CHAPTER V

THE THEORY OF THE PREDICATE CALCULUS WITH ONE AND TWO VARIABLES

5.1 The Theory of the Predicate Calculus with One Variable.

Basic symbols:

P,Q,R, ... are called Predicates

x,y,z, ... are called variables

a,b,c, ... are called fixed values of the variables.

$\forall_x$ is called the Universal Quantifier

$\exists_x$ is called the Existential Quantifier.

Formation rules.

P (a) is a statement.

$\forall_x$ P (x) is a statement ("For every x, P (x)").

$\exists_x$ P (x) is a statement ("There exists an x, such that P (x)").

Symbols of the form P (x) which are not complete statements, may be combined in the same way as statements with the symbols $\neg$, $\wedge$, $\vee$ .... (See example number 4 below).

Examples of interpretation.

(1) Let x denote any non-negative real number (any value of $x = u^2$, where u is real).

Let P denote the property of being greater than or equal to zero. Then P (x) means "$x \geq 0$" and $\forall_x$ P (x) means "For every x, $x \geq 0$" or "Every x is greater than or equal to zero."

(2) Let a be 3, then P (a) means "$3 \geq 0$."

(3) Suppose y denotes any number from a given set of real numbers.

Then $\neg \exists_y P(y)$ means "It is not the case that there exists a y such that $y \geqslant 0$."

(4) $\exists_y \left[\neg \{ P(y) \} \right]$ means "There exists a y such that $y \not\geqslant 0$."

(5) Let $a = -4$.

Then $\neg P(a)$ means "$-4 \not\geqslant 0$."

<u>Theory</u>

(1) Ax. $\neg \forall_x P(x) \equiv \exists_x [\neg P(x)]$

(2) Th. $\neg \exists_x P(x) \equiv \forall_x [\neg P(x)]$

(3) Ax. $\forall_x P(x) \vdash P(a)$

(4) Th. $P(a) \vdash \exists_x P(x)$

(5) Th. $\forall_x P(x) \vdash \exists_x P(x)$

(6) Ax. $\forall_x [P(x) \wedge Q(x)] \;.\equiv.\; [\forall_x P(x)] \wedge [\forall_x Q(x)]$

(7) Th. $[\exists_x P(x)] \vee [\exists_x Q(x)] \;.\equiv.\; \exists_x [P(x) \vee Q(x)]$

(8) Th. $[\forall_x P(x)] \wedge [\forall_x Q(x)] \vdash [\forall_x P(x)] \wedge [\exists_x Q(x)]$

(9) Th. $[\forall_x P(x)] \vee [\exists_x Q(x)] \vdash [\exists_x P(x)] \vee [\exists_x Q(x)]$

(10) Ax. $[\forall_x P(x)] \wedge [\exists_x Q(x)] \vdash \exists_x [P(x) \wedge Q(x)]$

(11) Th. $\forall_x [P(x) \vee Q(x)] \vdash [\forall_x P(x)] \vee [\exists_x Q(x)]$

(12) Ax. $\exists_x [P(x) \wedge Q(x)] \vdash [\exists_x P(x)] \wedge [\exists_x Q(x)]$

(13) Th. $[\forall_x P(x)] \vee [\forall_x Q(x)] \vdash \forall_x [P(x) \vee Q(x)]$

(14) Th. $[\forall_x P(x)] \wedge [\exists_x Q(x)] \vdash [\forall_x P(x)] \vee [\forall_x Q(x)]$

(15) Th. $[\exists_x P(x)] \wedge [\exists_x Q(x)] \vdash [\forall_x P(x)] \vee [\exists_x Q(x)]$

(16) Th. $[\forall_x P(x)] \Longrightarrow [\exists_x Q(x)] \;.\equiv.\; \exists_x [P(x) \Longrightarrow Q(x)]$

