CHAPTER IV

THE DERIVATION OF THE FORM OF THE FUNCTION f BY DIFFERENTIATION

Differentiating the equation

$$f\left\{uf(v) + vf(u)\right\} = f(u)f(v) - (1 - f(v)^2)u/v$$

with respect to u, holding v constant, we obtain

$$f'\left\{uf(v) + vf(u)\right\}\left\{f(v) + vf'(u)\right\} = f'(u)f(v) - (1 - f(v)^2)/v .$$

When $u = 0$ this becomes

$$f'\left\{vf(0)\right\}\left\{f(v) + vf'(0)\right\} = f'(0)f(v) - (1 - f(v)^2)/v .$$

From equation (3), Chapter II, we have $f(0) = 1$. Therefore the above equation is

$$f'(v)\left\{f(v) + vf'(0)\right\} = f'(0)f(v) - (1 - f(v)^2)/v. \quad (1)$$

The Maclaurin expansions of $f(v)$ and $f'(v)$ are

$$f(v) = f(0) + vf'(0) + (v^2/2)f''(0) + (v^3/3)f'''(0) + \dots\dots\dots\dots,$$

$$f'(v) = f'(0) + vf''(0) + (v^2/2)f'''(0) + \dots\dots\dots\dots\dots\dots$$

Substituting these expansions in (1) we get

$$(f'(0) + vf''(0) + \dots)(f(0) + vf'(0) + \dots + vf'(0))$$

$$= f'(0)(f(0) + vf'(0) + \dots) - \frac{(1 - (f(0) + vf'(0) + \dots)^2}{v}$$

and putting $f(0) = 1$ we have

$$f'(0) + 2v\left\{(f'(0)^2 + f''(0)\right\} + 2v^2 f'(0)f''(0) + \ldots\ldots\ldots$$

$$= f'(0) + v(f'(0)^2) + \ldots + 2f'(0) + v\left\{(f'(0)^2 + f''(0)\right\} +$$

$$+ \ldots\ldots\ldots\ldots$$

Putting $v = 0$, we obtain

$$f'(0) = f'(0) + 2f'(0).$$

Therefore $\quad f'(0) = 0.$

Then equation (1) becomes

$$f'(v)f(v) = (f(v)^2 - 1)/v \qquad (2)$$

or

$$\frac{f(v)df(v)}{f(v)^2 - 1} = \frac{dv}{v}\,. \qquad (3)$$

Integrating we get

$$\tfrac{1}{2}\ln\left|f(v)^2 - 1\right| = \ln|v| - \ln k\,,$$ where k is an arbitrary constant.

Solving for $f(v)$ we find

$$f(v) = \pm(1 \pm v^2/k^2)^{\frac{1}{2}}\,.$$

But $f(0) = 1$, therefore the minus sign outside the bracket does not hold, and $f(v) = (1 \pm v^2/k^2)^{\frac{1}{2}}$.

From equation (18), Chapter II,

$$w = vf(u) + uf(v).$$

Substituting for f we find

$$w = (1 \pm u^2/k^2)^{\frac{1}{2}}v + (1 \pm v^2/k^2)^{\frac{1}{2}}u \qquad (4)$$

which, on squaring each side, gives

$$w^2 = (1 \pm u^2/k^2)v^2 + 2uv(1 \pm u^2/k^2)^{\frac{1}{2}}(1 \pm v^2/k^2)^{\frac{1}{2}} + (1 \pm v^2/k^2)u^2.$$

But w^2 must be real , therefore

$$(1 \pm u^2/k^2)(1 \pm v^2/k^2) \geqslant 0 .$$

This is true for positive signs in the brackets.

If

$$(1 - u^2/k^2)(1 - v^2/k^2) \geqslant 0,$$

then $u^2 > k^2$ implies $v^2 \geqslant k^2$, and

$u^2 < k^2$ implies $v^2 \leqslant k^2$. But u and v are independent.

Therefore we reject the negative signs in the brackets, and conclude that

$$f(v) = (1 + v^2/k^2)^{\frac{1}{2}}. \qquad (5)$$

By expanding the right hand side as a power series in v, we obtain the same result as that in equation (8), Chapter III, when $a_2 = 1/2k^2$. The expression $(1 + v^2/k^2)^{\frac{1}{2}}$ is meaningful when k is infinite, and in this case we obtain the Galilean transformation.

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