CHAPTER II

THE FUNCTIONAL EQUATION FOR f

The basic assumptions for the new derivation of the Lorentz transformation are.

1. The transformation is linear.
2. The composition of two transformations is a transformation of the same form.
3. The transformation is symmetric.

We summerize first the usual application of the principle of relativity to two frames in uniform relative motion to find the form of the transformation. Motion and all measurements are in the x-direction.

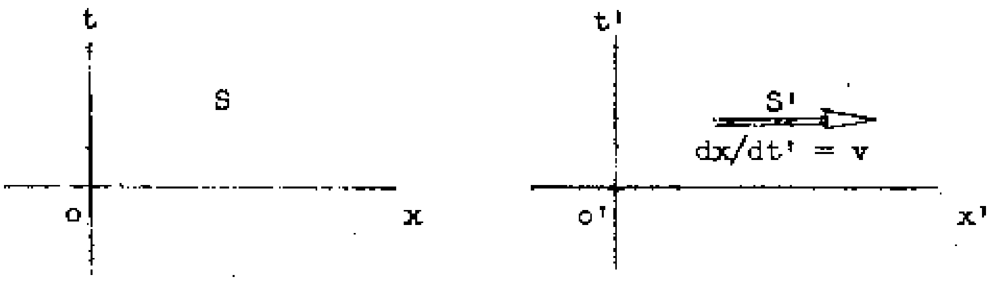

Figure 1. The Motion of S' Relative to S.

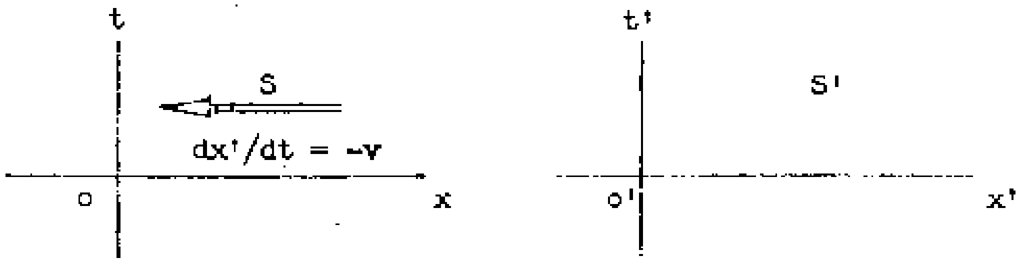

Fig. 2. The Motion of S Relative to S' for the Same System Shown in Fig. 1.

(a) Let $x' = x'(x,t,v)$, $t' = t'(x,t,v)$, (1)

where v is a constant equal to $dx(o')/dt'$ (see Figure 1.). It follows that $dx'(o)/dt = -v$.

$$dx' = x'_x dx + x'_t dt, \qquad dt' = t'_x dx + t'_t dt , \tag{2}$$

Assume that dx' is independent of x and t. Also set the clocks so that when $x = 0$, $t = 0$ we have $x' = 0$, $t' = 0$. These conditions demand that x' and t' must be linear in x and t and must satisfy equations of the form

$$x' = f(v)x + g(v)t, \tag{3}$$

$$t' = h(v)x + k(v)t. \tag{4}$$

(b) By symmetry (see Figure 2.)

$$x = f(-v)x' + g(-v)t', \tag{5}$$

$$t = h(-v)x' + k(-v)t'. \tag{6}$$

From the proper velocity of O' relative to S, we have

$$g(-v) = dx/dt' = v ,$$

and from the proper velocity of O relative to S', we have

$$g(v) = dx'/dt = -v.$$

Therefore

$$x' = f(v)x - vt , \qquad x = f(-v)x' + vt', \tag{7}$$

$$t' = h(v)x + k(v)t, \qquad t = h(-v)x' + k(-v)t', \tag{8}$$

Putting $x = 0$ in (7) and eliminating x' we obtain

$$t' = f(-v)t. \tag{9}$$

Putting $x' = 0$ in (7) and eliminating x we obtain

$$t = f(v)t'. \tag{10}$$

The factors $f(v)$ and $f(-v)$ in (9) and (10) should be the same (by the symmetry assumption). i.e. $f(v) = f(-v)$.