(17) Th. $[\forall_x P(x)] \Longrightarrow [\forall_x Q(x)] \vdash [\forall_x P(x)] \Longrightarrow [\exists_x Q(x)]$

(18) Th. $[\exists_x P(x)] \Longrightarrow [\exists_x Q(x)] \vdash [\forall_x P(x)] \Longrightarrow [\exists_x Q(x)]$

(19) Th. $\forall_x [P(x) \Longrightarrow Q(x)] \vdash [\forall_x P(x)] \Longrightarrow [\forall_x Q(x)]$

(20) Th. $\forall_x [P(x) \Longrightarrow Q(x)] \vdash [\exists_x P(x)] \Longrightarrow [\exists_x Q(x)]$

(21) Th. $[\exists_x P(x)] \Longrightarrow [\forall_x Q(x)] \vdash \forall_x [P(x) \Longrightarrow Q(x)]$

(22) Ax. $\vdash \forall_x T(x)$, where $T(x)$ has the form of a tautology.

(23) Th. $\forall_x [P(x) \Longrightarrow Q(x)], \forall_x [Q(x) \Longrightarrow R(x)] \vdash \forall_x [P(x) \Longrightarrow R(x)]$

Before proving these theorems we have to show that the set of axioms is consistent by using a model as follows.

Let $\forall_x P(x)$ correspond to $P_1 \wedge P_2$ where P_1 and P_2 are statements., and let $\exists_x P(x) \longleftrightarrow P_1 \vee P_2$

Consider Ax.(1) and the correspondences

$$\forall_x P(x) \longleftrightarrow P_1 \wedge P_2,$$

$$\neg \forall_x P(x) \longleftrightarrow \neg P_1 \vee \neg P_2,$$

and

$$\exists_x P(x) \longleftrightarrow P_1 \vee P_2,$$

$$\exists_x [\neg P(x)] \longleftrightarrow \neg P_1 \vee \neg P_2.$$

Since the statements corresponding to $\neg \forall_x P(x)$ and $\exists_x [\neg P(x)]$ are the same, Ax. (1): $\neg \forall_x P(x) \equiv \exists_x [\neg P(x)]$ corresponds to a true statement in the model.

Consider Ax. (10):

$$[\forall_x P(x)] \wedge [\exists_x Q(x)] \vdash \exists_x [P(x) \wedge Q(x)]$$

$$\equiv \vdash [\{\forall_x P(x)\} \wedge \{\exists_x Q(x)\}] \Rightarrow \exists_x [P(x) \wedge Q(x)]$$

$$\equiv \vdash \neg [\{\forall_x P(x)\} \wedge \{\exists_x Q(x)\}] \vee \exists_x [P(x) \wedge Q(x)]$$

$$\equiv \vdash [\neg \{\forall_x P(x)\} \vee \neg \{\exists_x Q(x)\}] \vee \exists_x [P(x) \wedge Q(x)]$$

$$\equiv \vdash [\{\exists_x (\neg P(x))\} \vee \{\forall_x (\neg Q(x))\}] \vee \exists_x [P(x) \wedge Q(x)]$$

$$\longleftrightarrow \vdash [(\neg P_1 \vee \neg P_2) \vee (\neg Q_1 \wedge \neg Q_2)] \vee [(P_1 \wedge Q_1) \vee (P_2 \wedge Q_2)]$$

But the truth value of the compound statement in the last line is

$$t\left[\left\{(\neg p_1 \vee \neg p_2) \vee (\neg q_1 \wedge \neg q_2)\right\} \vee \left\{(p_1 \wedge q_1) \vee (p_2 \wedge q_2)\right\}\right]$$

$$\equiv p_1' + p_2' + q_1' \cdot q_2' + (p_1 \cdot q_1) + (p_2 \cdot q_2)$$

$$\equiv (p_1' + p_1) \cdot (p_1' + q_1) + (p_2' + p_2) \cdot (p_2' + q_2) + q_1' \cdot q_2'$$

