{i}{\hbar}S\left(\vec{x}_1,...,\vec{x}_N;\vec{R}\right)\right]. \quad (4\text{-}31)$$

Since we can choose the many-body wave function in such a way that the dependence on $\vec{x}$ is entirely through $\vec{x}-\vec{R}$, the partial derivatives with respect to $\vec{R}$ can be replaced by a sum over partial derivatives with respect to the particle coordinate $\vec{x}_i$ and define the phase of many-body wave function of the system by the polar angle in cylindrical coordinate:

$$S\left(\vec{x}_1,...\vec{x}_N;\vec{R}\right)\equiv\hbar\sum_{i=1}^{N}\theta\left(\vec{x}_i-\vec{R}\right). \quad (4\text{-}32)$$

We can write equation (4-22) into the integral from as

$$\vec{F}_{Magnus}\equiv\hat{k}\times\dot{\vec{R}}\oint_{\text{over space}}N\rho\left(\vec{x};\vec{R}\right)\vec{P}\left(\vec{x};\vec{R}\right)\cdot d\vec{x} \quad (4\text{-}33)$$

and

$$\vec{P}\left(\vec{x};\vec{R}\right)= \quad \hbar\vec{\nabla}_x\theta\left(\vec{x}-\vec{R}\right). \tag{4-34}$$

We can define velocity field

$$\vec{v}_v \equiv \frac{\hbar}{m}\vec{\nabla}_x\theta\left(\vec{x}-\vec{R}\right) \tag{4-35}$$

where m is the mass of charged boson which equals to $2\times$mass of an electron. We obtain

$$\vec{v}_v = \left(\frac{\hbar}{m}\hat{k}\right)\times\frac{\vec{x}-\vec{R}}{\left(\vec{x}-\vec{R}\right)^2}. \tag{4-36}$$

The probability density $\rho\left(\vec{x};\vec{R}\right)$ must satisfy the boundary condition that the density $\rho\left(\vec{x};\vec{R}\right)$ vanish continuously at $\vec{x}=\vec{R}$ and approaches the background density ρ_0 as $\left|\vec{x}-\vec{R}\right|\to\infty$. So that equation (4-33) can be written as

$$\begin{aligned}\vec{F}_{\text{Magnus}} &= (N\rho_0)\hat{k}\times\dot{\vec{R}}\oint_{\text{over space}}\vec{v}_v\cdot d\vec{x}\\ &= 2\pi\rho_S\hbar\,\hat{k}\times\dot{\vec{R}}.\end{aligned} \tag{4-37}$$

Equation (4-36) and (4-37) look like equation (1-8) and (1-9) respectively in classical hydrodynamic. We can say that the conditions of the occurrence of the Magnus force in superconductor are that the circulation exists or a vortex has a local circulation around it and the vortex velocity is non-vanishing in the reference frame moving with the vortex velocity $\dot{\vec{R}}$.

There the net velocity flow of fluid past the vortex is $\vec{V} = \vec{v}_v - \dot{\vec{R}}$. The occurrence of the local circulation around a vortex concerns the result of transition amplitude of superconductor.

The transition amplitude of charged boson system from ground state to ground state happens by external parameter evolving in time. If we consider the adiabatic motion of vortex along close loop C, in the adiabatic approximation, we pick up the quantum transition only between the states with the same quantum number n. So that we can write equation (4-17) as

$$K_{n\,n} = \int D[\vec{P}]D[\vec{R}]\exp\left[\frac{i}{\hbar}\left(\left[\int_{t_a}^{t_b} dt\left(\left[\vec{P}\cdot\dot{\vec{R}} - H_V\right] - E_n(\vec{R})\right) + \hbar\Gamma_n\right]\right)\right]. \quad (4\text{-}38)$$

We are interested only the quantum transition between the ground state to the ground state. The second term in the exponent is immediately recognized as Berry's phase. We can write as

$$\Gamma_n = i\oint_C \left\langle \psi_n;\vec{R} \left| \vec{\nabla}_R \right| \psi_n;\vec{R} \right\rangle \cdot d\vec{R} \quad (4\text{-}39)$$

and equation (4-40) is essentially the same as the Berry phase in chapter three. Using the Stokes theorem equation (4-40) can be simplify to

$$\Gamma_n = i\int_S \vec{\nabla}_R \times \left\langle \psi_n;\vec{R} \left| \vec{\nabla}_R \right| \psi_n;\vec{R} \right\rangle \cdot \hat{n}dS, \quad (4\text{-}40)$$

where S (C) is the area enclosed by loop C and $\hat{n}$ is the unit vector which its direction in 2 dimension problem along the z direction.

