CHAPTER IV

TANGENT LINES

Definition IV.1

The line $y = m x + b$ is said to be the tangent line to the curve $y = f(x)$ at the point where $x = h$, if it is the only line which satisfies the following conditions.

1) $f(h) = m h + b$

2) there exists $\mu > o$ such that if $o < \partial < \mu$ then $f(h + \partial) - m(h + \partial) - b$ and $f(h - \partial) - m(h - \partial) - b$ are both positive or both negative.

Theorem IV.2

If $f(x)$ is a polynomial such that the line $y = m x + b$ is tangent to the curve $y = f(x)$ at the point where $x = h$, then $D_x^h f(x) = m$

Proof. Let $g(x) = f(x) - m x - b$

then $g(h) = f(h) - m h - b$

$= o$

and $\begin{cases} g(h + \partial) = f(h + \partial) - m(h + \partial) - b \\ g(h - \partial) = f(h - \partial) - m(h - \partial) - b \end{cases}$

$\therefore$ $g(h + \partial)$ and $g(h - \partial)$ are both positive or both negative

$\therefore$ there exists $\mu > o$, such that, if $o < \partial < \mu$ then either both $g(h + \partial)$ and $g(h - \partial)$ are greater than $g(h)$ or both $g(h + \partial)$ and $g(h - \partial)$ are less than $g(h)$

$\therefore$ $g(x)$ has a minimum or maximum value at $x = h$

$\therefore$ By theorem III.4 $D_x^h g(x) = 0$

$$\therefore \quad D_x^h \{ f(x) - m x - b \} = 0$$

$$\therefore \quad D_x^h f(x) - m = 0$$

i.e. $D_x^h f(x) = m$

Theorem IV.3

If $f(x)$ is a polynomial such that $f(x) - m x$ has a maximum or minimum value at $x = h$, then the line $y - f(h) = m(x - h)$ is the tangent to the curve $y = f(x)$ at the point where $x = h$.

Proof. Since $f(h)$ is a single-valued function, the line

$$y - f(h) = m(x - h) \quad \text{or} \quad y = m x + b,$$

where $b = f(h) - m h$ is unique

Now $m h + b = m h + f(h) - m h$

$= f(h)$ .... (1)

Since $f(x) - m x$ has a maximum or minimum value at $x = h$, there exists $\mu > 0$ such that if $0 < \partial < \mu$ then $f(h + \partial) - m(h + \partial)$ and $f(h - \partial) - m(h - \partial)$ are both less than or both greater than $f(h) - m h$ $(= b)$

i.e. $f(h + \partial) - m(a + \partial) - b$ and $f(h - \partial) - m(h - \partial) - b$ are both negative or both positive .... (2)

(1) and (2) satisfy the definiton IV.1

$\therefore$ the line $y - f(h) = m(x - h)$ is the tengent to the curve $y = f(x)$ at the point where $x = h$

Example IV.4

Find the equation of the line tangent to the curve

$$y = x^2 \quad \text{at} \quad (1, 1)$$

Let $$g(x) = f(x) - \left[D_x^1 f(x) \right] \cdot x$$

$$= x^2 - 2x$$

Let $$h(x) = g(x + 1)$$

$$= (x + 1)^2 - 2(x + 1)$$

$$= x^2 - 1$$

By theorem II.6 $h(x)$ has a minimum value at $x = 0$,

or $g(x)$ has a minimum value at $x = 1$.

$\therefore$ The tangent is $y - 1 = 2(x - 1)$