$$\equiv 1 \cdot (p_1' + q_1) + 1 \cdot (p_2' + q_2) + q_1' \cdot q_2'$$

$$\equiv (p_1' + q_1) + (p_2' + q_2) + q_1' \cdot q_2'$$

$$\equiv (q_1 + q_1') \cdot (q_1 + q_2') + q_2 + p_1' + p_2'$$

$$\equiv 1 \cdot (q_1 + q_2') + q_2 + p_1' + p_2'$$

$$\equiv (q_1 + q_2') + q_2 + p_1' + p_2'$$

$$\equiv q_1 + (q_2' + q_2) + p_1' + p_2'$$

$$\equiv q_1 + 1 + p_1' + p_2'$$

$$\equiv 1$$

Similarly we can show that the axioms (3), (6), (12), (22), correspond to true statements in the model. Therefore the set of axioms is consistent.

<u>Examples of interpretations.</u>

Let P mean "> 0" and let x be in some set of numbers. Then $\neg \forall_x P(x) \equiv \exists_x [\neg P(x)]$ means that the statement "It is not the case that for all x, $x > 0$" is identical with the statement "There exists an x, such that $x \not> 0$."

Suppose a denotes 3.

Then $\forall_x P(x) \vdash P(a)$ means from the statement "For all x, $x > 0$" we may deduce "$3 > 0$"

Example of proof: th. (2)

Let	$Q(x)$	$\equiv$	$\neg P(x)$	
Since	$\neg\forall_x Q(x)$	$\equiv$	$\exists_x[\neg Q(x)]$	(Ax. 1)
Therefore	$\neg\neg\forall_x Q(x)$	$\equiv$	$\neg\exists_x[\neg Q(x)]$	$(\neg\neg A \equiv A)$
and	$\forall_x[\neg P(x)]$	$\equiv$	$\neg\exists_x[\neg\neg P(x)]$	
		$\equiv$	$\neg\exists_x P(x)$	
Hence	$\forall_x[\neg P(x)]$	$\equiv$	$\neg\exists_x P(x)$	

Example of proof: th. (7)

Let	$R(x)$	$\equiv$	$\neg P(x)$	
and	$S(x)$	$\equiv$	$\neg Q(x)$	
then	$\forall_x[R(x)\wedge S(x)]$	$\equiv$	$[\forall_x R(x)]\wedge[\forall_x S(x)]$	(Ax. 6)
	$\neg\forall_x[R(x)\wedge S(x)]$	$\equiv$	$\neg[\{\forall_x R(x)\}\wedge\{\forall_x S(x)\}]$	
	$\exists_x[\neg\{R(x)\wedge S(x)\}]$	$\equiv$	$[\neg\forall_x R(x)]\vee[\neg\forall_x S(x)]$	
	$\exists_x[\neg R(x)\vee\neg S(x)]$	$\equiv$	$[\exists_x(\neg R(x))]\vee[\exists_x(\neg S(x))]$	
	$\exists_x[P(x)\vee Q(x)]$	$\equiv$	$[\exists_x P(x)]\vee[\exists_x Q(x)]$	

Example of proof: th. (23)

Consider: $\forall_x\left[\{(P(x)\Longrightarrow Q(x))\wedge(Q(x)\Longrightarrow R(x))\}\Longrightarrow(P(x)\Longrightarrow R(x))\right]$

$\vdash \forall_x[(P(x)\Longrightarrow Q(x))\wedge(Q(x)\Longrightarrow R(x))]\Longrightarrow\forall_x[P(x)\Longrightarrow R(x)]$ (Th. 19)

Since $[\{(P(x)\Longrightarrow Q(x))\wedge(Q(x)\Longrightarrow R(x))\}\Rightarrow(P(x)\Longrightarrow R(x))]$

is a tautology.