We find that the Berry phase of the ground eigenstate for closed loop C is

$$\Gamma_0 \quad = \quad -2\pi\rho_S \times S\,(C)\,. \tag{4-41}$$

The Berry phase Γ_0 for adiabatic motion of vortex around a closed loop is proportional to the number charged boson density ρ_S and S (C) is the area enclosed by loop C.

4.3 Environment

In recently, Professor Virulh Sa-yakanit studies the dynamic effect of the pinning potential on the original Magnus force by using the model Hamiltonian of Ao and Thouless to treat the problem of vortex tunneling in superconductor with pinning potential and dissipation [14]. He formulates this problem by using the real time path integral and calculated the propagator exactly by eliminating the x direction leaving only the path integral in y-direction. The result is the Magnus force in y direction and it can be written as

$$F(\tau) = \frac{M\omega\Omega_x}{2\sin\Omega_x t}\left[x_2\cos\Omega_x\tau - x_1\cos\Omega_x\tau\right] \tag{4-42}$$

In this section we derive the effect of environment on the original Magnus force. By using the Hamiltonian of Ao and Thouless [14], which vortex moves in 2 dimension.

The Hamiltonian is

$$H = \frac{1}{2M}\left(\vec{P} - q_V \vec{A}\right)^2 + V\left(\vec{R}\right) + \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \frac{m_i \omega_i^2}{2}\left(\vec{x} - \frac{c}{m_i \omega_i^2}\vec{R}\right)^2 . \quad (4\text{-}43)$$

where

M is the vortex mass, $V(\vec{R})$ is the vector potential, h the Planck constant, ρ_s the superfluid electron numbers density, $q_v = \pm(1)$ stating for the parallelism (amtiparallelism) in the z direction. The original Magnus force is

$$\vec{F} = \vec{V} \times \left(\vec{\nabla} \times \vec{A}\right) = q_V h \rho_s \vec{V} \times \hat{k} . \quad (4\text{-}44)$$

The first term, $\frac{1}{2M}\left(\vec{P} - q_V \vec{A}\right)^2 + V\left(\vec{R}\right)$, $\vec{P}$ and $\vec{R}$ represent momentum and position of vortex respectively. It is the system of vortex moving in superconductor at T = 0 $\hat{k}$ under the influence of pinning potential $V(\vec{R})$. The second term, $\sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \frac{m_i \omega_i}{2}\left(\vec{x}_i - \frac{c}{m_i \omega_i^2}\vec{R}\right)^2$, it is the Hamiltonian of the environment coupled to the vortex coordinate $\vec{R}$ harmonically.

To find the effect of environment on the original Magnus force, firstly we can simplify the problem by changing a set of N harmonic oscillators in Hamiltonian of Ao and Thouless, equation (4-43), to a single harmonic oscillator represented the environment. We can simplify the Hamiltonian of system, equation (4-43) become

$$H = \frac{1}{2M}\left(\vec{P} - q_V \vec{A}\right)^2 + V\left(\vec{R}\right) + \frac{p_x^2}{2m} + \frac{m\omega^2}{2}\left(x - \frac{c}{m\omega^2} R_x\right)^2 . \qquad (4\text{-}45)$$

Secondly, using the mathematical model as in the derivation of original Magnus force in the last section.

Fig 4-2

This is model of a single vortex in influence of a harmonic oscillator, which represented environment.