Therefore the transformation has the form

$$x' = f(v)x - vt, \qquad x = f(v)x' + vt'. \tag{11}$$

Solving simultaneously for t' and t we have

$$t' = \left(\frac{1 - f(v)^2}{v}\right)x + f(v)t, \tag{12}$$

$$t = \left(\frac{f(v)^2 - 1}{v}\right)x' + f(v)t'. \tag{13}$$

These results give $h(v)$ and $k(v)$ in equation (4) in terms of $f(v)$ and v.

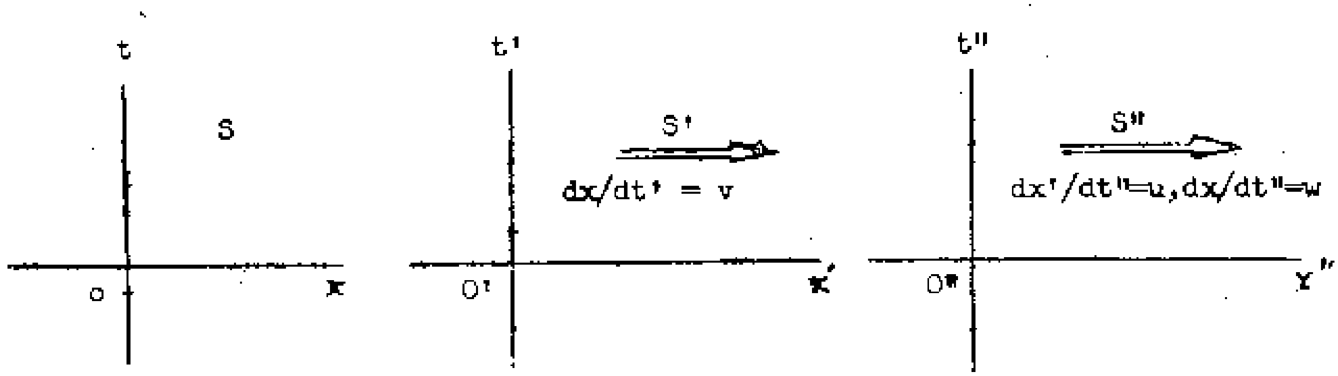

Fig. 3. The Relative Motion of S,S',and S''

Consider three frames in uniform relative motion in the x-direction as shown in Figure 3.

Let the proper velocity of S" relative to S' be dx'/dt" = u, and let the proper velocity of S" relative to S be dx/dt" = w.

By assumption 2. , the transformation from S' to S" must have the same form as that from S to S', as in equation (11). Therefore

$$x'' \quad = \quad f(u)x' - ut'. \qquad (14)$$

Similarly, the transformation from S to S" requires

$$x'' \quad = \quad f(w)x - wt. \qquad (15)$$

Now, substituting x' and t' from (11) and (12) into (14) we have

$$x'' \quad = \left\{ f(u)f(v) - \frac{1 - f(v)^2}{v}u \right\} x - \left\{ vf(u) + uf(v) \right\} t \; . \qquad (16)$$

From eqns. (15) and (16), comparing the coefficients of x we obtain

$$f(w) \quad = \quad f(u)f(v) - \; (1 - f(v)^2)u/v, \qquad (17)$$

and the coefficients of t is

$$w \quad = \quad vf(u) + uf(v). \qquad (18)$$

Therefore substituting w from (18) into (17) we obtain the equation which defines the function f in the form

$$f\left\{ vf(u) + uf(v) \right\} \quad = \quad f(u)f(v) \; - \; (1 - f(v)^2)u/v. \qquad (19)$$