Therefore by using Ax. (22) and Th. 1 for the theory of deduction (If $\vdash A$ and $A\vdash B$, then $\vdash B$) we obtain

$$\vdash \forall_x[(P(x)\Longrightarrow Q(x))\wedge(Q(x)\Longrightarrow R(x))]\Longrightarrow\forall_x[P(x)\Longrightarrow R(x)]$$

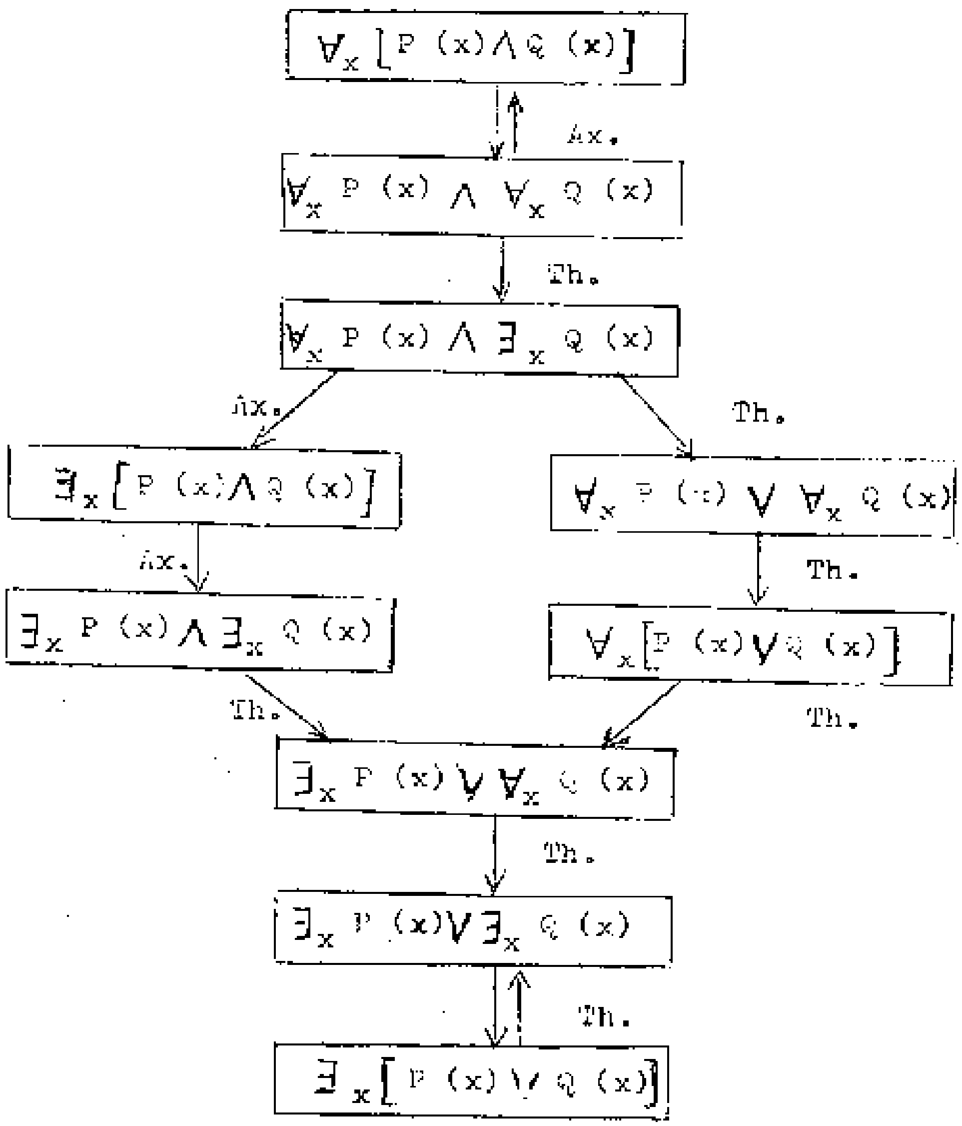
$$\equiv \forall_x[(P(x)\Longrightarrow Q(x))\wedge(Q(x)\Longrightarrow R(x))]\vdash\forall_x[P(x)\Longrightarrow R(x)]$$