4.3.1 Effect of Environment

We find the propagator of a single vortex coupled to a mutual interaction environment. The Hamiltonian opeator of the system is independent of time. The time-evolution operator is

$$U(t_b, t_a) \equiv e^{-\frac{i}{\hbar}H(t_b - t_a)}. \tag{4-46}$$

Using the mathematical method as in the section 4.1. Therefore, the propagator of this system is

$$K(t_b, t_a) = \sum_m \sum_n \psi_m\left(x_b - \frac{c}{m\omega^2}R_{xb}\right)\psi^*{}_n\left(x_a - \frac{c}{m\omega^2}R_{xa}\right)\int D[\vec{P}]D[\vec{R}]\, e^{-\frac{i}{\hbar}S^{eff}_{m,n}[\vec{R}(t),\vec{P}(t)]} \tag{4-47}$$

where

$S^{\ eff}_{m\ ,\ n} = \int_{t_a}^{t_b} dt \quad L^{\ eff}_{m\ ,\ n}$ is the effective action of the vortex

and $L^{eff}_{m,n} = \left[\vec{P}\cdot\dot{\vec{R}} - H_{Vortex}\right]\delta_{m,n} - E_m\delta_{m,n} + i\hbar\,\vec{A}_{m,n}\cdot\dot{\vec{R}}$ is the effective Lagrangian of a single vortex. We denote $\psi_m\left(x - \frac{c}{m\omega^2}R_x\right)$ to be the eigenfunction of a harmonic oscillator, which also depend on the external parameter $\vec{R}$, with the eigenvalue E_m.

The derivation of the eigenfunction and the eigenvalue of a harmonic oscillator, $\frac{\vec{p}^2}{2m} + \frac{m\omega^2}{2}\left(\vec{x} - \frac{c}{m\omega^2}\vec{R}\right)^2$, has been shown in Appendix [B].

We denote $\vec{A}_{m,n}$ to be

$$\begin{aligned}\vec{A}_{m,n} &= \left\langle\, m;\vec{R}\ \left|\vec{\nabla}_R\right|\ n;\vec{R}\,\right\rangle \\ &= \int d\vec{x}\ \psi^*{}_m\left(x-\frac{c}{m\omega^2}R_x\right)\vec{\nabla}_R\ \psi_n\left(x-\frac{c}{m\omega^2}R_x\right).\end{aligned} \tag{4-48}$$

The propagator can be separated into two parts. The first part depends on the initial and final state of a harmonic oscillator and the second part depends on the transition amplitude for system of a single vortex with an additional effect from motion of environment, $E_m\ \delta_{m,n}$ and $\vec{A}_{m,n}$. Therefore, the effective Lagrangian of a single vortex have three terms, the first term is bare Lagrangian of vortex, the second term corresponds with the scalar potential, which make a single vortex dynamic similar to that of a charged particles in a magnetic field, $\Delta\vec{B}_{m,n}=\vec{\nabla}_R\times\vec{A}_{m,n}$. It is effect of environment on the original Magnus force.

The properties of the wave function $\psi_m\left(x-\frac{c}{m\omega^2}R_x\right)$ and a harmonic oscillator are (i) a harmonic oscillator is invariant under the translation, (ii) wave function depends only on $x-\frac{c}{m\omega^2}R_x$. We can change Gradient $\vec{\nabla}_R\to-\frac{m\omega^2}{c}\vec{\nabla}_x$. So that we can write

$$\begin{aligned}i\hbar\vec{A}_{m,n} &= \left\langle\, m;\vec{R}\ \left|i\hbar\vec{\nabla}_R\right|\ n;\vec{R}\,\right\rangle\to\left\langle\, m;\vec{R}\ \left|i\hbar\vec{\nabla}_x\right|\ n;\vec{R}\,\right\rangle \\ &= \frac{m\omega^2}{c}\left\langle\,\hat{\vec{p}}\,\right\rangle_{m,n}\end{aligned}. \tag{4-49}$$

It is the matrix element of the $\hat{\vec{p}} = -i\hbar\vec{\nabla}_x$ momentum operator:

$$\begin{aligned} \left\langle \hat{\vec{p}} \right\rangle &= \left\langle m;\vec{R} \left| -i\hbar\vec{\nabla}_x \right| n;\vec{R} \right\rangle \\ &= i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)\hat{x} \ . \end{aligned} \tag{4-50}$$

Here $\hat{x}$ is the unit vector in **x** direction. The vector potential $\vec{A}_{m,n}$ is independent of the external coordinate $\vec{R}$. Therefore, $\Delta\vec{B}_{m,n} = \vec{\nabla}\times\vec{A}_{m,n} = 0$, the harmonic oscillator in **x** direction has no effect on the original Magnus force.