$$= \forall_x [P(x) \Longrightarrow Q(x)] \wedge \forall_x [Q(x) \Longrightarrow R(x)] \vdash \forall_x [P(x) \Longrightarrow R(x)]$$

$$= \forall_x [P(x) \Longrightarrow Q(x)], \forall_x [Q(x) \Longrightarrow R(x)] \vdash \forall_x [P(x) \Longrightarrow R(x)]$$

Hence the theorem is proved.

The relations between the statements in the above theory (numbers 6 to 15) may be illustrated by means of the lattice diagram below, where the arrows correspond to the $\vdash$ symbol.



If we consider $\forall_x [P(x) \wedge Q(x)]$ and $\forall_x P(x) \wedge \forall_x Q(x)$ to represent one and the same element in the diagram, and similarly for $\exists_x P(x) \vee \exists_x Q(x)$ and $\exists_x [P(x) \vee Q(x)]$, and we let the arrows represent a partial ordering of the elements in the diagram, then the system is a lattice because (1) The arrows in the system satisfy the three laws of partial ordering (p.336 Birkhoff and Mac Lane) (2) For any pair of elements in the system there is a least upper bound (l.u.b) and a greatest lower bound (g.l.b) (p.351 Birkhoff and Mac Lane)

To prove the arrows satisfy the three laws of partial ordering.

(i) Since $A \vdash A$ for any element A in the system, we can write $A \longrightarrow A$. It follows that the relation represented by the arrow is reflexive.

(ii) For any two elements A and B in the system such that $A \longrightarrow B$ and $B \longrightarrow A$, we have $A \equiv B$. It follows that the relation represented by the arrow is anti-symmetric.

(iii) For any three elements A, B and C in the system such that $A \longrightarrow B$ and $B \longrightarrow C$, we have $A \longrightarrow C$. It follows that the relation represented by the arrow is transitive.

To prove for any pair of elements in the system there is a l.u.b and a g.l.b.

There are 28 pairs of different elements in the system. For each pair (A,B) there exists a C such that $C \longrightarrow A$ and $C \longrightarrow B$, and for any other element D such that $D \longrightarrow A$ and $D \longrightarrow B$ we have $D \longrightarrow C$. C is the l.u.b of A and B.

Similarly there exists an E such that $A \longrightarrow E$ and $B \longrightarrow E$, and for any other element F such that $A \longrightarrow F$ and $B \longrightarrow F$ we have $E \longrightarrow F$. E is the g.l.b of A and B.

For example, if we let $\exists_x [P(x) \wedge Q(x)]$ be A and $\forall_x P(x) \vee \forall_x Q(x)$ be B, we have $\forall_x P(x) \wedge \exists_x Q(x)$ is C, $\forall_x P(x) \wedge \forall_x Q(x)$ is D, $\exists_x P(x) \vee \forall_x Q(x)$ is E and $\exists_x P(x) \vee \exists_x Q(x)$ is F.

In the same way if we let $\forall_x P(x) \vee \forall_x Q(x)$ be A and $\exists_x P(x) \vee \forall_x Q(x)$ be B, we have $\forall_x P(x) \vee \forall_x Q(x)$ is C, $\forall_x P(x) \wedge \exists_x Q(x)$ is D, $\exists_x P(x) \vee \forall_x Q(x)$ is E and $\exists_x P(x) \vee \exists_x Q(x)$ is F.

Similarly we can prove that for any pair of elements in the system there is a l.u.b and a g.l.b.