Chapter V

Discussion and Conclusions

In this thesis we derive the Magnus force acting on a quantized vortex in a charged superfluid or superconductor by following the model of Michael R. Geller, Carlos Wexler and David J. Thouless [16] but we shall neglect the current-current interaction. This problem was simplified by using the symmetry of the lattice. Here the charged boson embedded in a constant positive background is considered. This model is known as the “jellium model”. It has been the property of superfludity, which the current flows with out carried by charged boson. The interaction between charged boson is the Coulomb type. The generalization of this model to the case of superconductors is far from realistic, since it relies on the uniformity of the positive background, therefore there is no phonon interaction, so our results can only form a first step towards a plausible theory of the origin Magnus force.

The nature of the vortex state of a superconductor, it has been the axial symmetry along the magnetic flux, which is quantized in units of apply magnetic filed and the magnetic flux through a superconductor has be found to be quantized in units of $ch/2e$.

The shape of the vortex line corresponds to infinitely long straight solenoid that encloses a magnetic flux. The magnetic field vanishes outside of vortex but the vector potential dose not. The interaction between charged boson gas and vortex is like that of A-B effect. A single vortex has defined as a point particle tied up with magnetic flux quanta, which the effect of a Chern-Simons gauge field is to tie magnetic flux to point particle.

We formulate this problem using the real time path integral and calculate the propagator exactly. The effective Lagrangian are proposed is consistent which additive the effective Schrödinger equation for molecular physics [12]. In this method, the wave functions of the system are divided into 2 parts: fast and slow moving wave function. So we need to solve a matrix-valued operator for the slow wave function. The basic of mathematical methods in the derive the Magnus force is due to Kuratsji and Iide [14] and we use it to calculate all the quantum transition exactly. The result is that effective Lagrangian which additive the effect of quantum transition is similar to the vortex acting as a particle in a magnetic field and a scalar potential. The strong analogy between the behavior of an electron in strong magnetic field and a vortex in superconductor is that the transverse force is proportional to velocity. We can be define the components of Magnus force acting on the vortex as:

$$
\begin{aligned}
F_Y &= i\hbar\, V_X \left[\left\langle \frac{\partial \psi_m}{\partial Y} \middle| \frac{\partial \psi_n}{\partial X} \right\rangle - \left\langle \frac{\partial \psi_m}{\partial Y} \middle| \frac{\partial \psi_n}{\partial X} \right\rangle \right] \\
F_X &= -i\hbar V_Y \left[\left\langle \frac{\partial \psi_m}{\partial Y} \middle| \frac{\partial \psi_n}{\partial X} \right\rangle - \left\langle \frac{\partial \psi_m}{\partial Y} \middle| \frac{\partial \psi_n}{\partial X} \right\rangle \right].
\end{aligned}
\tag{5-1}
$$

In sense of quantum mechanics we can say that the origination of Magnus force is an effect of the transition amplitude of supercurrent retrospectively affecting the vortex motion. And the transition amplitude of charged boson system happens by the external parameter $\vec{R}$ evolving in time. For ground state contribution we had shown that Magnus force at the ground state, which is consistent with the method of E. Simanek.

In thermal equilibrium at the absolute zero temperature, all particles occupy in ground state energy as given by the Bose-Einstein distribution. We used time independent wave function which both amplitude and phase can vary in space and time,

$$
\psi_0\left(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}\right) = \tilde{\psi}_0\left(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}\right)\exp\left[\frac{i}{\hbar}S\left(\vec{x}_1,\ldots,\vec{x}_N;\vec{R}\right)\right] \tag{5-2}
$$

In case of the charged boson gas coupled to a vortex, we define phase of the many-body wave function of system as the polar angle in the cylindrical coordinate:

$$
S\left(\vec{x}_1,..,\vec{x}_N,\vec{R}\right) \equiv \sum_{i=1}^{N} \hbar\theta\left(\vec{x}_i - \vec{R}\right). \tag{5-3}
$$

The probability density of finding a particle at $\vec{x}$, must satisfy the boundary conditions: the density, $\rho\left(\vec{x};\vec{R}\right)$, vanishes continuously at

$\vec{x} = \vec{R}$ and approaches the background density ρ_0 as $\left|\vec{x} - \vec{R}\right| \to \infty$. By virtue of the property of the many-body wave function and the boundary condition, The Magnus force become

$$\vec{F}_{magnus} = 2\pi\rho_S \vec{V} \times \hat{k} . \qquad (5\text{-}4)$$

The above equation looks like the Magnus force in classical hydrodynamics, equation (1-16). From corresponding of the equations (4-36)-(4-37) and equations (1-8)-(1-9), we can be say that the exists of Magnus force in superconductor because the general property of local circulation around the vortex and the absence of the velocity of the vortex, $\dot{\vec{R}}$. Therefore the net velocity flow of fluid past the vortex in the reference frame which moving with the vortex velocity is $\vec{V} = \vec{v}_v - \dot{\vec{R}}$. The occurrence of local circulation around the vortex at the zero temperature concerns the result of transition amplitude of supercurrent.

The main claim of Ao and Thouless is that there is a universal exact expression for the total transverse force, on which does not depend on the presence of quasiparticles or impurities, environment [15]. This force derived from the concept of the geometrical phase, Berry phase, coincides with the superfluid Magnus force and therefore is proportional to the superfluid density. According to Ao and Thouless, there is no transverse force on vortex from quasiparticles or impurities, though they

might influence the value of the superfluid density and thereby influence the amplitude of the Magnus force. Therefore it is important to understand the Ao and Thouless theory is true or not.

We find that there is no influence of the environment on the existence of the Magnus force in superconductor. We start with the Hamiltonian of a single vortex from a model proposed by Ao and Thouless for treating the vortex tunneling in a superconductor with the pinning potential and the dissipation. We has been simplify problem by changed a set of N harmonic oscillator to be a harmonic oscillator represented the environment. The result is that the harmonic oscillator in the x-direction was treated as a dissipative environment and we found that it give not effects on the origination of the Magnus force.

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Appendices A

Gauge Transformation in Electromagnetism

Let us denote by $|\alpha\rangle$ the state ket in the presence of vector potential $\vec{A}$, the state ket for the same physical situation when

$$\tilde{\vec{A}} = \vec{A} + \vec{\nabla}\Lambda \tag{A.1}$$

is used in place of $\vec{A}$ is denoted by $|\tilde{\alpha}\rangle$. Here $\Lambda \in \mathrm{Re}$, as well as $\vec{A}$, is a function of the position operator $\vec{x}$. Our basis requirements are

$$\langle\alpha|\,\vec{x}\,|\alpha\rangle = \langle\tilde{\alpha}|\,\vec{x}\,|\tilde{\alpha}\rangle \tag{A.2}$$

and

$$\langle\alpha|\;\vec{p} - \frac{e}{c}\vec{A}\;|\alpha\rangle = \langle\tilde{\alpha}|\;\vec{p} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\,|\tilde{\alpha}\rangle \tag{A.3}$$

where $\vec{p}$ is the canonical momentum. In addition we require, as usual, the norm of the state ket to be preserved

$$\langle\alpha|\,\alpha\rangle = \langle\tilde{\alpha}|\,\tilde{\alpha}\rangle. \tag{A.4}$$