5.2 <u>The Theory of the Predicate Calculus with Two Variables.</u>

(1)	Ax.	$\forall_x \forall_y P(x,y)$	$=$	$\forall_y \forall_x P(x,y)$
(2)	Th.	$\exists_x \exists_y P(x,y)$	$=$	$\exists_y \exists_x P(x,y)$
(3)	Th.	$\forall_x \forall_y P(x,y)$	$\vdash$	$\exists_x \forall_y P(x,y)$
(4)	Ax.	$\exists_x \forall_y P(x,y)$	$\vdash$	$\forall_y \exists_x P(x,y)$
(5)	Th.	$\forall_x \forall_y P(x,y)$	$\vdash$	$\exists_x \exists_y P(x,y)$
(6)	Th.	$\forall_x \forall_y P(x,y)$	$\vdash$	$\forall_x \exists_y P(x,y)$
(7)	Th.	$\exists_x \forall_y P(x,y)$	$\vdash$	$\exists_x \exists_y P(x,y)$
(8)	Th.	$\forall_x \forall_y P(x,y)$	$\vdash$	$\forall_y P(a,y)$
(9)	Th.	$\forall_y P(a,y)$	$\vdash$	$P(a,b)$
(10)	Th.	$P(a,b)$	$\vdash$	$\exists_y P(a,y)$
(11)	Th.	$\exists_y P(a,y)$	$\vdash$	$\exists_y \exists_x P(x,y)$

Example of proof: th. (2)

Let $Q(x,y) \equiv \neg P(x,y)$

Since
$$\forall_x \forall_y Q(x,y) \equiv \forall_y \forall_x Q(x,y) \quad \text{(Ax. 1)}$$
$$\forall_x \forall_y [\neg P(x,y)] \equiv \forall_y \forall_x [\neg P(x,y)]$$
$$\forall_x [\neg \exists_y P(x,y)] \equiv \forall_y [\neg \exists_x P(x,y)]$$
$$\neg \exists_x \exists_y P(x,y) \equiv \neg \exists_y \exists_x P(x,y)$$
$$\exists_x \exists_y P(x,y) \equiv \exists_y \exists_x P(x,y)$$

Hence the theorem is proved.

Example of proof: th. (7)

Let $Q(y) \equiv \exists_x P(x,y)$

Since
$$\exists_x \forall_y P(x,y) \vdash \forall_y \exists_x P(x,y) \quad \text{(Ax. 4)}$$
$$\exists_x \forall_y P(x,y) \vdash \forall_y Q(y).$$

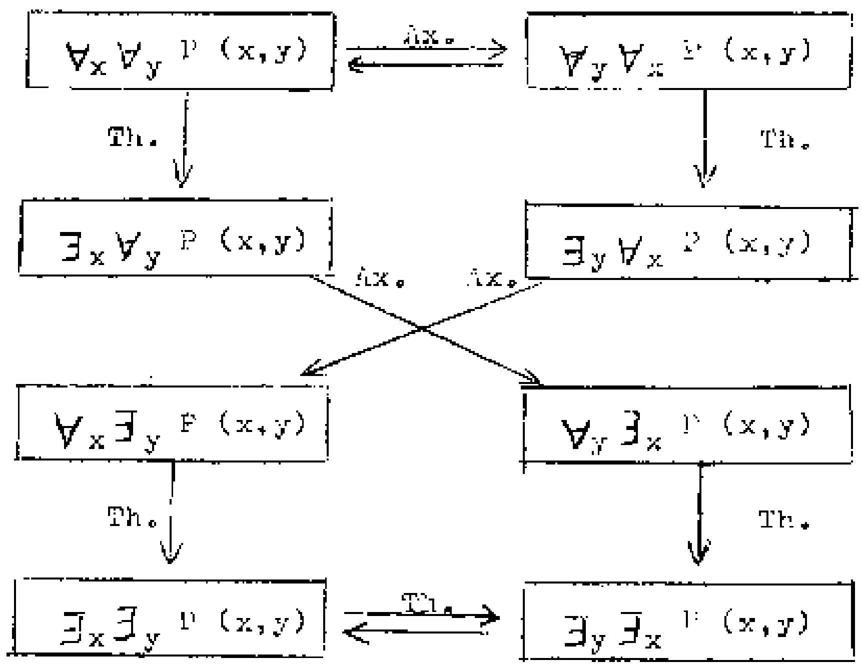
But
$$\forall_y Q(y) \vdash \exists_y Q(y) \quad \text{(Predicate calculus with one variable)}$$