We must construct an operator T that relates $|\alpha\rangle$ to $|\tilde{\alpha}\rangle$:

$$|\tilde{\alpha}\rangle = T|\alpha\rangle. \tag{A.5}$$

Invariance properties (A.2) and (A.3) are guaranteed if

$$T^{*}\,\vec{x}\,T \;=\; \vec{x} \tag{A.6}$$

and

$$T^{*}\left(\vec{p} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\right)T \;=\; \vec{p} - \frac{e}{c}\vec{A}. \tag{A.7}$$

We assert that

$$T = \exp\left[i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] \tag{A.8}$$

will do the job. First, T is unitary, so equation (A.4) is all right. Second, equation (A.6) is obviously satisfied because $\vec{x}$ commutes with any function of $\vec{x}$ .As for equation (A.7), just note that

$$\begin{aligned} \exp\left[-i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] \vec{p} \exp\left[i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] &= \exp\left[-i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] \left[\vec{p}, \exp\left[i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] \right] + \vec{p} \\ &= \exp\left[-i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] - i\hbar\vec{\nabla}\left[\exp\left[i\frac{e}{c\hbar}\Lambda(\vec{x}) \right] \right] + \vec{p} \\ &= \vec{p} + \frac{e}{c}\vec{\nabla}\Lambda \end{aligned} \tag{A.9}$$

The invariance of quantum mechanics under gauge transformation can also be demonstrated by looking directly at the Schrödinger equation. Let $|\alpha, t_0; t\rangle$ be a solution to the Schrödinger equation in the presence of $\vec{A}$.

$$\left[\frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A} \right]^2 + e\phi \right] |\alpha, t_0; t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle \tag{A.10}$$

The corresponding solution in the presence of $\tilde{\vec{A}}$ must satisfy

$$\left[\frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda \right]^2 + e\phi \right] |\tilde{\alpha}, t_0; t\rangle = i\hbar\frac{\partial}{\partial t}|\tilde{\alpha}, t_0; t\rangle \tag{A.11}$$

We see that if the new ket is taken be

$$|\tilde{\alpha}, t_0; t\rangle = \exp\left[i\frac{e}{c\hbar}\Lambda \right] |\alpha, t_0; t\rangle \tag{A.12}$$

in accordance with (A.8), then the new Schrödinger equation (A.11) will be satisfied, all we have to note is that

$$\exp\left[-i\frac{e}{c\hbar}\Lambda\right]\left[\vec{p}-\frac{e}{c}\vec{A}-\frac{e}{c}\vec{\nabla}\Lambda\right]^2\exp\left[i\frac{e}{c\hbar}\Lambda\right]=\left[\vec{p}-\frac{e}{c}\vec{A}\right]^2 \qquad (A.13)$$

which follows from applying (A.9) twice.

Equation (A.12) also implies that the corresponding wave equation are related via

$$\tilde{\varphi}(\vec{x},t)=\exp\left[i\frac{e}{c\hbar}\Lambda(\vec{x})\right]\varphi(\vec{x},t) \qquad (A.14)$$

Where $\Lambda(\vec{x})$ is now a real function of the position vector eigenvalue $\vec{x}$.

Appendices B

Harmonic Oscillator in Constant potential

We find the energy eigenkets and energy eigenvalues of the simple harmonic oscillator. The Hamiltonian of the simple harmonic oscillator is

$$\hat{H} = \frac{\hat{p}_x^{\,2}}{2m} + \frac{m\omega^2}{2}\left(\hat{x} - \frac{c}{m\omega^2}R_x\right)^2, \qquad \text{(B.1)}$$

where ω is the angular frequency of the classical oscillator related to the spring constant k in Hooke' law via $\omega = \sqrt{k/m}$. The operators $\hat{x}$ and $\hat{p}_x$ are Hermitian. R_x is the external parameter of the Hamiltonian. It is convenient to define two non-Hermitian operators, known as the annihilation operator and creation operator respectively.

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - R_x + i\frac{\hat{p}_x}{m\omega}\right), \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - R_x - i\frac{\hat{p}_x}{m\omega}\right). \qquad \text{(B.2)}$$

Using the canonical commutation relation, we readily obtain

$$\left[a, a^+\right] = \left(\frac{1}{2\hbar}\right)\left(-i\left[\hat{x}, \hat{p}_x\right] + i\left[\hat{p}_x, \hat{x}\right]\right) = 1. \qquad \text{(B.3)}$$

We also define the number operator

$$\hat{N} = a^+ a, \qquad \text{(B.4)}$$

which is obviously Hermitian. It is straightforward to show that

$$\begin{aligned} a^+ a &= \left(\frac{m\omega}{2\hbar}\right)\left((x - R_x)^2 + \frac{p_x^2}{m^2\omega^2}\right) + \left(\frac{i}{2\hbar}\right)\left[x, p_x\right] \\ &= \frac{\hat{H}}{\hbar\omega} - \frac{1}{2} \end{aligned} \qquad \text{(B.5)}$$

so we have an important relation between the number operator and the Hamiltonian operator:

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right). \tag{B.6}$$

Because $\hat{H}$ is just a linear function of $\hat{N}$, $\hat{N}$ can be diagonalized simultaneously with $\hat{H}$. We denote an energy eigenket of $\hat{N}$ by its eigenvalue n, so

$$\hat{N}|n\rangle = n|n\rangle. \tag{B.7}$$

Therefore, the energy eigenvalues of Hamiltonian (B.6) are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{B.8}$$

The operator method can be used to obtain the energy eigenfunctions in position space. Let us start the ground state define by

$$a|0\rangle = 0 \tag{B.9}$$

which, in the x-representation, reads

$$\langle x|\, a\,|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\left(\hat{x} + \frac{i\hat{p}_x}{m\omega}\right)|0\rangle = 0. \tag{B.10}$$

We can regard this as a differential equation for the ground-state wave function $\langle x|0\rangle$:

$$\left((x - R_x) + x_0^2\frac{d}{dx'}\right)\langle x|0\rangle = 0 \quad , \tag{B.11}$$

where we have introduced

$$x_0 \equiv \sqrt{\frac{\hbar}{m\omega}} \quad , \tag{B.12}$$

which set the length scale of the oscillator. We see that the normalized solution (B.11) is

$$\langle x|0\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right) \exp\left[-\frac{1}{2}\left(\frac{x-R_x}{x_0}\right)^2\right] \quad . \tag{B.13}$$

We can also obtain the energy eignfunction for excited states by evaluating

$$\begin{aligned}
\langle x|1\rangle &= \langle x|\, a^+|0\rangle = \left(\frac{1}{\sqrt{2}\ x_0}\right)\left((x-R_x) - x_0^2\frac{d}{dx}\right)\langle x|0\rangle, \\
\langle x|2\rangle &= \langle x|\, (a^+)^2|0\rangle = \left(\frac{1}{\sqrt{2!}}\right)\left(\frac{1}{\sqrt{2}\ x_0}\right)\left((x-R_x) - x_0^2\frac{d}{dx}\right)^2\langle x|0\rangle, \ldots
\end{aligned} \tag{B.14}$$

The ground-state wave function $\langle x|0\rangle$ only depends on $x - R_X$. So that, a differential equation for excited state (B.14) can be written

$$\begin{aligned}
\langle x|1\rangle &= \left(\frac{1}{\sqrt{2}\ x_0}\right)\left((x-R_x) - x_0^2\frac{d}{d(x-R_x)}\right)\langle x|0\rangle, \\
\langle x|2\rangle &= \left(\frac{1}{\sqrt{2!}}\right)\left(\frac{1}{\sqrt{2}\ x_0}\right)\left((x-R_x) - x_0^2\frac{d}{d(x-R_x)}\right)^2\langle x|0\rangle, \ldots
\end{aligned} \tag{B.15}$$

In general, we obtain

$$\langle x|n\rangle = \langle x|\, (a^+)^n|0\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{2^n n!}}\right)\left(\frac{1}{x_0^{n/2}}\right)\left((x-R_x) - x_0^2\frac{d}{d(x-R_x)}\right)^n\langle x|0\rangle \tag{B.16}$$

Curriculum Vitae

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