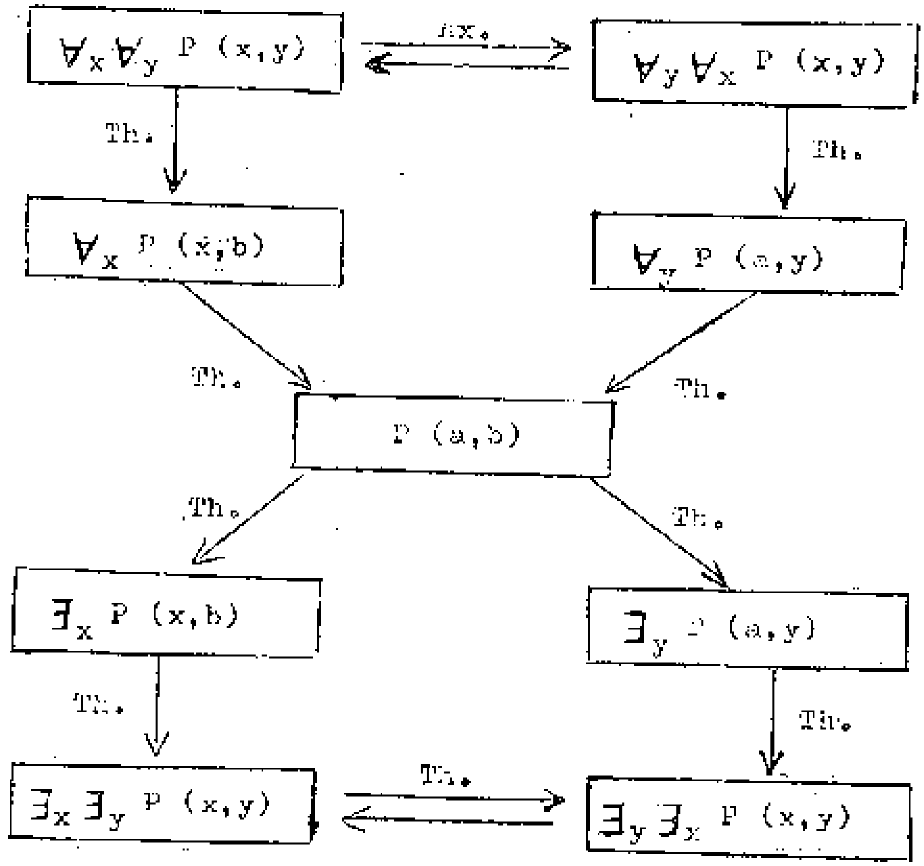
Hence
$$\exists_x \forall_y P(x,y) \vdash \exists_y Q(y) \quad \text{(Ax. for the Theory of Deduction)}$$

and
$$\exists_x \forall_y P(x,y) \vdash \exists_y \exists_x P(x,y),$$
$$\exists_x \forall_y P(x,y) \vdash \exists_x \exists_y P(x,y) \quad \text{(Th. 2)}$$

Hence the theorem is proved.

The relations between the statements in the above theory (numbers 1 to 11) may be illustrated by means of the two lattice diagrams below.





We can show that the systems of elements in the two diagrams above are lattices, by the methods used above for the predicate calculus with one variable.

Statements connected by arrows thus $\rightleftarrows$ are to be considered as one and the same element of the lattices.